Name SOLUTIONS

# 18.311, Principles of Applied Mathematics, Spring 2002, Bazant

## Final Exam – Monday, May 20, 2002

**Instructions:** Please write your name on every page (or you will lose points). This closed-book exam will last three hours. Point totals for each problem are given out of 100. Graded exams and solutions will be available after May 22.

1. (15 POINTS) Consider the initial traffic density for a red light turning green,

$$\rho(x,0) = \begin{cases} \rho_j & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

and assume a *parabolic* velocity-density relationship,

$$u(\rho) = u_{max}\left(1 - \left(\frac{\rho}{\rho_j}\right)^2\right)$$

in the Lighthill-Whitham theory of traffic flow.

(a) Write down a PDE for the density, $\rho(x,t)$, expressing the conservation of cars.

flux $q(\rho) = \rho u(\rho) = u_{max}\left(\rho - \frac{\rho^3}{\rho_j^2}\right)$

wave velocity $\boxed{c(\rho) = q'(\rho) = u_{max}\left(1 - 3\left(\frac{\rho}{\rho_j}\right)^2\right)}$

Conservation law (differential form)

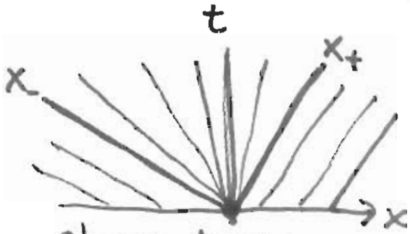
$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \boxed{\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0}$$

(b) What is the density cars at the traffic light, $\rho(0,t)$, after it turns green ($t > 0$)?

$c(\rho) = 0 \Rightarrow \boxed{\rho = \rho_j/\sqrt{3}}$ at $x = 0$ (since this density "does not move").

(c) Calculate the positions of weak discontinuities marking the edges of an expansion fan, $x_-(t) < x < x_+(t)$, such that $\rho = \rho_j$ for $x < x_-(t)$ and $\rho = 0$ for $x > x_+(t)$.

$$\boxed{\begin{aligned} x_-(t) &= c(\rho_j)t = -2u_{max}t \\ x_+(t) &= c(0)t = u_{max}t \end{aligned}}$$




Characteristics

(d) Solve for the traffic density inside the expansion fan (using the Hodograph Method or the Method of Characteristics), and plot the solution at some time $t > 0$.

Hodograph: $\frac{\partial x}{\partial t} = c(\rho) \Rightarrow x = c(\rho)t + \cancel{x_0(\rho)}^{\,0}$

$$x = u_{max}t\left(1 - 3(\rho/\rho_j)^2\right)$$

$$\boxed{\rho = \rho_j\sqrt{\tfrac{1}{3}\left(1 - \frac{x}{u_{max}t}\right)}} \quad \text{for } x_- < x < x_+$$

Characteristics: $\frac{d\rho}{dt} = 0$ on $\frac{dx}{dt} = c(\rho)$

$\rho = \rho_0(\xi)$ $\quad x = c(\rho)t \rightarrow$ same answer.

pick $\xi = \rho$ = density at $x=0, t=0$ as the characteristic label

2. (15 POINTS) Use the Method of Characteristics to solve the following river equation for $x > 0$ and $t > 0$,

$$A_t + \sqrt{A} A_x = -A, \quad A(x, 0) = x^2.$$

Plot $A(x, t)$ at some times $t > 0$, and sketch some characteristics in the $(x, t)$ plane.

$\frac{dA}{dt} = -A$ on characteristics, $\frac{dx}{dt} = \sqrt{A}$

$A = A(t=0)e^{-t}$

label characteristics by initial position $x(t=0) = \xi \geq 0$

Then $\boxed{A = \xi^2 e^{-t}}$

$\frac{dx}{dt} = \xi e^{-t/2}$

$\boxed{x = \xi(3 - 2e^{-t/2})}$

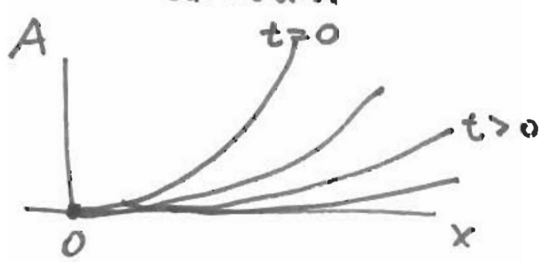
using $x(t=0) = \xi$

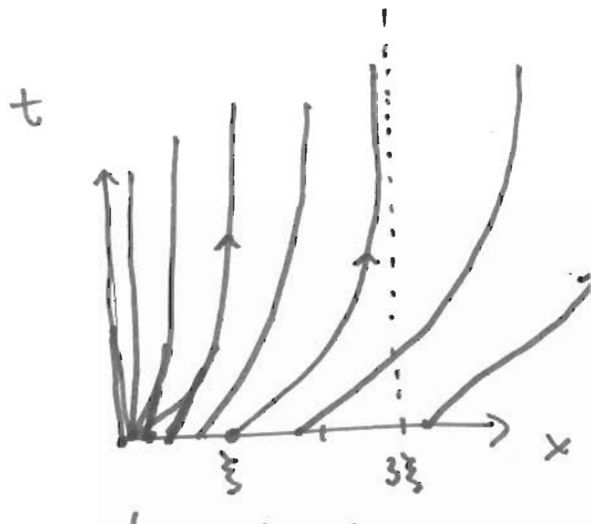
Combine:

$$A = \left(\frac{x}{3 - 2e^{-t/2}}\right)^2 e^{-t}$$

$$\boxed{A(x,t) = \left[\frac{e^{-t}}{(3 - 2e^{-t/2})^2}\right] x^2}$$

parabolae, with exponentially decreasing curvature:



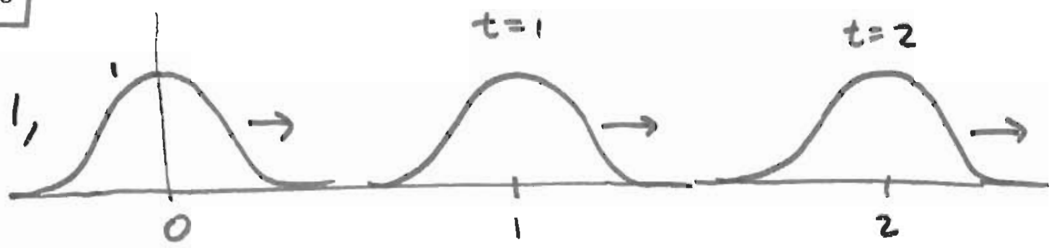




characteristics

3. (15 POINTS) Sketch the solution, $\rho(x,t)$, to each of the following PDEs for several times $t > 0$ for the initial condition, $\rho(x,0) = e^{-x^2}$. DO NOT SOLVE ANALYTICALLY.

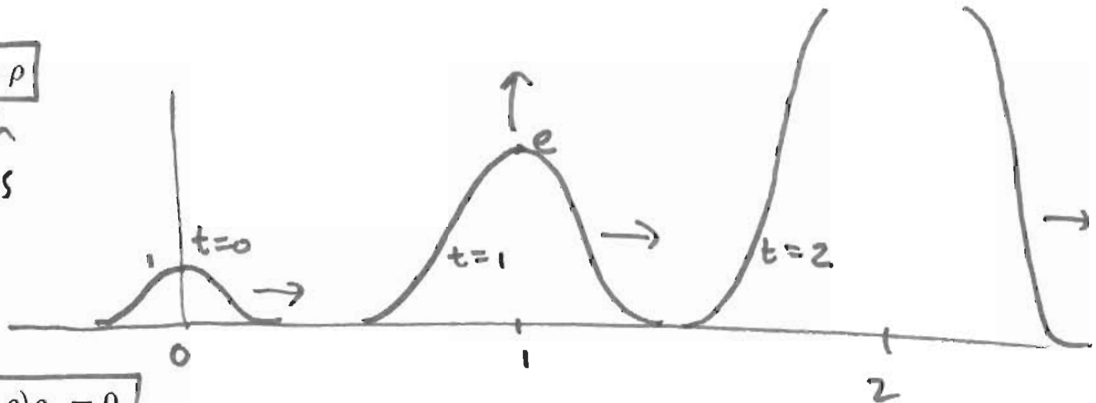
(a) $\boxed{\rho_t + \rho_x = 0}$

pulse moves a constant velocity = 1, keeping its shape.




(b) $\boxed{\rho_t + \rho_x = \rho}$

pulse moves with velocity = 1 and grows exponentially in magnitude (keeping its shape).

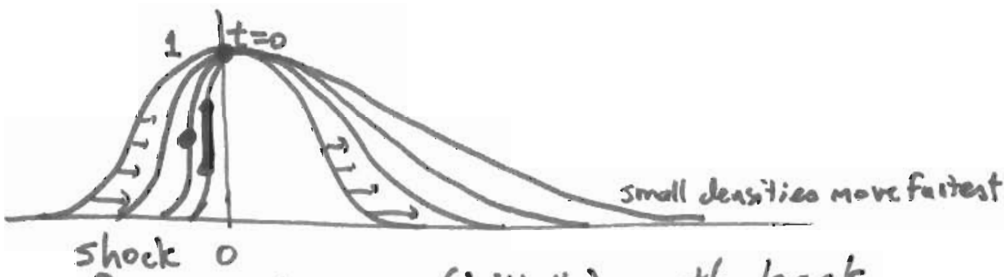



(c) $\boxed{\rho_t + (1-\rho)\rho_x = 0}$

$c(\rho) = 1 - \rho$

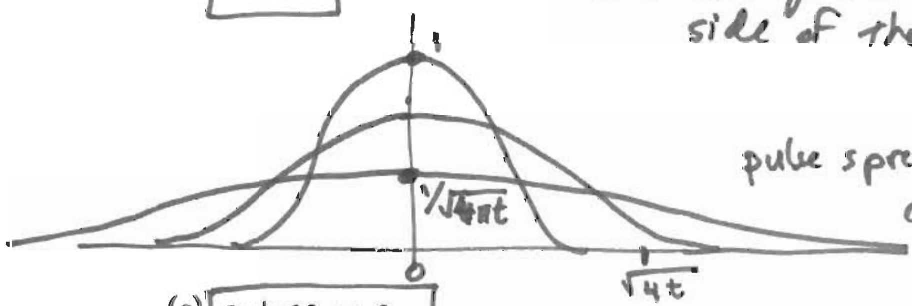
$c(1) = 0$

$c(0) = 1$




shock forms and grows (initially) on the back side of the wave as it "breaks"

(d) $\boxed{\rho_t = \rho_{xx}}$




pulse spreads with a width $\sqrt{t}$ and height $1/\sqrt{t}$, maintaining its shape and total area under the curve

(e) $\boxed{\rho_t + \rho\rho_x = \rho_{xx}}$

$c(\rho) = \rho$

$c(1) = 1$

$c(0) = 0$


Wave tries to break on its front side, but reaches a stable width (of order 1 = viscosity)

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(10 POINTS) Use a Green function to solve the linear diffusion equation, $\rho_t = D\rho_{xx}$, subject to the initial condition,

$$\rho(x,0) = \begin{cases} \rho_o & \text{if } |x| < \ell \\ 0 & \text{if } |x| \geq \ell \end{cases}$$

Express your answer in terms of the error function,

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy,$$

and sketch the solution at several times $t > 0$.

$$\rho(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\rho(y,0)\,dy$$

convolution of initial data with the Green function

$$= \int_{-\ell}^{\ell} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi D t}}\, \rho_o\, dy$$

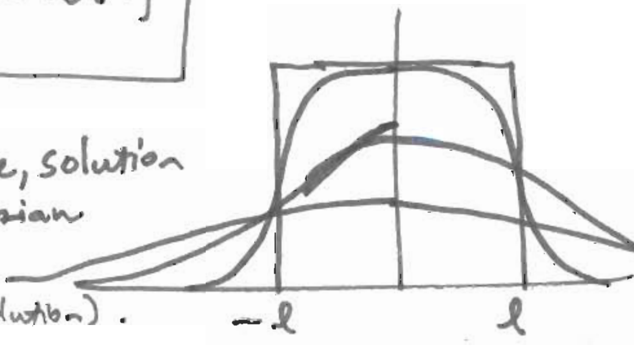
Green function derived in class (e.g. by Fourier transform)

$$= \frac{\rho_o}{\sqrt{4\pi D t}} \int_{\frac{-\ell-x}{\sqrt{4Dt}}}^{\frac{\ell-x}{\sqrt{4Dt}}} e^{-z^2}\, dz \cdot \sqrt{4Dt}$$

$$z = \frac{y-x}{\sqrt{4Dt}}$$

$$= \frac{\rho_o}{\sqrt{\pi}} \int_{-\frac{(\ell+x)}{\sqrt{4Dt}}}^{\ell-x/\sqrt{4Dt}} e^{-z^2} dz$$

$$\boxed{\rho = \frac{\rho_o}{2}\left[\text{erf}\left(\frac{\ell-x}{2\sqrt{Dt}}\right) - \text{erf}\left(\frac{\ell+x}{2\sqrt{Dt}}\right)\right]}$$




after long time, solution approaches a gaussian spreading like $\sqrt{t}$ (Green function similarity solution).

5. (15 POINTS) Find a similarity solution to the nonlinear diffusion equation

$$\rho_t = t(\rho\rho_x)_x \qquad (1)$$

of the form

$$\rho(x,t) = \frac{1}{t^\alpha} F\left(\frac{x}{t^\beta}\right) \qquad (2)$$

for the case of "spreading" of a concentration profile satisfying,

$$\int_{-\infty}^{\infty} \rho(x,1)dx = 1 \qquad (3)$$

Note that solution will have "fronts" (weak discontinuities) beyond which $\rho = 0$.

(a) Show that total mass is conserved,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho(x,t)dx = 0,$$

and use this fact to relate $\alpha$ and $\beta$.

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \rho\, dx = \int_{-\infty}^{\infty} \rho_t\, dx = t\int_{-\infty}^{\infty} (\rho\rho_x)_x dx = t\,(\rho\rho_x)\Big|_{-\infty}^{\infty} = 0$$

since $\rho \to 0$ and $\rho_x \to 0$ (beyond the fronts).

This implies $\boxed{\alpha=\beta}$ since $\int_{-\infty}^{\infty} \rho(x,t)dx = \frac{1}{t^\alpha}\int_{-\infty}^{\infty} F\left(\frac{x}{t^\beta}\right)dx$

$$= \frac{t^\beta}{t^\alpha}\underbrace{\int_{-\infty}^{\infty} F(z)dz}_{\substack{\text{constant} \\ = 1 \text{ using (3)}}} = 1$$

(b) Determine $\beta$ and derive and ODE for $F(\xi)$.

Substitute (2) into (1); using $\alpha=\beta$:

$$-\frac{\beta}{t}\frac{1}{t^\beta}\left(F(\xi) + \xi F'(\xi)\right) = t\,\frac{1}{t^{2\beta}}\cdot\frac{1}{t^{2\beta}}\left(F(\xi)F'(\xi)\right)'$$

$\therefore\ \beta + 1 = 4\beta - 1$

$$\boxed{\alpha = \beta = 2/3}$$

$$\boxed{\begin{gathered} -\frac{2}{3}\left(F + \xi F'\right) = \left(FF'\right)' \\ F(\infty) = F'(\infty) = 0, \quad \xi = \frac{x}{t^{2/3}} \end{gathered}}$$

The spreading exponent is larger than in "normal" linear diffusion ($\beta = 1/2$). Apparently, the time-dependence (D increases like $t$) beats the concentration dependence ($D \to 0$ as $\rho \to 0$).

(c) Solve for the scaling function, $F(\xi)$.

$$-\frac{2}{3}(F + \xi F') = (FF')'$$

$$= -\frac{2}{3}(\xi F)'$$

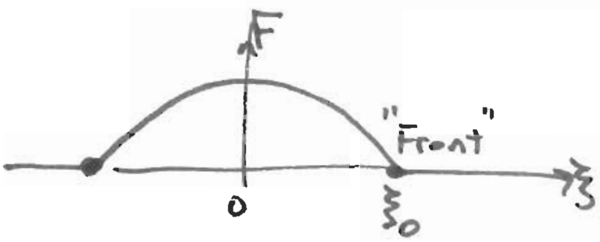
Integrate: $$-\frac{2}{3}\xi F = FF' + \underbrace{\cancel{C}}_{=0 \text{ from B.C.}}$$

$$-\frac{2}{3}\xi = F'$$

$$\therefore \quad F(\xi) = C' - \frac{1}{3}\xi^2 \qquad \text{parabola}$$

Write the constant in terms of the front location $\xi_0$:

$$\boxed{F(\xi) = \begin{cases} \frac{1}{3}(\xi_0^2 - \xi^2) & \text{if } |\xi| < \xi_0 \\ 0 & \text{if } |\xi| > \xi_0 \end{cases}}$$




Scaling function (parabola)

To determine $\xi_0$:

$$1 = \int_{-\infty}^{\infty} F(\xi)\,d\xi$$

$$= 2\int_0^{\xi_0} \frac{1}{3}(\xi_0^2 - \xi^2)\,d\xi$$

$$= \frac{2}{3}\xi_0^2 - \frac{2}{9}\xi_0^3$$

$$\boxed{\frac{9}{2} = (3 - \xi_0)\xi_0^2}$$

This cubic eqn. determines $\xi_0 > 0$.

6. (15 POINTS) The evolution of a rough crystal surface due to surface diffusion satisfies, $h_t = h_{xxxx}$, where $h$ is the surface height (in dimensionless form).

(a) Find the Fourier transform of the Green function, which solves $G_t = G_{xxxx}$ subject to $G(x,0) = \delta(x)$.

$$\hat{G}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} G(x,t)\,dx, \quad G(x,t) = \int_{-\infty}^{\infty} e^{ikx} \hat{G}(k,t)\,\frac{dk}{2\pi}$$

$$G_t = G_{xxxx}$$

$$\hat{G}_t = (ik)^4 \hat{G} = k^4 \hat{G}$$

$$\therefore \boxed{\hat{G}(k,t) = e^{k^4 t}}$$

since $G(x,0) = \delta(x)$
$\Rightarrow \hat{G}(k,0) = 1$

(b) Change variables in the inverse Fourier transform integral to show that the Green function can be expressed as a similarity solution,

$$G(x,t) = \frac{1}{t^\alpha} F\left(\frac{x}{t^\alpha}\right)$$

for some $F(\xi)$ (defined by an integral). What is $\alpha$?

$$G(x,t) = \int_{-\infty}^{\infty} e^{ikx} e^{k^4 t} \frac{dk}{2\pi}$$

let $\omega = k t^{1/4}$

$$= \int_{-\infty}^{\infty} e^{i\frac{\omega}{t^{1/4}}x + \omega^4} \frac{d\omega}{2\pi t^{1/4}}$$

$$= \frac{1}{t^{1/4}} F\left(\frac{x}{t^{1/4}}\right)$$

where

$$\boxed{F(\xi) = \int_{-\infty}^{\infty} e^{i\omega\xi + \omega^4} \frac{d\omega}{2\pi}, \quad \hat{F}(\omega) = e^{\omega^4}}$$


$$\boxed{\alpha = 1/4}$$

spreads more slowly than "normal diffusion" ($\alpha = 1/2$).

7. (15 POINTS) Consider the boundary-value problem,

$$-\epsilon y'' + y = x + 1, \quad y(0) = 0, \quad y(1) = 2.$$

The leading-order "outer approximation", $y \sim \bar{y}(x)$, as $\epsilon \to 0$, is clearly, $\bar{y} = x + 1$, which satisfies the boundary condition at $x = 1$, but not the one at $x = 0$. Therefore, we expect a boundary layer near $x = 0$.

(a) Introduce a scaled variable, $\xi = x/\epsilon^\alpha$, for an appropriate choice of $\alpha$, and derive the leading-order "inner approximation", $y \sim \tilde{y}(\xi)$. Apply asymptotic matching.

$$\xi = \frac{x}{\epsilon^\alpha} \Rightarrow -\epsilon \frac{1}{\epsilon^{2\alpha}} \tilde{y}'' + \tilde{y} = x + 1 = \epsilon^\alpha \xi + 1$$

In order to get a boundary layer which decays as $\xi \to \infty$ (to a constant) as required for asymptotic matching, we need $\boxed{\alpha = 1/2}$.

Then $-\tilde{y}'' + \tilde{y} = \sqrt{\epsilon}\,\xi + 1$

At leading order ($\epsilon \to 0$): $-\tilde{y}'' + \tilde{y} \sim 1$

Integrate using $\tilde{y}(0) = 0$ and matching condition

$$\tilde{y}(\infty) = \bar{y}(0) = 1 \quad (*)$$

we get $\boxed{\tilde{y}(\xi) \sim 1 - e^{-\xi}}$

(b) Combine the leading-order inner and outer approximations to construct a uniformly valid asymptotic approximation. Sketch the solution.

From matching (*), the "overlap" between the inner and outer approximations is 1.

$$\therefore \quad y(x) \sim \bar{y}(x) + \tilde{y}(x/\sqrt{\epsilon}) - 1 = x + 1 + 1 - e^{-x/\sqrt{\epsilon}} - 1$$

$$= \boxed{x + 1 - e^{-x/\sqrt{\epsilon}}}$$