COMMUNICATION SCIENCES

AND

ENGINEERING

XVI. STATISTICAL COMMUNICATION THEORY*

Academic and Research Staff

Prof. Y. W. Lee
Prof. A. G. Bose
Prof. J. D. Bruce
Prof. A. V. Oppenheim
Prof. D. E. Nelsen

Graduate Students

D. K. Campbell
T. Huang
V. Nedzelnitsky
P. E. Perkins
S. D. Personick

Undergraduate Students

J. P. Morgenstein
E. Rosenfeld

A. WORK COMPLETED

1. A THREE-STATE AMPLIFICATION SYSTEM

This study has been completed by P. E. Perkins. In May 1968, he submitted the results to the Department of Electrical Engineering, M. I. T., as a thesis in partial fulfillment of the requirements for the degrees of Electrical Engineer and Master of Science.

A. G. Bose

2. AN EXPERIMENTAL INVESTIGATION OF SPECTRAL FLUCTUATIONS IN NONSTATIONARY ACOUSTICAL NOISE

This study has been completed by W. A. Taylor. In May 1968, he submitted the results to the Department of Electrical Engineering, M. I. T., as a thesis in partial fulfillment of the requirements for the degree of Master of Science.

A. G. Bose

3. NOISE DUE TO TIME-VARYING CURRENT EXCITATION OF CARBON RESISTORS

This study has been completed by S. D. Personick. In May 1968, he submitted the results to the Department of Electrical Engineering, M. I. T., as a thesis in partial fulfillment of the requirements for the degree of Master of Science.

D. E. Nelsen

*This work was supported in part by the Joint Services Electronics Programs (U. S. Army, U. S. Navy, and U. S. Air Force) under Contract DA 28-043-AMC-02536(E), and in part by the National Aeronautics and Space Administration (Grant NGR-22-009-304).

4. AN INVESTIGATION OF JITTER IN A SILICON-CONTROLLED RECTIFIER

This study has been completed by J. P. Morgenstein. In May 1968, he submitted the results to the Department of Electrical Engineering, M. I. T., as a thesis in partial fulfillment of the requirements for the degree of Bachelor of Science.

D. E. Nelsen

B. SMALL-SIGNAL METHOD FOR DETERMINING JITTER IN REGENERATIVE SWITCHING CIRCUITS

1. Introduction

When a ramp is applied to the input of a regenerative switching circuit, the circuit will switch approximately when the ramp crosses a threshold level that is characteristic of the switch. If this experiment is repeated many times, the switching instants will be found to be randomly distributed about some mean value. We have found a small-signal method for relating the switching-time randomness, or jitter, to noise generated within the components of the circuit or present at the input. This method should be widely applicable to different types of regenerative switches. When applied to the tunnel diode switch, which has been investigated previously by this writer,[1,2] not only are the results obtained consistent with the previous results, but new insight into the jitter mechanism is gained.

For the tunnel diode switch we could previously only relate the jitter statistics to either very wideband or very low frequency noise; with the new approach, the jitter statistics (variance and mean) can be simply expressed in terms of noise having an arbitrary power spectrum, provided that the noise is small enough.

In the previous analysis of jitter in the tunnel diode, we first obtained the dynamic equilibrium equation for the circuit which was valid during and immediately before the switching transient. This equation, which we call the "switching equation," is a nonlinear differential equation relating the tunnel diode voltage to the input ramp slope, circuit parameters, and noise in the circuit. This equation was then solved for the mean, variance, and distribution of the jitter by using Monte Carlo methods with a digital computer. The solution was obtained for white noise.

The new method of analysis obtains a solution of the switching equation by using small-signal perturbation techniques. The result, although limited to noise of low amplitude, is valid for noise having an arbitrary spectrum and yields much more insight into the mechanism of coupling between noise and jitter.

2. Switching Equation for the Tunnel Diode Switch

We shall apply the small-signal technique to the switching equation describing the tunnel diode switch. The method should, however, be usable with switching equations for other types of regenerative switches.

The derivation of the switching equation for the tunnel diode is described in detail in other reports by this writer.[1,2] Briefly, the equation is obtained by writing the equilibrium equation for a tunnel diode model that is valid near the peak of the diode's i-v characteristic. The nonlinear shape of the i-v curve at its peak is preserved by approximating it with a parabola. The small-signal noise sources that are valid in the vicinity of the peak are included in the model. The switching equation is

$$C \frac{dv'}{dt'} - kv'^2 = \alpha t' + n'(t'), \tag{1}$$

where C is the diode's junction capacitance in the peak region, k is one-half the magnitude of the second derivative of the i-v curve (evaluated at the peak), α is the slope of the input ramp, and $n'(t')$ is noise present when operating near the peak of the i-v curve. This noise was predominantly shot noise. By substituting

$$v = \left(\frac{k^2}{C\alpha}\right)^{1/3} v', \qquad t = \left(\frac{k\alpha}{C^2}\right)^{1/3} t', \qquad n = \left(\frac{k}{C^2\alpha^2}\right)^{1/3} n' \tag{2}$$

in Eq. 1, the switching equation in dimensionless form,

$$\frac{dv}{dt} - v^2 = t + n(t), \tag{3}$$

results. The dimensionless power spectrum $\Phi_{nn}(\omega)$ of the noise n(t) is expressible in terms of its dimensional counterpart $\Phi_{nn}(\omega')$ as

$$\Phi_{nn}(\omega) = \left(\frac{k}{C^2\alpha}\right) \Phi'_{nn}\left[\omega\left(\frac{C^2}{k\alpha}\right)^{1/3}\right]. \tag{4}$$

In this report, variables will be primed if they are dimensional, otherwise they will be unprimed.

3. Small-Signal Solution of the Switching Equation

With the noise n(t) set to zero, the solution of the switching equation (3) is shown in Fig. XVI-1. The initial condition used, $v_o = -\sqrt{|t_o|}$ for $t_o \ll 0$, places the operating point on the i-v relation at time $t_o \ll 0$.

Soon after the input ramp, t, crosses through zero at t = 0, the voltage goes to infinity at time T_{so}. (If the actual S-shaped tunnel diode characteristic were used instead of the parabola, v(t) would have saturated out at some higher level slightly before time T_{so}.)

If noise were present, the entire waveform v(t) would fluctuate randomly; the switching time T_s (T_{so} denotes the switching time with the noise n(t) = 0), would be a

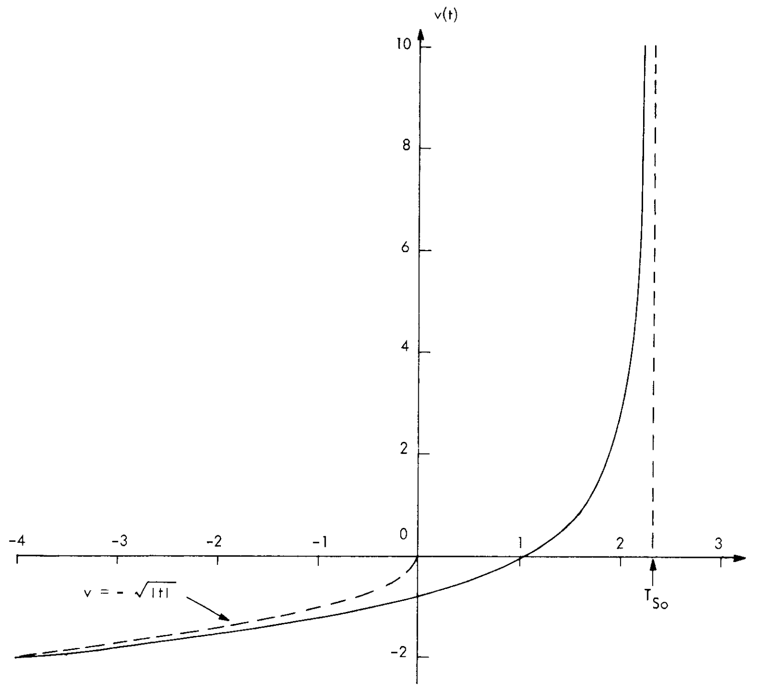

Fig. XVI-1. Solution of the dimensionless switching equation (3) with $n(t) = 0$. Initial condition was $v_o = -\sqrt{|t_o|}$ for $t_o \ll 0$. The switching time T_{so} is shown.

random variable and depend upon the entire past history of the noise before switching occurs. One would expect noise fluctuations at times long before the switching time T_s to have a negligible effect on the final switching time. Furthermore, perturbations very close to the final switching time T_s would have a small effect on T_s. This can be seen by writing Eq. 3 in the form

$$\frac{dv}{dt} = v^2 + t + n(t). \tag{5}$$

When v becomes large, $v^2 \gg t + n(t)$, and (5) asymptotically becomes

$$\frac{dv}{dt} \cong v^2, \tag{6}$$

which is independent of the noise.

If the noise is small enough, it is reasonable to assume that the change in the switching time, $T_s - T_{so}$, because of the noise, will depend "linearly" on the noise that occurred before time T_{so}. More specifically, we would expect that

$$(T_s - T_{so}) \cong \int_{-\infty}^{T_{so}} h_1(\tau)\, n(\tau)\, d\tau, \tag{7}$$

where $h_1(\tau)$ is a weighting function, and $n(t)$ is the noise signal that appeared in the switching equation (3). This means that the perturbation of the switching time is a linear superposition of the "past" of the noise before T_{so}.

If $n(t) = u_o(t-t_1)$ is substituted in Eq. 7, then $(T_s - T_{so}) = h_1(t_1)$. This means that $h_1(t_1)$ is the change in $T_s - T_{so}$ resulting from a unit impulse occurring at time t_1. Thus the integral of Eq. 7 is equivalent to the superposition integral of linear system theory.

The form of Eq. 7 suggests that $T_s - T_{so}$ might be expressed more accurately as a Volterra series defined over $n(t)$ for $t < T_{so}$. Then $T_s - T_{so}$ would be

$$T_s - T_{so} = \int_{-\infty}^{T_{so}} h_1(\tau)\, n(\tau)\, d\tau + \int_{-\infty}^{T_{so}} \int_{-\infty}^{T_{so}} h_2(\tau_1, \tau_2)\, n(\tau_1)\, n(\tau_2)\, d\tau_1 d\tau_2$$

$$+ \dots + \int_{-\infty}^{T_{so}} \dots \int_{-\infty}^{T_{so}} h_n(\tau_1, \dots, \tau_n)\, n(\tau_1) \dots n(\tau_n)\, d\tau_1 \dots d\tau_n + \dots, \tag{8}$$

where the kernel $h_n(\tau_1, \dots \tau_n)$ is an n-dimensional weighting function.

Since perturbations long before T_{so} and immediately before T_{so} have negligible effect on the switching time, $h_n(\tau_1, \dots \tau_n) \to 0$ as any of $\tau_1, \tau_2 \dots \tau_n \to -\infty$, or any of $\tau_1, \tau_2 \dots \tau_n \to T_{so}$. We can simplify limits on the integrals of Eq. 8 by defining $h_n(\tau_1, \dots, \tau_n) = 0$ for any of $\tau_1, \tau_2, \dots \tau_n > T_{so}$, and replace the T_{so} limits by $+\infty$ to obtain

$$T_s - T_{so} = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots \tau_n)\, n(\tau_1) \dots n_n(\tau_n)\, d\tau_1 \dots d\tau_n. \tag{9}$$

We shall find it comforting that only the first two terms of this very formidable series are needed to accurately describe the mean and variance of T_s over a wide range of operation of the tunnel diode switch.

4. Properties of the Kernels, $h_n(\tau_1, \dots, \tau_n)$

Some properties of the kernels of the integrals in Eq. 9 can be easily derived. First, $h_1(t) \le 0$ for all t. If in the switching equation (3) $n(t)$ is a small positive perturbation occurring at time t_1, such as $\epsilon u_o(t-t_1)$ where ϵ is small, the switching time will always occur earlier. This means that $h_1(t) \le 0$.

Another property is obtained by letting $n(t)$ be the constant, γ. Then Eq. 9 becomes

$$T_s - T_{so} = \sum_{n=1}^{\infty} \gamma^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n)\, d\tau_1 \dots d\tau_n. \tag{10}$$

Substituting $n(t) = \gamma$ in the switching equation (3), we obtain

$$\frac{dv}{dt} - v^2 = t + \gamma. \tag{11}$$

Observe that this simply causes switching to occur γ "seconds" earlier, or

$$T_s = T_{so} - \gamma. \tag{12}$$

By equating coefficients of γ^n in Eqs. 10 and 12, we obtain

$$\int_{-\infty}^{\infty} h_1(\tau)\, d\tau = -1 \tag{13}$$

and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots \tau_n)\, d\tau_1 \ldots d\tau_n = 0 \qquad \text{for } n > 1. \tag{14}$$

If $n'(t') = \gamma$ were substituted in the dimensional switching equation (1), the dimensional result corresponding to (13) would be

$$\int_{-\infty}^{\infty} h_1'(\tau')\, d\tau' = -\frac{1}{a}, \tag{15}$$

where a is the slope of the input ramp.

5. Mean and Variance of T_s for White Noise

The mean $\overline{T_s}$ and variance σ_T^2 of T_s when $n(t)$ is white noise will now be obtained. This case is important because the predominant noise source in the tunnel diode model is shot noise (which can be modeled as white noise over the diode's bandwidth.)

The mean $\overline{T_s}$ and variance σ_T^2 of T_s can be obtained by taking the appropriate statistical averages of Eq. 9. If $n(t)$ is white with autocorrelation $\phi_{nn}(\tau) = N_o u_o(\tau)$, $\overline{T_s}$ and σ_T^2 become, keeping only the lowest terms containing N_o,

$$\overline{T_s} = T_{so} + N_o \int_{-\infty}^{\infty} h_2(\tau, \tau)\, d\tau \tag{16}$$

and

$$\sigma_T^2 = N_o \int_{-\infty}^{\infty} h_1^2(\tau)\, d\tau. \tag{17}$$

If we make the additional assumption that the noise is Gaussian, then all of the higher order terms (which involve statistics of order higher than two) can be evaluated. Of interest is inclusion of the second term in the expansion of σ_T^2, which is

$$\sigma_T^2 = N_o \int_{-\infty}^{\infty} h_1^2(\tau)\, d\tau + 2N_o^2 \int_{-\infty}^{\infty} h_2^2(\tau, \tau)\, d\tau. \tag{18}$$

If $h_2(t,t)$ ($h_2(t_1,t_2)$ evaluated along the diagonal $\tau_1=\tau_2=t$) can be evaluated, then the size of the second term can be evaluated as a check on the accuracy of Eq. 17.

We might comment here that the dimensionless spectral height, N_o, is related to the dimensional height N_o' by

$$N_o = \left(\frac{k}{C^2 a}\right) N_o'. \tag{19}$$

Even if the actual noise in the circuit, N_o', is constant, the effective noise, N_o, can be increased by decreasing C or a, or by increasing k. Therefore we would expect the small-signal expressions (16) and (17) for the jitter to break down if the input slope, a, is made too small.

6. Computation of $h_1(t)$ and $h_2(t,t)$

Since the results of Eqs. 16 and 17 are similar in form to those obtained in our previous analysis[1,2] of the switching equation, we calculated with a digital computer $h_1(t)$, $h_2(t,t)$, and the integrals in Eqs. 16-18 involving these functions. With these computed quantities, we could compare the statistics of Eqs. 16 and 17 with those previously obtained by other methods, thereby checking the validity of this method.

The functions can be determined by substituting in Eq. 9, $n(t) = \epsilon u_o(t-t_1)$. Then (9) becomes

$$T_s(\epsilon, t_1) = T_{so} + \epsilon h_1(t_1) + \epsilon^2 h_2(t_1, t_1) + \dots . \tag{20}$$

The kernels $h_1(t_1)$ and $h_2(t_1,t_1)$ are readily obtained as

$$h_1(t_1) = \left[\frac{dT_s(\epsilon, t_1)}{d\epsilon}\right]_{\epsilon=0}$$

and

$$h_2(t_1, t_1) = \frac{1}{2}\left[\frac{d^2 T_s(\epsilon, t_1)}{d\epsilon^2}\right]_{\epsilon=0}. \tag{21}$$

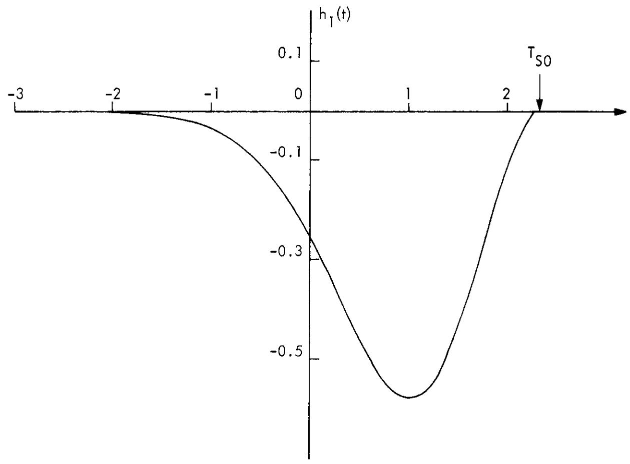

Fig. XVI-2. Graph of (dimensionless) $h_1(t)$. Points were calculated on a digital computer.

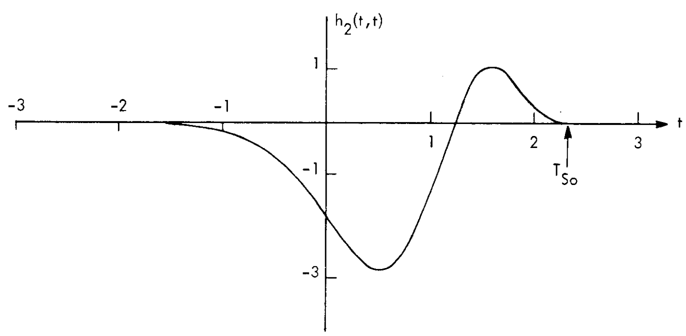

Fig. XVI-3. Graph of (dimensionless) $h_2(t,t)$, the second-order kernel, $h_2(\tau_1, \tau_2)$, evaluated along the line $\tau_1 = \tau_2 = t$.

To perform the computation, we first determined $v(t)$ and T_{so} (shown in Fig. XVI-1) for the case $n(t) = 0$, with the initial condition $v_o = -\sqrt{|t_o|}$, $t_o \ll 0$. Then, to find $T_s(\epsilon, t_1)$, we again solved the noiseless equation, this time using as the initial condition, $t_o = t_1$ and $v_o = v(t_1) + \epsilon$. Then $h_1(t_1)$ and $h_2(t_1, t_1)$ were straightforwardly obtained by calculating the first and second derivatives of $T_s(\epsilon, t_1)$ in Eq. 20. Graphs of $h_1(t)$ and $h_2(t,t)$ are shown in Figs. XVI-2 and XVI-3. Integrals of $h_1(t)$, $h_1^2(t)$, $h_2(t,t)$ and $h_2^2(t,t)$ were computed and found to be

$$\int_{-\infty}^{\infty} h_1(t)\, dt = -1.000 \qquad \int_{-\infty}^{\infty} h_1^2(t)\, dt = 0.417$$

$$\int_{-\infty}^{\infty} h_2(t,t)\, dt = -0.279 \qquad \int_{-\infty}^{\infty} h_2^2(t,t)\, dt = 0.074 \tag{22}$$

Substituting these in Eqs. 16 and 18, we obtain

$$\overline{T_s} = 2.33 - 0.279\ N_o \tag{23}$$

and

$$\sigma_T^2 = 0.417\ N_o + 0.148\ N_o^2. \tag{24}$$

The corresponding solutions previously obtained by using direct Monte Carlo methods can be approximated algebraically by

$$T_s = 2.33 - 0.30\ N_o \tag{25}$$

and

$$\sigma_T^2 = 0.42\ N_o \qquad \text{for } 0 < N_o < 2. \tag{26}$$

Results involving the first-order $h_1(t)$ agree remarkably well with the results of previous analyses: the first term of Eq. 24 agrees with Eq. 26 within the accuracy of (26), and $\int_{-\infty}^{\infty} h(t)\, dt = -1$ to four significant figures. Equation 23 involving $h_2(t.\,t)$ is not as accurate – probably on account of round-off error in the computation of the second derivative of $T_s(\epsilon;t_1)$, most of which could have been avoided by using double-precision accuracy in the computation (we used single-precision). The amount of computer time used for the small-signal method was much less than that used with the Monte Carlo method.

7. Statistics $\overline{T_s}$ and σ_T^2 for Noise with Arbitrary Power Spectrum (or Autocorrelation)

If n(t) is a zero-mean stationary random process having an arbitrary autocorrelation function $\phi_{nn}(\tau)$, we can write, in terms of the lowest order nonzero terms,

$$\overline{T_s} = T_{so} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)\ \phi_{nn}(\tau_1 - \tau_2)\ d\tau_1 d\tau_2 \tag{27}$$

and

$$\sigma_T^2 = \int_{-\infty}^{\infty} \phi_{hh}(\tau)\ \phi_{nn}(\tau)\ d\tau, \tag{28}$$

where

$$\phi_{hh}(\tau) = \int_{-\infty}^{\infty} h_1(t)\ h_1(t+\tau)\ dt.$$

Expressed in the frequency domain, Eqs. 27 and 28 would be

$$\sigma_T^2 = \int_{-\infty}^{\infty} |H_1(\omega)|^2 \, \Phi_{nn}(\omega) \frac{d\omega}{2\pi} \tag{29}$$

and

$$\overline{T_s} = T_{so} + \int_{-\infty}^{\infty} H_2(\omega, -\omega) \, \Phi_{nn}(\omega) \frac{d\omega}{2\pi}, \tag{30}$$

where

$$H_1(\omega) = \int_{-\infty}^{\infty} h(t) \, e^{-j\omega t} \, dt$$

and

$$H_2(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) \exp(j\omega_1\tau_1 + j\omega_2\tau_2) \, d\tau_1 d\tau_2.$$

8. Physical Interpretation of These Results

To gain maximum insight into the results just obtained, they should be transformed back to the dimensional domain, so that the parameter dependences will become evident. By using the transformations in Eq. 2, $h_1(t)$ becomes

$$h'(t') = \left(\frac{k}{C^2 a^2}\right)^{1/3} h\left[t' \left(\frac{ka}{C^2}\right)^{1/3}\right]. \tag{31}$$

The function $h'(t')$ (primed variables are dimensional) can be visualized as an indicator of the sensitivity of the switch to noise. The larger h' is at time t_1', the larger the effect of a perturbation at time t_1'.

An increase in C has the effect of making h' wider in time and smaller in amplitude, keeping the area under h' constant, equal to $1/a$. Since h' becomes wider in time and smaller in amplitude, a longer sample of the noise will be averaged, thereby resulting in a smaller amount of jitter. Thus there is a trade-off between the switching speed and jitter in a tunnel diode switch, since switching speed varies inversely with C.

An increase in slope a causes h' to become narrower in time and smaller in amplitude. Thus, as one would intuitively expect, the amount of jitter decreases with increasing slope.

The dimensional form of Eqs. 28 is

$$\sigma'^2_T = \left(\frac{k}{C^2 a^5}\right)^{1/3} \int_{-\infty}^{\infty} \phi_{hh}\left[\tau'\left(\frac{ka}{C^2}\right)^{1/3}\right] \phi'_{nn}(\tau')\, d\tau', \tag{32}$$

where the dimensional $\phi'_{hh}(\tau')$ is expressed in terms of the dimensionless $\phi_{hh}(\tau)$ as

$$\phi'_{hh}(\tau') = \left(\frac{k}{C^2 a^5}\right)^{1/3} \phi_{hh}\left[\tau'\left(\frac{ka}{C^2}\right)^{1/3}\right].$$

The dimensional form of Eq. 29 is

$$\sigma'^2_T = \frac{1}{a^2} \int_{-\infty}^{\infty} \left| H\left[\omega'\left(\frac{C^2}{ka}\right)^{1/3}\right]\right|^2 \Phi'_{nn}(\omega') \frac{d\omega'}{2\pi}, \tag{33}$$

where the dimensional $H'(\omega')$ is expressed in terms of the dimensionless $H(\omega)$ as

$$H'(\omega') = \frac{1}{a} H\left[\omega'\left(\frac{C^2}{ka}\right)^{1/3}\right].$$

When the bandwidth of the noise is much narrower than that of the switch $[\Phi'_{nn}(\omega')$ is much narrower than $|H'(\omega')|^2]$, then

$$\sigma'_T = \frac{\sqrt{\phi_{nn}(0)}}{a}. \tag{34}$$

This form also results when $a \to \infty$, since the width of $|H'(\omega')|^2$ increases with a.

When the noise bandwidth is much wider than that of the switch, then

$$\sigma'_T = \frac{Ak^{1/6}(\Phi_{nn}(0))^{1/2}}{a^{5/6}\, C^{1/3}}, \tag{35}$$

where $A = \int_{-\infty}^{\infty} h_1^2(t)\, dt = 0.417$. This form also occurs for a fixed noise spectrum when a is decreased, since $|H'(\omega')|$ becomes narrow and a is decreased. Equation 35 describes only the lowest order term of the Volterra series; higher order terms become more significant as a is decreased. Therefore (35) will not continue to be valid if the noise is too large, or if a is decreased too much.

The results (34) and (35) were obtained in the previous analysis[1,2] of the tunnel diode switching equation. Equation 35 describes the experimentally observed tunnel diode jitter over a wide operating range.

For a typical 1-mA tunnel diode with C = 5 pF, $k = 1/7\ A/V^2$, and $a = 1$ A/sec, one dimensionless unit of time would correspond to 56 nsec wide, $h_1'(t')$ would be approximately 150 nsec wide, and the width of $H'(\omega')$ (an effective "bandwidth" of the switch) would be approximately 13 mHz. If a were 100 A/sec, then $h_1'(t_1)$ would be approximately 30 nsec wide, and the "bandwidth" would be approximately 60 mHz.

9. Conclusions

We have presented a small-signal method for determining jitter in regenerative switching circuits. When applied to the tunnel diode switch, the method quite dramatically provided new insight into the mechanism whereby noise fluctuations cause jitter. Although, in practice, this method is only applicable when the noise is "small enough," the fact that the method works over a wide operating range for the tunnel diode switch indicates that the noise in that switch is indeed "small enough" over that range.

It appears that this method for determining jitter in switches is a very powerful technique; application of the technique to other regenerative switching circuits may yield valuable circuit design techniques for minimizing jitter in those circuits.

D. E. Nelsen

References

1. D. E. Nelsen, "Statistics of Switching-Time Jitter for a Tunnel Diode Threshold-Crossing Detector," 1966 IEEE International Convention Record, Part 7, pp. 288-295.
2. D. E. Nelsen, "Statistics of Switching-Time Jitter for a Tunnel Diode Threshold-Crossing Detector," Technical Report 456, Research Laboratory of Electronics, M. I. T., Cambridge, Mass., August 31, 1967.
3. D. E. Nelsen, "Investigation of Switching Jitter in Physical Electronic Switches," Quarterly Progress Report No. 86, Research Laboratory of Electronics, M. I. T., July 15, 1967, pp. 215-219.