XXIII. DETECTION AND ESTIMATION THEORY*

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RESEARCH OBJECTIVES AND SUMMARY OF RESEARCH

The work of our group can be divided into three major areas.

1. Sonar

The central problem is still the development of effective processing techniques for the output of an array with a large number of sensors. We have developed a state-variable formulation for the estimation and detection problem when the signal is a sample function from a nonstationary process that has passed through a dispersive medium. Work has continued in this area and we are attempting to find effective solution procedures for the equations that result. Iterative techniques to measure the interference and modify the array processor are also being studied. The current work includes both the study of fixed and adaptive arrays.

2. Communications

a. Digital Systems

We have continued to work on the problem of evaluating the performance in the problem of detecting Gaussian signals in Gaussian noise. The results are being applied to a number of design problems in the radar and sonar fields. The problem of suboptimal receiver design is being studied, at the present time.

The study of digital and analog systems operating when there is a feedback channel available from receiver to transmitter continues. The performance of several suboptimal systems has been computed, and work continues on the design of optimal systems and related problem design.

b. Analog Systems

Work continues on the problem of estimating continuous waveforms in real time. We are using a state-variable approach based on Markov processes. Several specific problems have been studied experimentally. In order to investigate the accuracy obtainable and the complexity required in these systems by allowing the state equation to be nonlinear, we may also include interesting problems such as parameter estimation. This modification has been included and several parameter estimation problems are being studied experimentally.

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3. Random Process Theory and Applications

a. State-Variable and Continuous Markov Process Techniques

Previously, we have described an effective method for obtaining solutions to the Fredholm integral equation. As a part of this technique we found the Fredholm determinant. Subsequent research has shown that a similar determinant arises in a number of problems of interest. Specifically, we have been able to formulate several interesting design problems and carry through the solution. Work in this area continues.

b. System Identification Problem

The system identification problem is still an item of research. Applications of interest include measurement of spatial noise fields, random-process statistics, and linear system functions.

c. Detection Techniques

Various extensions of the Gaussian detection problem are being studied. A particular topic of current interest is the detection of non-Gaussian Markov processes.

H. L. Van Trees

A. BARANKIN BOUND ON THE VARIANCE OF ESTIMATES OF THE PARAMETERS OF A GAUSSIAN RANDOM PROCESS

1. Introduction

In many problems of interest, one wants to determine how accurately he can estimate parameters that are imbedded in a random process. Often this accuracy is expressed in terms of a bound, usually the Cramer-Rao bound.[1] Unless, however, one can verify the existence of an efficient or possibly an asymptotically efficient estimate for this bound, the tightness of the result remains a question. This is particularly true in the threshold region.

The issue of tighter bounds then remains. Probably the most common response to this is the Bhattacharyya bound.[1] One can derive, however, a bound that is optimum, in that it is the tightest possible. This is the Barankin bound.[2]

Two forms for this bound have appeared. We present them briefly here. Barankin has shown that for any unbiased estimate $\hat{a}(R)$, the greatest lower bound on the variance is given by

$$\operatorname{Var}\left[\hat{a}(\underline{R})-A\right] \geq \max_{\{a_i\},\{h_i\}} \frac{\left[\sum_{i=1}^{n} a_i h_i\right]^2}{E\left\{\left[\sum_{i=1}^{n} a_i \frac{p_{\underline{r}|a}(\underline{R}|A+h_i)}{p_{\underline{r}|a}(\underline{R}|A)}\right]^2\right\}}. \tag{1}$$

The maximization is done over the set $\{a_i\}$ of arbitrary dimension, n, and the set $\{h_i\}$ such that the ratio $p_{\underline{r}|a}(\underline{R}|A+h_i)/p_{\underline{r}|a}(\underline{R}|A) \in L_2$ and A + h is contained in the parameter space.

Keifer has put the bound as expressed in Eq. 1 in functional form.[3,4]

$$\text{Var}\left[\hat{a}(R)-A\right] \geq \max_{[f(H)]} \frac{\left[\int Hf(H)\,dH\right]^2}{E\left\{\left[\int \frac{p_{\underline{r}|a}(\underline{R}|A+H)}{p_{\underline{r}|a}(\underline{R}|A)} f(H)\,dH\right]^2\right\}}. \tag{2}$$

[Note: Barankin considered the bound for the s^{th} absolute central moment, while Keifer considered the variance only; however, his derivation is easily generalized by applying a Minkowski inequality rather than the Schwarz inequality. Keifer also assumed that f(H) could be expressed in terms of the difference of two densities. Although the optimum choice of f(H) has this property, there is no apparent reason to assume this a priori.]

Several applications of this bound to estimating the parameters of processes have appeared in the literature.[5-8] These all consider the problem of estimating the parameter when it is imbedded in the mean of a Gaussian random process, that is,

$$r(t) = m(t, A) + n(t), \quad T_o \leq t \leq T_f, \tag{3}$$

where m(t, A) is conditionally known to the receiver, and n(t) is an additive noise that is a Gaussian random process whose statistical description does not depend on A.

In this report we want to consider the situation when one has

$$r(t) = s(t:A) + n(t), \quad T_o \leq t \leq T_f, \tag{4}$$

where s(t:A) is a Gaussian random process whose covariance, $K_s(t, u:A)$, depends on A, and n(t) is an additive Gaussian noise. For simplicity, we assume: (a) that s(t:A) has a zero mean, and (b) n(t) is a white process of spectral height $N_o/2$. Putting a finite mean simply introduces terms whose analysis has already been treated.[5-8] If n(t) is nonwhite, it can be reduced to the case above by the customary whitening arguments.

We shall use Keifer's expression in Eq. 2 in our development. It is useful to write Eq. 2 in a slightly different form by exchanging the squaring and expectation operation in the denominator. This yields

$$\text{Var}\left[\hat{a}(R)-A\right] \geq \max_{f(H)} \left\{\frac{\left[\int Hf(H)\,dH\right]^2}{\iint f(H_1)\,G(H_1, H_2:A)\,f(H_2)\,dH_1 dH_2}\right\}, \tag{5}$$

where all integrals are over the domain of definition for $G(H_1, H_2:A)$, and

$$G(H_1, H_2:A) = E\left\{\frac{p_{\underline{r}|a}(\underline{R}|A+H_1)}{p_{\underline{r}|a}(\underline{R}|A)} \frac{p_{\underline{r}|a}(\underline{R}|A+H_2)}{p_{\underline{r}|a}(\underline{R}|A)}\right\}$$

$$= \int\!\!\cdots\!\!\int_{\underline{R}} \left\{\frac{p_{\underline{r}|a}(\underline{R}|A+H_1)\, p_{\underline{r}|a}(\underline{R}|A+H_2)}{p_{\underline{r}|a}(\underline{R}|A+H_2)}\right\} d\underline{R}. \tag{6}$$

In this form there are essentially three relatively distinct issues in calculating the bound:

1. an effective computational method for evaluating the function $G(H_1, H_2:A)$;
2. the optimal choice of the function f(H) so as to maximize the right side of Eq. 5; and
3. effective computational alogrithms for implementing the bound.

We shall focus our discussion on the evaluation of $G(H_1, H_2:A)$, since this is the manner in which applying the bound for estimating the parameters of process essentially differs from what has been done previously. We shall then make some comments on the second and third issues as they relate to applying the bound.

2. Evaluation of the Function $G(H_1, H_2:A)$

When the parameter to be estimated is imbedded simply in the mean of the observed process, the calculation of $G(H_1, H_2:A)$ is relatively straightforward.[5, 6] We want to consider the situation when the parameter is imbedded in the covariance function. Essentially, our approach is first to assume that we are working with a sampled version of the signal, next to derive the resulting bound, and then to let the sampling interval become vanishingly small.

Let us assume that we are using a finite time averaging for our sampler. Then we observe samples of the form

$$r_n = \frac{1}{\Delta T} \int_{n\Delta T}^{(n+1)\Delta T} r(\tau)\, d\tau. \tag{7}$$

We can consider the r_n to form a vector $\underline{r}$. If ΔT is small compared with any correlation times of s(t:A), then the elements of the covariance matrix of $\underline{r}$, which we denote by $K_{\underline{r}}(A)$, is given by

$$[K_{\underline{r}}(A)]_{nm} = K_s(n\Delta T, m\Delta T:A) + \frac{N_o}{2} \frac{\delta_{nm}}{\Delta T}, \tag{8}$$

so that we have

$$p_{\underline{r}|a}(\underline{R}|A) = \frac{1}{(2\pi)^{N/2} \det^{1/2}[K_{\underline{r}}(A)]} \exp\left[-\frac{1}{2}\underline{R}^T K_{\underline{r}}^{-1}(A)\underline{R}\right]. \tag{9}$$

Substituting this expression in that for $G(H_1, H_2:A)$, Eq. 6, we obtain

$$G(H_1, H_2:A) = \frac{1}{(2\pi)^{N/2}} \frac{\det^{1/2}[K_{\underline{r}}(A)]}{\det^{1/2}[K_{\underline{r}}(A+H_1)]\det^{1/2}[K_{\underline{r}}(A+H_2)]}$$

$$\times \int_R \exp\left[-\frac{1}{2}\underline{R}^T\left[K_{\underline{r}}^{-1}(A+H_1) + K_{\underline{r}}^{-1}(A+H_2) - K_r^{-1}(A)\right]\underline{R}\right] d\underline{R}. \tag{10a}$$

Two observations are important here.

1. For the integral to be convergent, the matrix

$$K_r^{-1}(A+H_1) + K_r^{-1}(A+H_2) - K_r^{-1}(A)$$

must be positive definite. It is easy to construct examples for which this would not be true; consequently, there is often an inherent limitation on our choice of H_1 and H_2 for any particular example.

2. The matrix above is symmetric; therefore, if the integral does exist, that is, it is convergent, it can be integrated conveniently by putting it into the form of a multivariate Gauss density. Doing this yields

$$G(H_1, H_2:A) = \left\{\frac{\det[K_{\underline{r}}(A)]\det\left[K_{\underline{r}}^{-1}(A+H_1) + K_{\underline{r}}^{-1}(A+H_2) - K_{\underline{r}}^{-1}(A)\right]^{-1}}{\det[K_{\underline{r}}(A+H_1)]\det[K_{\underline{r}}(A+H_2)]}\right\}^{1/2}. \tag{10b}$$

We now use some of the properties of determinants to obtain

$$G(H_1, H_2:A) = \left\{\frac{\det\left[K_{\underline{r}}(A+H_1) + K_{\underline{r}}(A+H_2) - K_{\underline{r}}(A+H_1)K_{\underline{r}}(A+H_2)K_{\underline{r}}^{-1}(A)\right]}{\det[K_{\underline{r}}(A)]}\right\}^{-1/2}$$

$$= \det[K_{\underline{r}}(A)]\det^{-1/2}\{(K_{\underline{r}}(A+H_1) + K_{\underline{r}}(A+H_2))K_{\underline{r}}(A) - K_{\underline{r}}(A+H_1)K_{\underline{r}}(A+H_2)\}. \tag{11}$$

We now want to let the sampling interval approach zero so as to collapse this to

functional form. First, we separate the white-noise component. Substituting Eq. 8 in Eq. 11 gives

$$G(H_1, H_2:A) = \det\left[\underline{K_s}(A) + \frac{N_o}{2}\frac{I}{\Delta T}\right]$$

$$\times \det^{-1/2}\left[\underline{K_s}(A+H_1)\,\underline{K_s}(A) + \frac{N_o}{2}\frac{1}{\Delta T}(\underline{K_s}(A+H_1)+\underline{K_s}(A)) + \left(\frac{N_o}{2}\right)^2\frac{I}{(\Delta T)^2}\right.$$

$$+ \underline{K_s}(A+H_2)\,\underline{K_s}(A) + \frac{N_o}{2}\frac{1}{\Delta T}(\underline{K_s}(A+H_2)+\underline{K_s}(A)) + \left(\frac{N_o}{2}\right)^2\frac{I}{(\Delta T)^2}$$

$$\left. - \underline{K_s}(A+H_1)\,\underline{K_s}(A+H_2) - \frac{N_o}{2}\frac{1}{\Delta T}(\underline{K_s}(A+H_1)+\underline{K_s}(A+H_2)) + \left(\frac{N_o}{2}\right)^2\frac{I}{(\Delta T)^2}\right]$$

$$= \det\left[I + \frac{2}{N_o}\underline{K_s}(A)\,\Delta T\right]$$

$$\times \det^{-1/2}\left[I + \frac{2}{N_o}\left\{\left[\frac{2}{N_o}\underline{K_s}(AH_1)\,\underline{K_s}(A)\,\Delta T + \frac{2}{N_o}\underline{K_s}(A+H_2)\,\underline{K_s}(A)\,\Delta T\right.\right.\right.$$

$$\left.\left.\left. - \frac{2}{N_o}\underline{K_s}(A+H_1)\,K_s(A+H_2)\,\Delta T\right] + 2\underline{K_s}(A)\right\}\right]. \tag{12}$$

Next, we define the kernel

$$\mathscr{L}[t, \tau:H_1, H_2, A] \triangleq \frac{2}{N_o}\left[\int_{T_o}^{T_f} K_s(t, u:A+H_1)\,K_s(u, \tau:A)\,du\right.$$

$$+ \int_{T_o}^{T_f} K_s(t, u:A+H_2)\,K_s(u, \tau:A)\,du$$

$$\left. - \int_{T_o}^{T_f} K_s(t, u:A+H_1)\,K_s(u, \tau:A+H_2)\,du\right] + 2K_s(t, \tau). \tag{13}$$

If we examine the second determinant in Eq. 12, we find that we have the identity matrix plus $2/N_o$ times $L(H_1, H_2, A)$, which is the sampled version of the kernel $\mathscr{L}(t, u:H_1, H_2, A)$; that is, we have

$$G(H_1, H_2:A) = \det\left[I + \frac{2}{N_o}\underline{K_s}(A)\,\Delta T\right] \times \det^{-1/2}\left[I + \frac{2}{N_o}L(H_1, H_2:A)\,\Delta T\right]. \tag{14}$$

If we now let the sample interval approach zero, we observe that each of the determinants becomes a Fredholm determinant evaluated at $2/N_o$, or

$$\lim_{\Delta T \to 0} \det\left[I + \frac{2}{N_o} \underline{K_s}(A)\, \Delta T\right] = \mathscr{D}_{\mathscr{F}}\left(\frac{2}{N_o} : K_s(t,u:A)\right), \tag{15}$$

$$\lim_{\Delta T \to 0} \det\left[I + \frac{2}{N_o}\left\{\left[\frac{2}{N_o} \underline{K_s}(A+H_1)\, \underline{K_s}(A)\, \Delta T + \frac{2}{N_o} \underline{K_s}(A+H_2)\, \underline{K_s}(A)\, \Delta T - \frac{2}{N_o} \underline{K_s}(A+H_1)\, \underline{K_s}(A+H_2)\, \Delta T\right] + 2\underline{K_s}(A)\right\}\right]$$

$$= \lim_{\Delta T \to 0} \det\left[I + \frac{2}{N_o} L(H_1, H_2, A)\right] = \mathscr{D}_{\mathscr{F}}\left(\frac{2}{N_o} : \mathscr{L}(t,u:H_1,H_2:A)\right). \tag{16}$$

[We note that the Fredholm determinant has the familiar form

$$\mathscr{D}_{\mathscr{F}}(z:K(t,u:A)) = \prod_{i=1}^{\infty} (1+z\lambda_i(A))$$

with the $\lambda_i(A)$ being the eigenvalues of the homogeneous Fredholm integral equation associated with the kernel $K(t,u:A)$.] The expression for $G(H_1, H_2:A)$ then becomes

$$G(H_1, H_2:A) = \mathscr{D}_{\mathscr{F}}\left(\frac{2}{N_o} : K_s(t,u:A) \times \mathscr{D}_{\mathscr{F}}^{-1/2}\left(\frac{2}{N_o} : \mathscr{L}(t,u:H_1,H_2:A)\right)\right). \tag{17}$$

Equation 17 is our basic result. Unless the computation of each of these Fredholm determinant expressions is tractable, we have not made much progress. Some observations are useful. The first Fredholm determinant is a constant with respect to H_1 and H_2 and it is often encountered in problems in communication theory. Its evaluation is well understood. The operator $\mathscr{L}[t,u:H_1,H_2:A]$ does not have any of the properties with which we are accustomed to working. It is generally nonsymmetric; and although we can guarantee that the determinant is real and positive for any value of its argument $z > 2/N_o$, the operator itself is not necessarily positive definite.

There are two important situations in which we can evaluate the determinants by methods that do not involve a direct computation of the eigenvalues of the kernels. The first concerns stationary processes observed over long time intervals, while the second is for the case in which the random process s(t:A) has a state variable of its generation. The former is easier both to derive and describe, and we can discuss it conveniently here. The latter is straightforward, but lengthy; therefore, we simply point out the essential points and material needed to derive its realization, and defer the actual derivation to a reference.

We assume that both s(t:A) and n(t) are stationary processes and that the interval length, $T = T_f - T_o$, is long compared with the effective correlation time of s(t:A). Under these assumptions, K and $\mathscr{L}$ both approach being a stationary operator and their eigenvalues have the same distribution as their respective spectra. For the first determinant we have

$$\ln \mathscr{D}_{\mathscr{F}}\left(\frac{2}{N_o} : K(t,u:A)\right) = \sum_j \ln\left(1 + \frac{2}{N_o}\lambda_j(A)\right)$$

$$\approx T\int_{-\infty}^{\infty} \ln\left(1 + \frac{2}{N_o} S_s(\omega:A)\right)\frac{d\omega}{2\pi}. \tag{18}$$

It is easy to verify by simple Fourier transform operations that the spectrum associated with $\mathscr{L}[t,u:H_1,H_2:A]$ is given by

$$S_L(\omega:H_1,H_2:A) = \frac{2}{N_o}\left\{\left[S_s(\omega:A+H_1) + S(\omega:A+H_2)\right] S_s(\omega:A)\right.$$

$$\left. - S_s(\omega:A+H_1)\, S_s(\omega:A+H_2)\right\} + 2S_s(\omega:A). \tag{19}$$

Therefore, we have

$$\ln \mathscr{D}_{\mathscr{F}}\left(\frac{2}{N_o} : \mathscr{L}[t,u:H_1,H_2:A]\right) = T\int_{-\infty}^{\infty} \ln\left(1 + \frac{2}{N_o} S_L(\omega:H_1,H_2:A)\right)\frac{d\omega}{2\pi}. \tag{20}$$

Exponentiating and substituting in the expression for $G(H_1,H_2:A)$, we obtain

$$G(H_1,H_2:A) = \exp\left\{T\int_{-\infty}^{\infty}\left[\ln\left(1 + \frac{2}{N_o} S_s(\omega:A)\right) - \frac{1}{2}\ln\left(1 + \frac{2}{N_o} S_L(\omega:H_1,H_2:A)\right)\right]\frac{d\omega}{2\pi}\right\} \tag{21}$$

with $S_L(\omega:H_1,H_2:A)$ given by Eq. 19.

We can obtain one simple check of the above result. It is straightforward to demonstrate that our original bound given by Eq. 2 reduces to the Cramer-Rao bound if we choose f(H) to be a doublet, $u_1(H)$. This yields

$$\text{Var}\left[\hat{a}(R)-A\right] \geq \frac{1}{\left.\dfrac{\partial^2 G(H_1,H_2:A)}{\partial H_1 \partial H_2}\right|_{H_1=H_2=0}} = \frac{1}{E\left\{\left[\dfrac{\partial \ln p_{\underline{r}|a}(\underline{R}|A)}{\partial A}\right]^2\right\}}. \tag{22}$$

Straightforward differentiation and noting that

$$1 + \frac{2}{N_o} S_L(\omega:0, 0:A) = \left(1 + \frac{2}{N_o} S_s(\omega:A) \right)^2 \tag{23}$$

yields

$$\left. \frac{\partial^2 G(H_1, H_2:A)}{\partial H_1 \partial H_2} \right|_{H_1=H_2=0} = \frac{T}{2} \int_{-\infty}^{\infty} \left\{ \frac{\frac{2}{N_o} \frac{\partial S_s(\omega:A)}{\partial A}}{1 + \frac{2}{N_o} S_s(\omega:A)} \right\}^2 \frac{d\omega}{2\pi}$$

$$= \frac{T}{2} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial A} \ln \left(1 + \frac{2}{N_o} S_s(\omega:A) \right) \right\}^2 \frac{d\omega}{2\pi}. \tag{24}$$

Substituting this in Eq. 22 yields the Cramer-Rao bound for estimating the parameters of a random process.[10]

We wish to obtain realization of the bound when s(t:A) is generated by a system described by state variables. We already know how to evaluate the Fredholm determinant associated with the operation K(t, u:A).[11, 12] The calculation for the determinant associated with $\mathscr{L}(t, u:H_1, H_2:A)$ can be derived in a manner that is essentially analogous to the derivation for K(t, u:A) performed previously.[11] We outline this derivation briefly now.

1. Referring to Eq. 13 which defines the operator $\mathscr{L}[t, u:H_1, H_2:A]$, we see that we can implement its operation with the block diagram in Fig. XXIII-1.

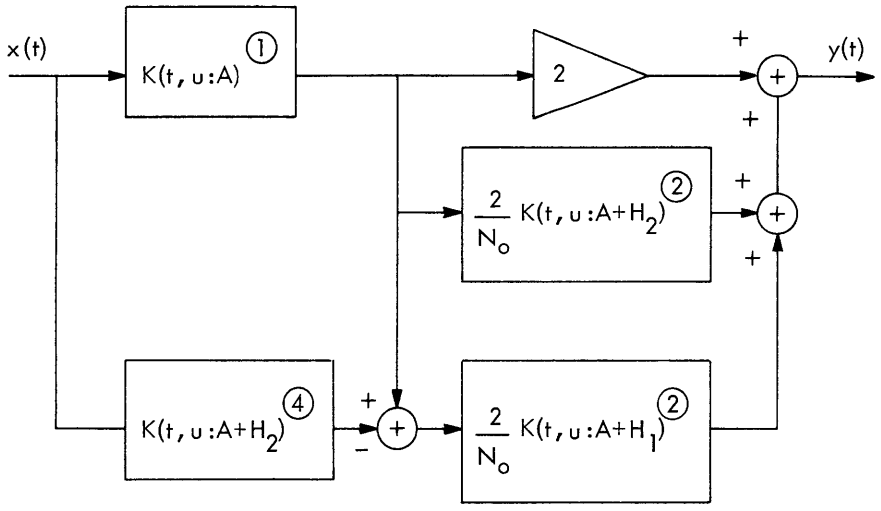

Fig. XXIII-1. Realization of the operator $\mathscr{L}[t, u:H_1, H_2:A]$ in terms of covariance operations.

Using previous results[11], we know how to specify a state-variable description of each of the blocks in this diagram; hence the total system has a state-variable description.

2. Using this description and imposing the eigenfunction condition

$$\int_{T_o}^{T_f} \mathscr{L}[t, u:H_1, H_2:A]\, \phi(u)\, du = \lambda\phi(t), \qquad T_o \leq t \leq T_f, \tag{25}$$

allows us to specify a transcendental equation in terms of λ whose roots are the eigenvalues of Eq. 25. We apply a normalization to this equation and evaluate it at $\lambda = -2/N_o$ to yield the desired Fredholm determinant. We emphasize that given the state-variable realization of the operator, the steps in the derivation are essentially parallel to those developed previously.[11] The specific results are given elsewhere.[13]

The major difficulty with this approach is that the system in Fig. XXIII-1 implies an 8N dimensional system, where N is the dimension of the system generating s(t:A). This large dimensionality imposes stringent computational demands for even relatively small N, especially when the time interval, T, is large. We hope that in many particular applications this dimensionality can be reduced; for as it currently stands, an approach using sampling, for example, evaluation of Eq. 11 directly, may be more expedient from a practical computational viewpoint. We also comment that the state-variable realization of the Cramer-Rao inequality can be derived from these results.

3. Discussion

We have discussed a method of using the Barankin bound for bounding estimates of the parameters of Gaussian random processes. We focused our attention on calculating $G(H_1, H_2:A)$. The optimum choice of f(H) can be shown[6] to be

$$f(H_1) = \int G^{-1}(H_1, H_2:A)\, H_2\, dH_2,$$

where $G^{-1}(H_1, H_2:A)$ is the inverse kernel associated with $G(H_1, H_2:A)$, so that the bound is given by

$$\mathrm{Var}\left[\hat{a}(R)-A\right] \geq \iint H_1 G^{-1}(H_1, H_2:A)\, H_2\, dH_1 dH_2.$$

Due to the complexity of $G(H_1, H_2:A)$ as a function of H_1 and H_2, one must resort to numerical methods for calculating either $G^{-1}(H_1, H_2:A)$ or the bound. This introduces the problem of effective computational procedures. We shall defer this issue until we have obtained more complete numerical results than we have at present. Some of our comments, however, have been summarized elsewhere.[13]

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