COMMUNICATION SCIENCES

AND

ENGINEERING

XV. PROCESSING AND TRANSMISSION OF INFORMATION*

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A. ERROR BOUNDS FOR THE TURBULENT OPTICAL CHANNEL (I)

Subject to some reasonable assumptions, the atmospheric optical channel can be modeled[1] as shown in Fig. XV-1. We suppose that this channel is used to transmit one of M orthogonal, equal-energy, equi-probable, (complex) waveforms $S_j(t)$; $j = 1, \dots, M$; and that the receiver is to decide, with minimum error probability, which waveform was transmitted. It is known[2] that, in the notation of Fig. XV-1, the appropriate receiver evaluates the quantities

$$L_k = \sum_{i=1}^{D} \ln \left\{ \int du\ p(u)\ I_o(u|y_{ik}|/N_o) \exp\left[-u^2 E_k/N_o\right] \right\} \quad \text{for } k = 1, \dots, M \tag{1}$$

where p(u) is the lognormal density[3]

$$p(u) = (2\pi\sigma^2 u^2)^{-1/2} \exp[-(\sigma^2 + \ln u)^2/2\sigma^2] \tag{2}$$

$$y_{ik} = \int dt\ y_i(t)\ S_k^*(t) \qquad i = 1, \dots, D; \qquad k = 1, \dots, M \tag{3}$$

$$E_k = \frac{Ac}{2} \int |S_k(t)|^2\ dt \qquad k = 1, \dots, M. \tag{4}$$

and $I_o(\cdot)$ is the modified Bessel function. The transmitted waveform is then presumed to be that one, say n, for which

$$L_n \geq L_k \qquad k = 1, \dots, M. \tag{5}$$

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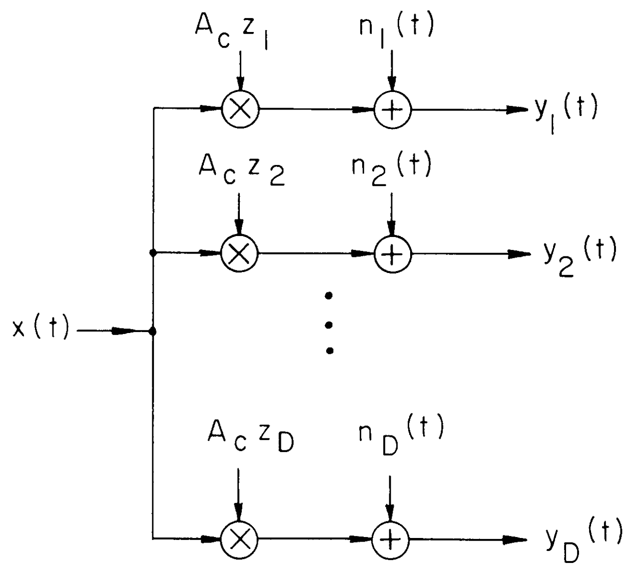

Fig. XV-1. Diversity representation of the turbulent optical channel.

Our objective is to establish the following bound to the error probability $P[\epsilon]$

$$P[\epsilon] \leq 2^{-KE}, \tag{6a}$$

where

$$E = \max_{0\leq\rho\leq 1} \left[\frac{\beta}{\ln 2} E_o(\rho) - \rho\right] \tag{6b}$$

with

$$E_o(\rho) = -\frac{1+\rho}{\alpha_\rho} \ln \left[\int dy\, e^{-y} \left\{\int du\, p(u)\, I_o(2u\sqrt{y\alpha_\rho}) \exp -u^2\alpha_\rho\right\}^{1/(1+\rho)}\right], \tag{6c}$$

where

$$K = \log_2 M \tag{7a}$$

$$\alpha_p = \frac{E_k}{N_o} \tag{7b}$$

$$\beta = \frac{D\alpha_p}{K}. \tag{7c}$$

As a first step in the derivation, we note the following statistical properties of the y_{ik} conditioned upon the knowledge that the n^{th} waveform is transmitted. First, all of the y_{ik} are statistically independent of each other. Second, for $k \neq n$, the y_{ik} are zero-mean complex Gaussian random variables with

$$\overline{(y_{ik})^2} = 0 \tag{8a}$$

and

$$\overline{|y_{ik}|^2} = 4E_kN_o. \tag{8b}$$

Third, conditioned upon a knowledge of z_i, y_{in} is a complex Gaussian random variable with

$$\overline{y}_{in} = 2z_iE_n \tag{9a}$$

$$\overline{(y_{in}-\overline{y}_{in})^2} = 0 \tag{9b}$$

and

$$\overline{|y_{in}-\bar{y}_{in}|^2} = 4E_kN_o. \tag{9c}$$

The preceding properties imply that, conditioned upon the knowledge of the z_i and of the transmitted message, the L_k are statistically independent of each other. Moreover, for $k \neq n$ the L_k are identically distributed. Finally, the distribution for the L_k for $k \neq n$ and also the distribution of L_n are independent of n. Consequently, the error probability conditioned upon the z_i is

$$P[\epsilon|\vec{z}] = 1 - \int dx\, p_o(x|\vec{z}) \left[\int_{-\infty}^{x} dy\, p_i(y)\right]^{M-1}, \tag{10}$$

where $p_o(x|\vec{z})$ is the probability density of the random variable of Eq. 1 when the y_{ik} are conditionally Gaussian with the moments of Eq. 9, and $p_i(x)$ is the density of that variable when the y_{ik} are conditionally Gaussian with the moments of Eq. 8.

To upper-bound $P[\epsilon|\vec{z}]$, we first note that, for $0 \leq \rho \leq 1$,

$$P[\epsilon|\vec{z}] \leq \int dx\, p_o(x|\vec{z}) \left\{1 - \left[1 - \int_x^{\infty} dy\, p_1(y)\right]^{(M-1)}\right\}^{\rho} \tag{11a}$$

$$\leq \int dx\, p_o(x|\vec{z}) \left[(M-1) \int_x^{\infty} dy\, p_1(y)\, dy\right]^{\rho}. \tag{11b}$$

Or, upon introducing the Chernov bound,

$$\int_x^{\infty} dy\, p_1(y) \leq \exp -[tx-\gamma_1(t)] \qquad t \geq 0 \tag{12a}$$

with

$$\gamma_1(t) = \ln\left[\int dy\, p_1(y) \exp ty\right], \tag{12b}$$

we obtain

$$P[\epsilon|\vec{z}] < M^{\rho} \int dx\, p_o(x|\vec{z}) \exp -[\rho tx - \rho\gamma_1(t)]. \tag{13}$$

Averaging Eq. 13 over the random variables z_i and defining

$$\gamma_o(s) = \ln\left\{\int p_{\vec{z}}(\vec{z})\, d\vec{z} \int dx\, p_o(x|\vec{z}) \exp sx\right\} \tag{14}$$

yields

$$P[\epsilon] < M^{\rho} \exp[\rho\gamma_1(t) + \gamma_o(-t\rho)]. \tag{15}$$

To complete the derivation, we require more explicit expressions for $\gamma_1(t)$ and $\gamma_o(s)$. These can be obtained from Eqs. 1, 8, 9, 12, and 14 in conjunction with the properties of the z_i. The result is

$$\gamma_1(t) = D \ln\left\{\int_0^\infty dy\ e^{-y}\left[\int_0^\infty du\ p(u)\ I_o(2u\sqrt{\alpha_p y})\ \exp -u^2\alpha_p\right]^t\right\} \tag{16a}$$

and

$$\gamma_o(s) = D \ln\left\{\int_0^\infty dy\ e^{-y}\left[\int_0^\infty du\ p(u)\ I_o(2u\sqrt{\alpha_p y})\ \exp -u^2\alpha_p\right]^{1+s}\right\}. \tag{16b}$$

We next set $t = (1+\rho)^{-1}$ and combine Eqs. 15 and 16 to obtain

$$P[\epsilon] \leqslant M^\rho \exp(1+\rho)\ \gamma_1\left(\frac{1}{1+\rho}\right)$$

or, by virtue of Eqs. 16a and 6c,

$$P[\epsilon] \leqslant M^\rho \exp -K\beta E_o(\rho).$$

Finally, we express M as 2^K, change from base e to base 2, and maximize the negative of the exponent to obtain the upper bound of Eq. 6a.

R. S. Kennedy, E. V. Hoversten

References

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3. J. Aitchison and J. A. C. Brown, The Lognormal Distribution (Cambridge University Press, London, 1963).