COMMUNICATION SCIENCES

AND

ENGINEERING

VIII. PROCESSING AND TRANSMISSION OF INFORMATION*

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A. MULTIPLE FAULT DETECTION IN COMBINATIONAL LOGIC NETWORKS

A Multiple Fault Test Set (MFTS) for a combinational logic network is a set V of input vectors which detects all multiple stuck-at-0 (s-a-0) or s-a-1 faults; that is, for any disjoint sets of leads L_0 and L_1, there exists an input vector $\overline{v} \in V$ such that, when $\overline{v}$ is applied to the network, the output of the network, when all leads in L_0 are s-a-0 and leads in L_1 are s-a-1, is different from the output of the correctly operating network. Minimum MFTS's can be found by using the usual fault table approach; however, this is not feasible for even relatively small networks, since both dimensions of the fault table grow exponentially (the number of multiple faults in an m-lead network is 3^m-1). An alternative approach is to first find a test set that detects all single s-a-0, s-a-1 faults (an SFTS), and then add input vectors to it to form an MFTS. In this report we shall summarize results that justify this approach by indicating that an SFTS usually detects most multiple faults, and then discuss the progress that has been made toward finding methods to determine those input vectors that must be added to an SFTS to form an MFTS.

1. Primitive Faults

In order to reduce the number of multiple faults which must be considered, we define a primitive multiple fault. A multiple fault is primitive if

1. No lead that is an input to an OR gate is s-a-1.
2. No lead that is an input to an AND gate is s-a-0.
3. No lead that is an input to an inverter is stuck.
4. For every stuck lead ℓ, there is a path from ℓ to a primary output containing no stuck leads.
5. There are no gates with all inputs stuck.

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It can be shown that any test set that detects all primitive multiple faults is an MFTS, since any multiple fault that is not primitive can be shown to be equivalent to some primitive fault (that is, it causes the same terminal behavior for the network).

The number of primitive multiple faults in a fanout-free network can be shown to be exactly 2^n, where n is the number of input leads to the network. This number is a lower bound for networks with fanout. Some primitive multiple faults may also be equivalent; however, it can be shown that a network has at least 2^i nonequivalent multiple faults, where i is the number of primary inputs.

2. Critical Faults

It can be shown that certain multiple faults are detected by any SFTS. A critical multiple fault is defined as a multiple fault that is not detected by at least one SFTS. Thus, in extending an SFTS into an MFTS, one need only consider critical multiple faults. The following multiple faults have been shown to be noncritical.

1. All multiple faults in a nonredundant one- or two-level network (a network is nonredundant if all single faults in it can be detected), and all multiple faults in any one- or two-level nonredundant, single-output, fanout-free portion of a larger network.
2. All multiple faults in a nonredundant cascade network, or in any nonredundant, fanout-free cascade portion of a larger network.
3. Any fault in a fanout-free network consisting of fewer than 4 stuck leads.

3. Extending an SFTS to an MFTS

Poage[1] has described an algorithm for computing the set of critical faults for a network; unfortunately, this is also too time-consuming to be practical for large networks. A simpler problem is to find the multiple faults not detected by a specific SFTS. Also, a specific SFTS will usually detect many critical multiple faults. In fact, for those examples that have been tried, most SFTS's detect all multiple faults, and those that do not detect all but one or two primitive multiple faults (these examples have been fairly small, but even small networks have a large number of multiple faults).

Several algorithms have been investigated for finding the multiple faults not detected by a given SFTS. A simple simulation approach has been programmed for research purposes, and works well for small networks (10-15 leads). Certain simplifications are taken advantage of for fanout-free networks, to permit analysis of somewhat larger networks of this type (10-15 input leads). The exponential growth of computing time with the number of leads makes this impractical for larger networks, but the program has proved useful for evaluating other methods.

Another method, which at one time appeared very promising, has been investigated in detail. It works extremely well for some networks and test sets, but very badly for others. "Passing" the analysis performed on a test set turns out to be a sufficient condition for the set to be an MFTS, but not a necessary condition. Test sets have been found that do not "pass" the analysis, but can be shown by simulation to be MFTS's. The method does give some insight into how to make an SFTS more likely to detect many multiple faults, and has led to the proof of several results, including: A fanout-free network with all inputs independent has at least one minimum SFTS which is also an MFTS.

Another approach that is being investigated is to eliminate as many multiple faults as possible before using a simulation method. For example, the noncritical faults listed above can be eliminated immediately. Also, it can be shown that any input vector in an SFTS detects at least 2^{m-1} multiple faults (where m is the number of leads in the network), and these can be determined relatively easily from a single simulation of the network for this input vector. The problem is to find some way of doing this without first constructing a table of all multiple faults and then "marking off" those known to be detected, since storing just this table may be prohibitive.

After finding the multiple faults that are not detected by an SFTS, the input vectors for detecting these multiple faults must be found. This is done most easily by using a sensitized path algorithm such as described by Roth.[2] A program for finding SFTS's using this approach has been written, and the modifications for finding input vectors that detect multiple faults are not difficult.

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References

1. J. F. Poage, "Derivation of Optimal Tests to Detect Faults in Combination Circuits," Mathematical Theory of Automata (Polytechnic Press, New York, 1963), p. 483.
2. J. P. Roth, "Diagnosis of Automata Failures, A Calculus and a Method," IBM J., July 1966, p. 278.

B. OPTIMAL MEAN-SQUARE ESTIMATION IN Q-M CHANNELS

1. Introduction

This report is concerned with single-parameter minimum mean-square estimation in channels that must be modeled in the framework of quantum mechanics. Previous work in the area has been published by Helstrom[1] and Liu,[2] and are suggested as references for situations in which the quantum aspects of a channel are important. We shall derive the optimal mean-square estimator (which is assumed to be Hermitian) and the associated minimum mean-square error. This will be followed by a derivation of the quantum-mechanical equivalent of the Cramér-Rao bound for the estimation of a random variable.

2. Optimal Hermitian Operator

Let A be a random variable with a priori probability distribution given by $\phi(u)$, that is, Pr $(u \leqslant a \leqslant u+du) = \phi(u)\, du$.

Let H be the Hilbert space representing the state (quantum field) in the volume of ordinary space in which the receiver may operate. Let ρ^a be defined as the density operator associated with H, given that "a" is the value of the unknown parameter A.

Conditioned on the true value of A, the expected value of the squared error associated with any Hermitian estimator G is given by $E(G-A)^2\big|_a = \mathrm{TR}\ \rho^a(G-aI)^2$.

Thus the average squared error of the estimator G is

$$E(G-A)^2 = \int \phi(a)\ \mathrm{TR}\ (\rho^a[G-aI]^2)\ da = \mathscr{E}^2(G).$$

Let $\mathscr{E}^2(\hat{G})$ be the minimum of $\mathscr{E}^2(G)$ over all Hermitian operators G, that is,

$$\mathscr{E}^2(\hat{G}) \leqslant \mathscr{E}^2(G) \quad \text{for all } G.$$

Let $L = G + \gamma\Delta$, where Δ is a Hermitian operator, and γ is a real number.

Then $\mathscr{E}^2(L) \geqslant \mathscr{E}^2(\hat{G})$, that is,

$$\int \phi(a)\ \mathrm{TR}\ [\rho^a(\hat{G}-aI)^2]\ da \leqslant \int \phi(a)\ \mathrm{TR}\ [\rho^a(\hat{G}-aI+\gamma\Delta)^2]\ da.$$

In other words

$$\mathscr{E}^2(\hat{G}) \leqslant \int \phi(a)\ \mathrm{TR}\ [\rho^a\{(\hat{G}-aI)^2+(\gamma\Delta)^2+\gamma(\hat{G}-aI)\Delta+\gamma\Delta(\hat{G}-aI)\}]\ da.$$

Since the trace is a linear operation,

$$\gamma \int \phi(a)\ \mathrm{TR}\ [\rho^a\{(\hat{G}-aI)\Delta+\Delta(\hat{G}-aI)\}]\ da \geqslant -\gamma^2 \int \phi(a)\ \mathrm{TR}\ [\rho^a\Delta^2]\ da$$

for all Hermitian operators Δ, which implies that the operator $\hat{G}$ must satisfy

$$\int \phi(a)\ \mathrm{TR}\ \{\rho^a[(\hat{G}-aI)\Delta+\Delta(\hat{G}-aI)]\}\ da = 0, \qquad \text{for all } \Delta.$$

Let $\Gamma = \int \phi(a)\ \rho^a\ da \quad \eta = \int a\phi(a)\ \rho^a\ da$. We have $\mathrm{TR}\ [\Gamma\hat{G}\Delta+\Gamma\Delta\hat{G}] = \mathrm{TR}\ [2\eta\Delta]$ for all Hermitian operators Δ. This equation is satisfied only if $\Gamma\hat{G} + \hat{G}\Gamma = 2\eta$. If Γ is positive definite, then the solution to this equation exists and is unique. That the solution is

$$\hat{G} = 2\int_0^{\infty} e^{-\Gamma\alpha}\, \eta\, e^{-\Gamma\alpha}\, d\alpha$$

can be checked by direct substitution and integration by parts. It is clear that $\hat{G}$ is indeed the optimal operator. Note that $\hat{G}$ is Hermitian, as required.

3. Quantum Equivalent of the Cramér-Rao Bound

Helstrom[3] has derived the quantum-mechanical equivalent of the Cramér-Rao bound for minimum-variance estimation of an unknown parameter. Following is the quantum equivalent for the minimum mean-square error estimate of a random variable.

Let A be a random variable with distribution $\phi(a)$. Let ρ^a be the density operator for the receiver Hilbert space H. Let $\hat{A}$ be any Hermitian operator, and

$$B(a) = \mathrm{TR}\ \rho^a(\hat{A}-aI).$$

We shall assume that $B(a)\ \phi(a)\big|_{-\infty \text{ or } +\infty} \doteq 0.$

$$\frac{\partial}{\partial a} B(a)\ \phi(a) = \frac{\partial\phi(a)}{\partial a}\ [\mathrm{TR}\ \rho^a(\hat{A}-aI)] + \phi(a)\ \mathrm{TR}\left[\left(\frac{\partial}{\partial a}\rho^a\right)(\hat{A}-aI)\right] - \phi(a)\ \mathrm{TR}\ (\rho^a).$$

Integrate the equation above over "a", and we obtain

$$1 = \int_{-\infty}^{+\infty} \frac{\partial\phi(a)}{\partial a}\ [\mathrm{TR}\{\rho^a(\hat{A}-aI)\}] + \phi(a)\left[\mathrm{TR}\ \left(\frac{\partial}{\partial a}\rho^a\right)(\hat{A}-aI)\right] da.$$

Let

$$\frac{\partial}{\partial a}\rho^a = \frac{1}{2}\,(L\rho^a+\rho^a L),$$

where L is the symmetrized logarithmic derivative

$$1 = \int \phi(a)\ \mathrm{RL}\left[\mathrm{TR}\left\{\left(\frac{\partial \ln\phi(a)}{\partial a} + L\right)\ \rho^a(\hat{A}-aI)\right\}\right] da,$$

where RL stands for "real part of." But

$$|\mathrm{TR}\,AB^+|^2 \leq \mathrm{TR}\ (AA^+)\ \mathrm{TR}\ (BB^+)$$

and

$$\left|\int AB\, da\right| \leqslant \left[\int A^2\, da \int B^2\, da\right]^{1/2}$$

$$1 = \left[RL \int \phi(a)\, TR \left\{\left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)\rho^a(\hat{A}-aI)\right\} da\right]^2 \leqslant \left|\int \phi(a)\, TR \left[\frac{\partial}{\partial a} (\ln \phi(a) + L)\rho^a(\hat{A}-aI)\right] da\right|^2$$

$$1 \leqslant \left[\int \phi(a) \left|TR\left[\left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)\rho^a(\hat{A}-aI)\right]\right| da\right]^2$$

$$1 \leqslant \left[\int \phi(a) \left\{TR\left[\left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)(\rho^a)\left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)^+\right]\right\}^{1/2} \right.$$
$$\left. \{TR\, [\rho^a(\hat{A}-aI)(\hat{A}-aI)^+]\}^{1/2}\, da\right]^2$$

$$1 \leqslant \int \phi(a)\; TR\left[\left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)^+ \left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)\rho^a\right] da$$

$$\cdot \int \phi(a)[TR\; \rho^a(\hat{A}-aI)(\hat{A}-aI)^+]\; da$$

with equality if and only if

1. $TR\left[\left(\frac{\partial}{\partial a} \ln \phi a + L\right)\rho^a(\hat{A}-aI)\right]$ is real and positive.
2. $\frac{\partial}{\partial a} \ln \phi(a) + L = k(a)(\hat{A}-aI)$, $k(a)$ a function of "a".
3. $TR\left[\left(\frac{\partial}{\partial A} \ln \phi a + L\right)^+ \left(\frac{\partial}{\partial a} \ln \phi(a) + L\right)\rho^a\right] = B\; TR\; \rho^a(\hat{A}-aI)(\hat{A}-aI)^+$.

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3. C. W. Helstrom, "Minimum Various Estimates in Quantum Signal Detection," IEEE Trans., Vol. IT-14 No. 2, pp. 234-241, March 1968.

C. BOUNDS AND APPROXIMATIONS FOR SOME INTEGRAL EXPRESSIONS INVOLVING LOG-NORMAL STATISTICS

1. Introduction

During research on the use of the turbulent atmosphere as an optical communication channel, several integral expressions, involving the log-normal probability density, have been repeatedly encountered.[1-4] These expressions cannot be evaluated exactly in closed form; however, some of them have been previously tabulated by using numerical techniques.[1,2] In order to obtain additional insight, and analytically tractable expressions, several bounds and approximations have been derived.

The expressions of interest are all averages of functions of the log-normal random variable u, where the probability density of u is

$$p(u) = \frac{1}{\sigma u \sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2} (\ln u - m)^2\right]. \tag{1}$$

Occasionally, it is convenient to replace m by $-\sigma^2$, which is equivalent[5] to setting $\overline{u^2} = 1$. This substitution can sometimes be justified physically by a conservation-of-energy argument; it also is mathematically convenient, as it leads to a single-parameter expression for p(u).

The expressions considered in this report are outlined as follows.

i. The moment-generating function of u,

$$M(s) = \overline{e^{su}}, \qquad s \leq 0. \tag{2}$$

ii. The probability density of a sample, y_m, of the received field in the aperture plane,

$$p(y_m) = \frac{1}{N_o} \exp\left[-\frac{1}{2N_o} |y_m|^2\right] \mathrm{Fr}\left(\frac{E}{2N_o}, \frac{|y_m|}{\sqrt{2N_o}} : m, \sigma\right), \tag{3}$$

where

$$\mathrm{Fr}\,(\alpha, \beta: m, \sigma) \equiv \overline{I_o(2\beta u\sqrt{\alpha})\, e^{-\alpha u^2}}^{\,u} . \tag{4}$$

Here the sample is formed by crosscorrelation with the transmitted signal, where the received field is the sum of the fading-signal field and white noise of spectral density $N_o/2$. Also, the average received signal energy is E.

iii. The single transmission probability of error for an optimum incoherent (unknown phase) receiver for binary orthogonal signals on a log-normal channel[6]:

$\frac{1}{2}\text{Fr}\left(\frac{E}{2N_o}, 0: m, \sigma\right)$

$$P[\epsilon] = \overline{\frac{1}{2}\exp\left[-\frac{E}{2N_o}u^2\right]}^{u} \equiv \frac{1}{2}e^{L}, \tag{5}$$

where L is the corresponding error exponent.

iv. The binary error probability for an optimum coherent (known phase) receiver, using orthogonal signals on a log-normal channel[7]:

$$P[\epsilon] = \overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u} \tag{6}$$

where $Q(a) = \Pr[x \geq a]$, with x the normalized Gaussian random variable, N(0, 1).

2. Bounds Resulting from Modifications in the Domain of Integration

The log-normal random variable u may be conveniently expressed in terms of the normalized Gaussian random variable x according to

$$u = e^{\sigma x + m} \tag{7}$$

$$= e^{\sigma(x-\sigma)}, \qquad \text{if } m = -\sigma^2. \tag{8}$$

Then using (8), we obtain

$$\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-x^2/2} \int_{\sqrt{\frac{E}{N_o}}\, e^{-\sigma^2} e^{\sigma x}}^{\infty} dy\, e^{-y^2/2}, \tag{9}$$

which is simply the integral of the joint density of two independent, normalized, Gaussian random variables, over the convex domain crosshatched in Fig. VIII-1.

Integrating the joint density above a tangent to f(x) yields an upper bound, Q(d), for (9), where d is the perpendicular distance from the tangent line to the origin. Obviously, the tightest such bound results from maximizing d; this amounts to choosing the point of tangency on f(x) as the unique point (x_o, y_o) closest to the origin:

$$\begin{aligned} x_o &= -\sigma \frac{E}{N_o} e^{-2\sigma^2} e^{2\sigma x_o} \\ y_o &= \sqrt{\frac{E}{N_o}}\, e^{-\sigma^2} e^{\sigma x_o}. \end{aligned} \tag{10}$$

Thus

$$\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u} \leq Q(d_{max}) = Q\left(\sqrt{x_o^2 + y_o^2}\right). \tag{11}$$

A lower bound to $\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u}$ of the form $Q(-x^*)\ Q(y^*)$ is clearly obtained by choosing a point (x^*, y^*) on $f(x)$, and integrating over the domain $\{(x, y) \ni x \leq x^*, y \geq y^*\}$.

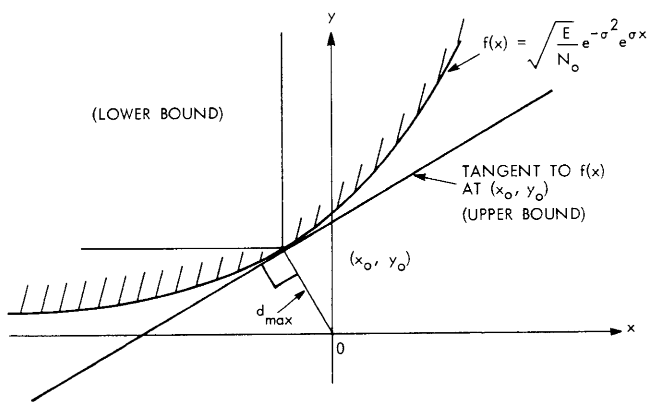

Fig. VIII-1. Geometry of domains of integration for bounds on $\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u}$.

Because of the exponential behavior of the Q-function, this bound is approximately maximized by choosing (x^*, y^*) to minimize $(x^{*2}+y^{*2})$. This requires $x^* = x_o$, $y^* = y_o$. Therefore

$$\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u} \geq Q(-x_o)\ Q(y_o). \tag{12}$$

The lower bound in (12) is also a lower bound for (5), since $Q\left(u\sqrt{\frac{E}{N_o}}\right) \leq \frac{1}{2}\exp\left[-\frac{E}{2N_o}u^2\right]$ ($u \geq 0$ always).[8] In Fig. VIII-2 the upper bound in (11) is plotted against E/N_o, for various values of σ.

3. Bounds Resulting from Modifications of the Integrand

Upper and lower bounds will now be derived for the exponent L in (5). Using (7) to convert from log-normal to normalized Gaussian statistics, we have

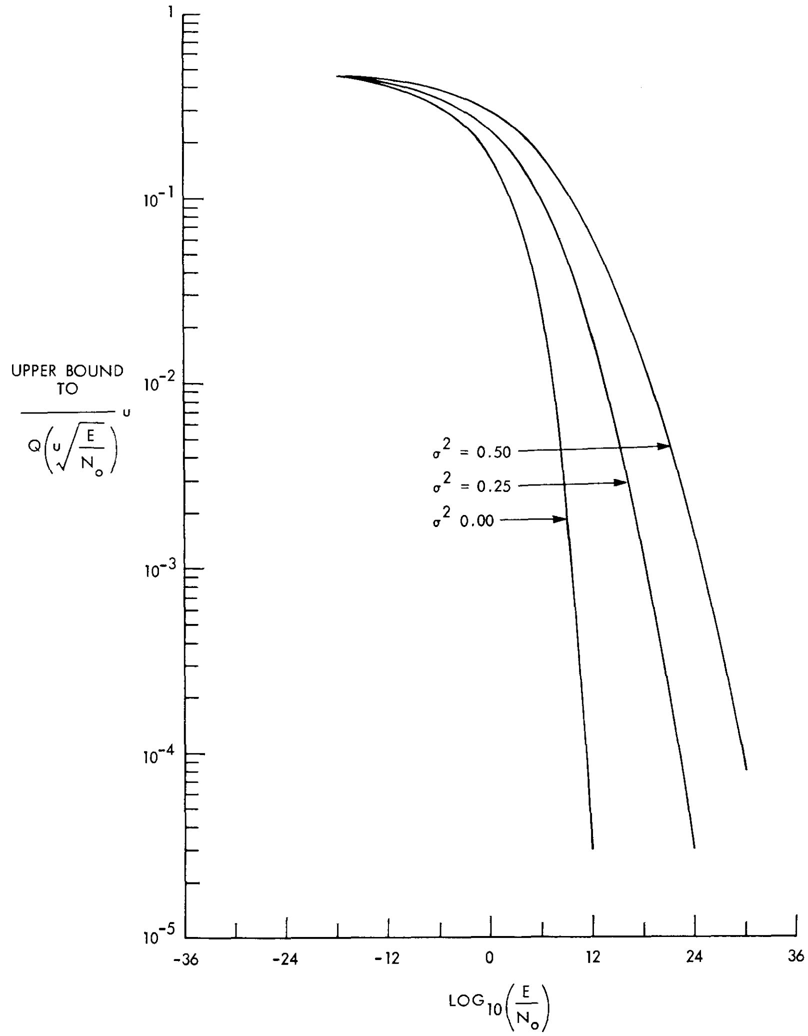

Fig. VIII-2. Upper bound for $\overline{Q\left(u\sqrt{\frac{E}{N_o}}\right)}^{u}$ vs $\log_{10}\left(\frac{E}{N_o}\right)$.

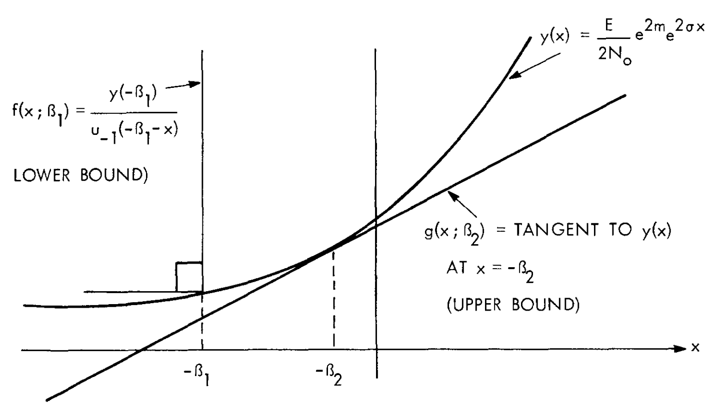

Fig. VIII-3. Bounds for $y(x) = \frac{E}{2N_o} e^{2m} e^{2\sigma x}$.

$$L = \ln \left[\overline{\exp\left(-\frac{E}{2N_o} u^2\right)}^{\,u} \right] = \ln \left[\overline{e^{-y(x)}}^{\,x} \right]. \tag{13}$$

where

$$y(x) = \frac{E}{2N_o} e^{2m} e^{2\sigma x}. \tag{14}$$

To obtain a lower bound on L, use the upper bound for y(x), denoted $f(x;\beta_1)$ in Fig. VIII-3. Thus

$$L \geq \max_{\beta_1} \left\{ \ln \left[\overline{e^{-f(x;\beta_1)}}^{\,x} \right] \right\}$$

$$= \max_{\beta_1} \left\{ \ln \left[Q(\beta_1)\right] - \frac{E}{2N_o} e^{-2(\beta_1\sigma - m)} \right\} \equiv L_1. \tag{15}$$

Differentiating the term in braces in (15) with respect to β_1, and setting the result to zero, we find that the optimum β_1 denoted by β_1^* satisfies the relation

$$\frac{E}{2N_o} e^{-2\left(\beta_1^*\sigma - m\right)} = \frac{1}{2\sigma\sqrt{2\pi}\; Q\left(\beta_1^*\right) \exp\left[\frac{1}{2}\left(\beta_1^*\right)^2\right]}. \tag{16}$$

Therefore we can relate the lower bound L_1 and the energy-to-noise ratio $E/2N_o$ in terms of the parameter β_1^*:

$$L_1 = \ln\left[Q\left(\beta_1^*\right)\right] - \frac{1}{\sigma M\left(\beta_1^*\right)} \tag{17}$$

and

$$\frac{E}{2N_o} = \frac{e^{2\left(\beta_1^*\sigma - m\right)}}{\sigma M\left(\beta_1^*\right)}, \tag{18}$$

where

$$M(\beta_1) \equiv 2\sqrt{2\pi}\; Q(\beta_1)\, e^{\beta_1^2/2}. \tag{19}$$

When m is replaced by $-\sigma^2$, (18) becomes

$$\frac{E}{2N_o} = \frac{e^{2\sigma\left(\beta_1^* + \sigma\right)}}{\sigma M\left(\beta_1^*\right)}. \tag{20}$$

The upper bound on L results from using the lower bound $g(x;\beta_2)$ for $y(x)$:

$$g(x;\beta_2) = y(-\beta_2) + y'(-\beta_2)(x+\beta_2)$$

$$= \frac{E}{2N_o} e^{2m} e^{-2\sigma\beta_2} (1+2\sigma\beta_2) + \frac{E}{2N_o} e^{2m} e^{-2\sigma\beta_2} 2\sigma x. \tag{21}$$

Therefore

$$L \leqslant \min_{\beta_2} \left\{ \ln \left[\overline{e^{-g(x;\beta_2)}}^{x} \right] \right\}$$

$$= \min_{\beta_2} \left\{ -\frac{E}{2N_o} e^{-2(\beta_2\sigma - m)} \left[1 + 2\beta_2\sigma - 2\sigma^2 \frac{E}{2N_o} e^{-2(\beta_2\sigma-m)} \right] \right\}$$

$$\equiv L_2. \tag{22}$$

Differentiating the term in braces in (22) with respect to β_2, and setting the result to zero, we find that the relation that must be satisfied by the optimum β_2, denoted β_2^*, is

$$\frac{E}{2N_o} e^{-2\left(\beta_2^*\sigma - m\right)} = \frac{\beta_2^*}{2\sigma}. \tag{23}$$

Thus, the upper bound, L_2, is expressed as a function of $E/2N_o$ in terms of the parameter β_2^*:

$$L_2 = \frac{-\beta_2^*}{2\sigma} - \frac{\left(\beta_2^*\right)^2}{2} \tag{24}$$

and

$$\frac{E}{2N_o} = \frac{\beta_2^*}{2\sigma} e^{2\left(\beta_2^*\sigma - m\right)}. \tag{25}$$

Again, if $m = -\sigma^2$, (25) becomes

$$\frac{E}{2N_o} = \frac{\beta_2^*}{2\sigma} e^{2\sigma\left(\beta_2^* + \sigma\right)}. \tag{26}$$

L_1, L_2, and L are plotted against $E/2N_o$ in Fig. VIII-4 for $\sigma = 0.2$ and $\sigma = 1.0$. The values of L vs $\frac{E}{2N_o}$ were computed numerically.[9]

Now, with $m = -\sigma^2$, let us consider the asymptotic behavior of the bounds. For large β_1^* and β_2^*, we can compare the upper and lower bounds. For $\beta_1 \gtrsim 3$,

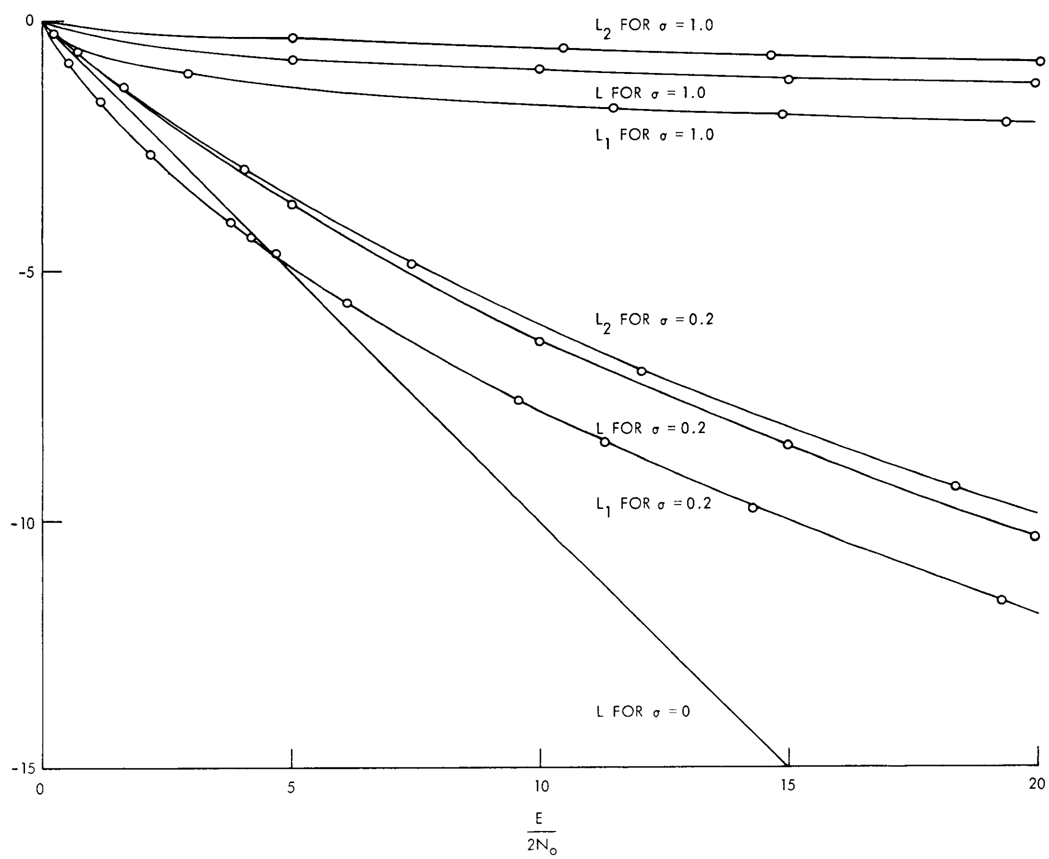

Fig. VIII-4. Error exponent $L = \ln \left[\overline{\exp\left(-\frac{E}{2N_o} u^2\right)}^{u}\right]$, and lower and upper bounds L_1 and L_2, for some values of σ, when $m = -\sigma^2$.

$Q(\beta_1) \approx \frac{1}{\beta_1 \sqrt{2\pi}} \exp\left[-\frac{1}{2}\beta_1^2\right]$;[8] then (17), (19), and (20) become

$$\beta_1^* \gtrsim 3 \begin{cases} M\left(\beta_1^*\right) \approx \dfrac{2}{\beta_1^*} & (27) \\[2ex] L_1 \approx -\dfrac{\beta_1^*}{2\sigma} - \dfrac{\left(\beta_1^*\right)^2}{2} - \ln\left(\beta_1^* \sqrt{2\pi}\right) & (28) \\[2ex] \dfrac{E}{2N_o} \approx \dfrac{\beta_1^*}{2\sigma} e^{2\sigma\left(\beta_1^* + \sigma\right)} & (29) \end{cases}$$

But, after comparing (29) with (26), it is clear that we can set $\beta_1^* \cong \beta_2^* = \beta$ in this range, and express L_1, L_2, and $E/2N_o$ in terms of this common parameter

$$\beta \gtrsim 3 \begin{cases} \dfrac{E}{2N_o} \approx \dfrac{\beta}{2\sigma} e^{2\sigma(\beta+\sigma)} & (30) \\[2ex] L_2 = -\dfrac{\beta}{2\sigma} - \dfrac{\beta^2}{2} & (31) \\[2ex] L_1 \approx L_2 - \ln\left(\beta\sqrt{2\pi}\right). & (32) \end{cases}$$

Since $E/2N_o$ in (30) is monotonically increasing in β, for a given σ, the condition $\beta \gtrsim 3$ is equivalent to

$$\frac{E}{2N_o} \gtrsim \frac{3}{2\sigma} e^{2\sigma(3+\sigma)} \equiv K_3(\sigma), \tag{33}$$

where $K_3(\sigma)$ is plotted in Fig. VIII-5. Thus, when (33) is satisfied, L_1 and L_2 diverge as $\ln(\beta\sqrt{2\pi})$ as $E/2N_o$ increases; however, $\ln(\beta\sqrt{2\pi})$ is small relative to L_2 in this range of $E/2N_o$.

From Fig. VIII-5, it is evident that for small σ, the asymptotic results of (30)-(33) do not apply over a large range of $E/2N_o$. Yet, from (13) and (14), $L \xrightarrow[\sigma\to o]{} -(E_s/2N_o)$. Therefore, consider the asymptotic behavior of L_1 and L_2 as $\sigma \to o$, for a given $E/2N_o$.

To show that $L_1 \xrightarrow[\sigma\to o]{} -(E_s/2N_o)$, let $\beta_1^* = -\sqrt{2\ln\left(\frac{\alpha_1}{\sigma}\right)}$, for some constant α_1. Then it can be readily verified, by using (17) and (20), that

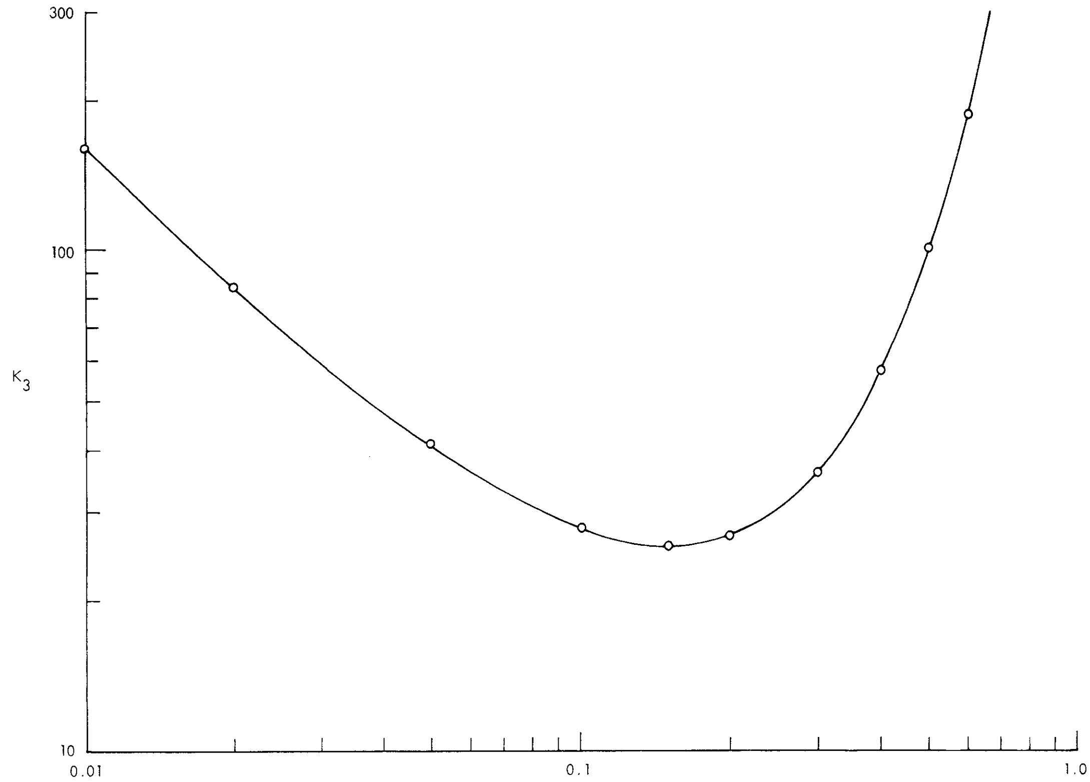

Fig. VIII-5. Plot of $K_3(\sigma) \equiv \frac{3}{2\sigma} e^{2\sigma(3+\sigma)}$. For $\frac{E}{2N_o} > K_3(\sigma)$, $|L_1 - L_2| \approx \ln(\beta\sqrt{2\pi})$.

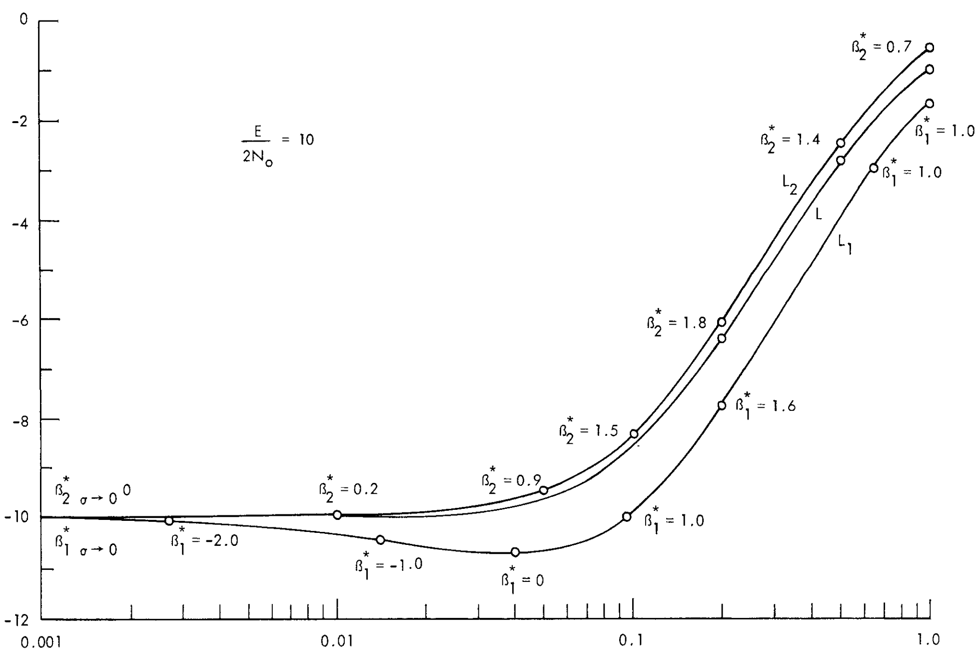

Fig. VIII-6. Asymptotic behavior of L_1 and L_2 for $E/2N_o = 10$ as $\sigma \to 0$.

$$L_1 \xrightarrow[\sigma\to o]{} -\frac{1}{2\sqrt{2\pi}\,\alpha_1}$$

$$\frac{E}{2N_o} \xrightarrow[\sigma\to o]{} +\frac{1}{2\sqrt{2\pi}\,\alpha_1}. \tag{34}$$

Similarly, setting $\beta_2^* = \alpha_2\sigma$, for some constant α_2, and using (24) and (26), we obtain

$$L_2 \xrightarrow[\sigma\to o]{} -\frac{\alpha_2}{2}$$

$$\frac{E}{2N_o} \xrightarrow[\sigma\to o]{} +\frac{\alpha_2}{2} \tag{35}$$

so that $L_2 \xrightarrow[\sigma\to o]{} -(E/2N_o)$. These last results are illustrated in Fig. VIII-6, wherein $E/2N_o = 10$, and L_1 and $L_2 \xrightarrow[\sigma\to o]{} -10$.

4. Second-Degree Approximation of the Integrand Exponent (Halme)

A way to approximate the function $Fr\,(\alpha,\beta\colon m,\sigma)$ of (4) is to use a method similar to saddle-point integration.[10,11] Setting $m = -\sigma^2$, we write

$$Fr[\alpha,y\colon\sigma] \equiv Fr[\alpha,y\colon m,\sigma]\Big|_{m=-\sigma^2} = \int_0^\infty \frac{du}{\sqrt{2\pi}\,\sigma u}\, I_o(2uy\sqrt{\alpha})\, \exp\left[-u^2\alpha-(\ln u+\sigma^2)^2/2\sigma^2\right]$$

$$= \int_0^\infty \frac{du}{\sqrt{2\pi}} \exp[-h(u)] \approx \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left[-h_o - \frac{h_2}{2}(u-u_o)^2\right] du$$

$$= \frac{e^{-h_o}}{\sqrt{h_2}}\left(1-Q(\sqrt{h_2}\,u_o)\right) \tag{36}$$

where

$$h_o = u_o^2\alpha + \frac{\left(\ln u_o+\sigma^2\right)^2}{2\sigma^2} + \ln \sigma u_o - \ln I_o(2u_oy\sqrt{\alpha}) \tag{37}$$

$$h'(u_o) = -\frac{I_1(2u_oy\sqrt{\alpha})}{I_o(2u_oy\sqrt{\alpha})}\,2y\sqrt{\alpha} + 2u_o\alpha + \frac{\ln u_o}{\sigma^2 u_o} + \frac{2}{u_o} = 0 \tag{38}$$

$$h_2 = 2\alpha + \frac{\ln u_o - 1}{\sigma^2 u_o^2} - \frac{2}{u_o^2} - \left(1 - \frac{I_1^2(2u_o y\sqrt{\alpha})}{I_o^2(2u_o y\sqrt{\alpha})} - \frac{1}{2u_o y\sqrt{\alpha}}\,\frac{I_1(2u_o y\sqrt{\alpha})}{I_o(2u_o y\sqrt{\alpha})}\right) 4y\alpha. \tag{39}$$

The idea is to solve (38) for u_o, then compute α_o and α_2 from (37) and (39), and substitute the results in (36). For interesting special cases, the following approximate results are true:

1. $\alpha \gg 1$, $\sigma^2 \approx 1$, $y > \sqrt{\alpha} \Rightarrow u_o \approx y/\sqrt{\alpha}$, $h_2 \approx -2\alpha$, $\mathrm{Fr}\,(\alpha, y{:}\,\sigma) = e^{-y^2}$

2. $\sigma^2 \ll 1 \Rightarrow u_o \approx 1$, $h_o \approx \ln I_o(2y\sqrt{\alpha}) - \alpha \ln \sigma$, $h_2 \approx 2(\alpha-1) + 1/\sigma^2$.

For large h_2, $Q(\sqrt{h_2}\, u_o) \ll 1$, and can be neglected.

The calculation of the moment-generating function (2) and binary error probability

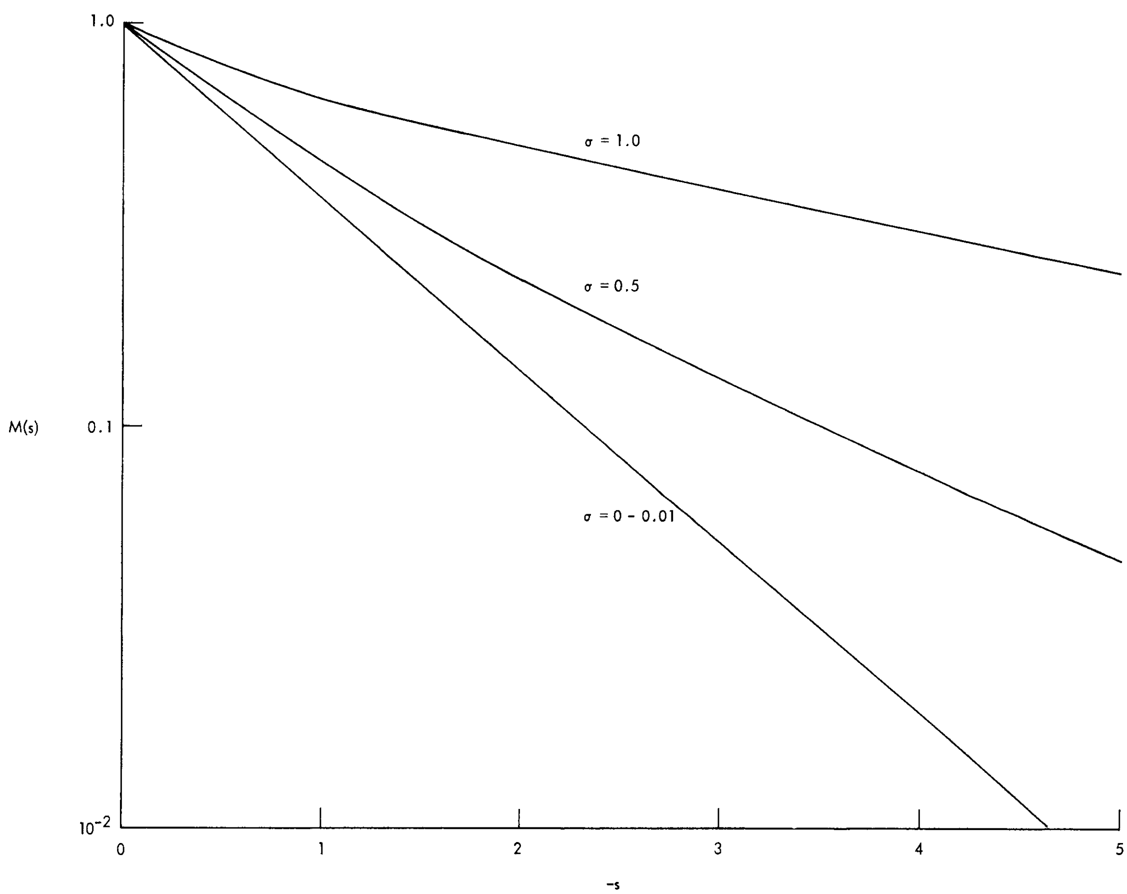

Fig. VIII-7. Plot of moment-generating function $M(s) = \overline{e^{su}}$ against $-s$.

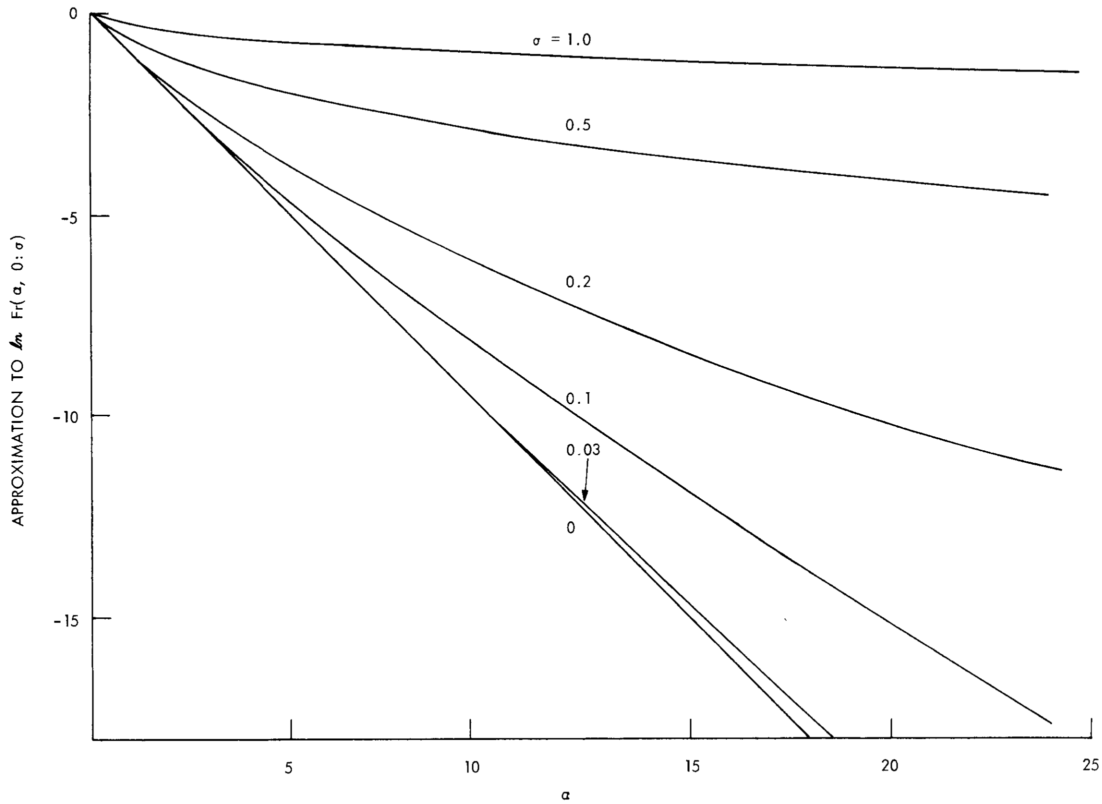

Fig. VIII-8. Plot of approximation to $\ln\left[\mathrm{Fr}\,(\alpha, o\colon \sigma)\right] = \ln\left[\frac{u}{e^{-\alpha u^2}}\right]$ against α for various values of σ.

can be approached in a similar way after an appropriate change of variables. Then M(s) takes the form

$$M(s) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - s\, e^{z\sigma}\, e^{-\sigma^2}\right). \tag{40}$$

Computing quantities similar to those of (37), (38), and (39), we obtain

$$M(s) \approx \frac{\exp\left[-\frac{x}{\sigma^2}\left(1+\frac{x}{2}\right)\right]}{\sqrt{1+x}}, \tag{41}$$

where $-s = x\, e^x\, e^{\sigma^2}/\sigma^2$.

Similarly,

$$\mathrm{Fr}\ (\alpha, o{:}\sigma) \approx \frac{\exp\left[-\frac{x}{4\sigma^2}\left(1+\frac{x}{2}\right)\right]}{\sqrt{1+x}}, \tag{42}$$

where $\alpha = x\, e^x\, e^{2\sigma^2}/4\sigma^2$. The results of (41) and (42) are plotted in Figs. VIII-7 and VIII-8, respectively. It turns out that the approximation is indeed surprisingly close. Another interesting point is that the exponent in (42) is exactly one upper bound to Fr $(\alpha, o{:}\sigma)$ as computed in (24). This is seen by setting $x = 2\sigma\beta_2$.

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