Name $\underline{\text{SOLUTIONS}}$

18.311, Principles of Applied Mathematics, Spring 2004, Bazant

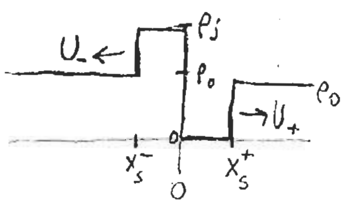
Final Exam – Wednesday, May 19, 2004

Instructions: Please write your name on every page. This *closed-book* exam will last three hours. Point totals for each problem are given out of 100. Graded exams and solutions will be available starting tomorrow. I hope you enjoyed the class. – MZB

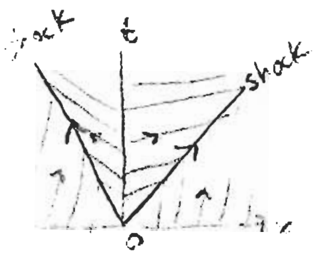
1. (10 POINTS) Consider a steady initial flow of traffic through a green light at uniform density, $\rho(x,0) = \rho_0$, where $0 < \rho_0 < \rho_j$. Assuming $u(\rho) = u_{max}(1 - \rho/\rho_j)$, solve the PDE

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \qquad q = u(\rho)\rho$$

with boundary conditions, $\rho(0^-, t) = \rho_j$ and $\rho(0^+, t) = 0$ for $t > 0$ after the light turns red.



The solution involves two $\underline{\text{shocks}}$ which move at constant velocities, $U_\pm = \dfrac{[q]}{[\rho]}$:



$$U_+ = \frac{q(\rho_0)}{\rho_0} = u(\rho_0) = \boxed{u_{max}(1 - \rho_0/\rho_j) = U_+}$$

$$U_- = \frac{q(\rho_j) - q(\rho_0)}{\rho_j - \rho_0} = \frac{1}{2}\left[c(\rho_j) + c(\rho_0)\right] \quad \text{where } c(\rho) = u_{max}\left(1 - \frac{2\rho}{\rho_j}\right)$$

since $u(\rho)$ is linear

$$= u_{max}\left(1 - \frac{\rho_0 + \rho_j}{\rho_j}\right) = \boxed{-\frac{u_{max}\rho_0}{\rho_j} = U_-}$$

$$\boxed{\rho(x,t) = \begin{cases} \rho_0 & , \quad x < U_- t \\ \rho_j & , \quad U_- t < x < 0 \\ 0 & , \quad 0 < x < U_+ t \\ \rho_0 & , \quad x > U_+ t \end{cases}}$$

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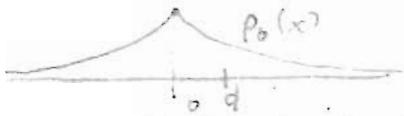
(25 POINTS) Consider again traffic flow with $u(\rho) = u_{max}(1-\rho/\rho_j)$ for the initial condition,



$$\rho(x,0) = \frac{\rho_j/2}{1+|x|/d} \quad \rho_0(x)$$

(a) Solve for $\rho(x,t)$ for $0 < t < t_s$, where t_s is the shock formation time. (You may leave your solution in implicit form.)

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \quad \Rightarrow \quad \frac{d\rho}{dt} = 0 \quad \text{on characteristics} \quad \frac{dx}{dt} = c(\rho)$$

$$\rho = \rho_0(\xi) \qquad x = c(\rho)t + \xi$$

labeled by $x(0) = \xi$.

$$\rho = \rho_0(x - c(\rho)t)$$

$$\boxed{\rho = \frac{\rho_j/2}{1 + |x - c(\rho)t|/d} \quad \text{where} \quad c(\rho) = u_{max}\left(1 - \frac{2\rho}{\rho_j}\right)}$$

to get an explicit solution, consider separately

$\underline{\xi > 0}$

$$\rho = \frac{\rho_j/2}{1 + (x - c(\rho)t)/d}$$

$$\rho\left(d + x - u_{max}\left(1 - \frac{2\rho}{\rho_j}\right)t\right) = d\rho_j/2$$

$$\left(\frac{2u_{max}t}{\rho_j}\right)\rho^2 + (d + x - u_{max}t)\rho - \frac{d\rho_j}{2} = 0$$

$$\rho = \left[-(d + x - u_{max}t) + \sqrt{(d+x-u_{max}t)^2 + 4du_{max}t}\right] \Big/ \frac{4u_{max}t}{\rho_j}$$

$$\rho = \frac{\rho_j}{4}\cdot\left[1 - \frac{d+x}{u_{max}t} + \sqrt{\left(1 - \frac{d+x}{u_{max}t}\right)^2 + \frac{4d}{u_{max}t}}\right]$$

$\underline{\xi < 0}$

$$\rho = \frac{\rho_j/2}{1 - (x - c(\rho)t)/d}$$

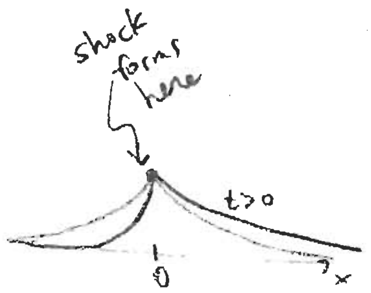
(b) Find the time, t_s, and the position, x_s, where a shock forms.

Initial characteristic velocities:

$$c_0(x) = c(\rho(x,0)) = u_{max}\left(1 - \frac{2\rho_0(x)}{\rho_j}\right)$$

$$= u_{max}\left(1 - \frac{1}{1+|x|/d}\right)$$

$$c_0'(x) = \frac{\mathrm{sgn}(x)}{(1+|x|/d)^2}\frac{u_{max}}{d} \qquad \mathrm{sgn}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$$



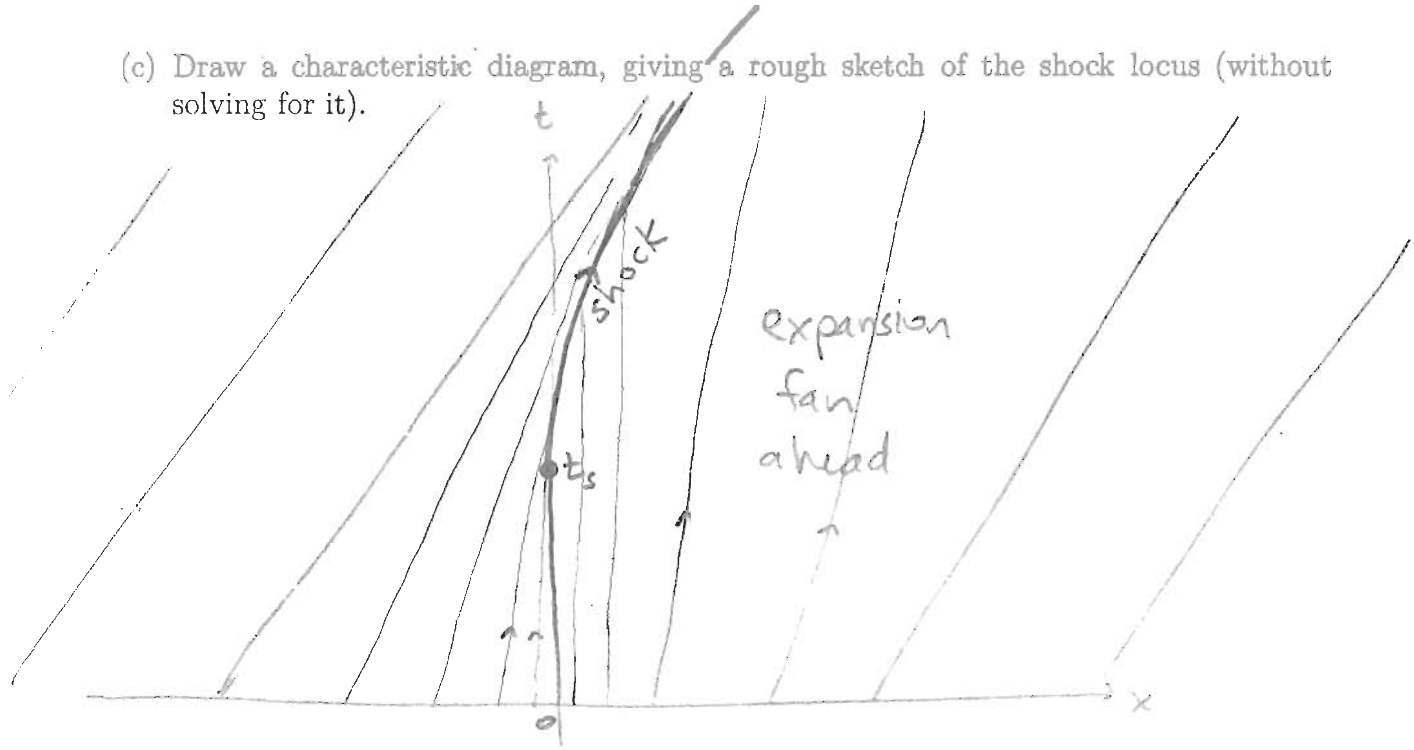
clearly, a shock will form on the backside of the wave, $x \le 0$.

$$t_s = \min_{x\le 0}\left\{-\frac{1}{c_0'(x)}\right\} = \min_{x \le 0}\left\{\frac{(1+|x|/d)^2 d}{u_{max}}\right\} = \boxed{\frac{d}{u_{max}}}$$

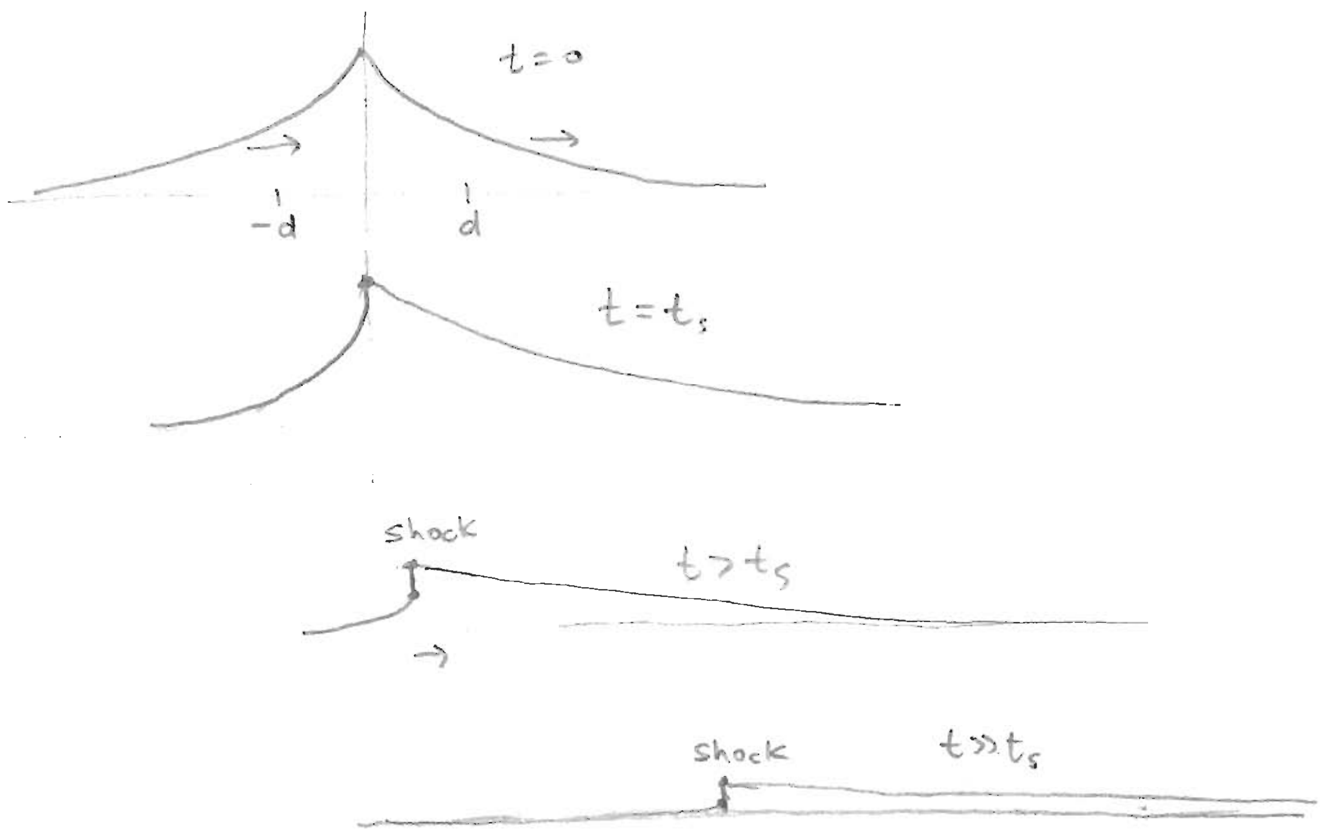
The shock forms at $\boxed{x_s = 0}$.

(which we could have guessed by dimensional analysis).

(c) Draw a characteristic diagram, giving a rough sketch of the shock locus (without solving for it).



(d) Sketch the solution, $\rho(x,t)$, at various times before and after shock formation. (Try this even if you did not solve the other parts.)



3. (15 POINTS) Use a Fourier transform (in x) to solve the linear reaction-diffusion equation:

$$\frac{\partial \rho}{\partial t} = D\,\frac{\partial^2 \rho}{\partial x^2} - r\,\rho$$

subject to the initial condition, $\rho(x,0) = \delta(x)$.

$$\hat{\rho}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} \rho(x,t)\,dx$$

$$\frac{\partial \hat{\rho}}{\partial t} = (-Dk^2 - r)\,\hat{\rho}$$

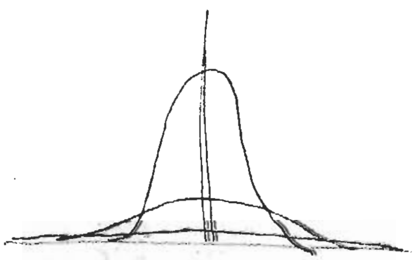
$$\Rightarrow \quad \boxed{\hat{\rho}(k,t) = e^{-(r+Dk^2)t}}$$

using $\hat{\rho}(k,0) = 1 = \int e^{-ikx}\delta(x)\,dx$

$$= e^{-rt}\,\underbrace{e^{-Dk^2 t}}$$

FT of Green function $G(x,t)$ for the diffusion equation ($r=0$) from class

$$\boxed{\begin{aligned} \rho(x,t) &= e^{-rt}\,G(x,t) \\ &= \frac{e^{-\frac{x^2}{4Dt} - rt}}{\sqrt{4\pi D t}} \end{aligned}}$$



4. (10 POINTS) Find the dispersion relation, $\omega(k)$, for the forced wave equation,

$$\frac{\partial^2 \rho}{\partial t^2} = c_0^2 \frac{\partial^2 \rho}{\partial x^2} - \omega_0^2\,\rho,$$

and show that $u_p u_g = c_0^2$, where $u_p(k)$ is the phase velocity and $u_g(k)$ is the group velocity.

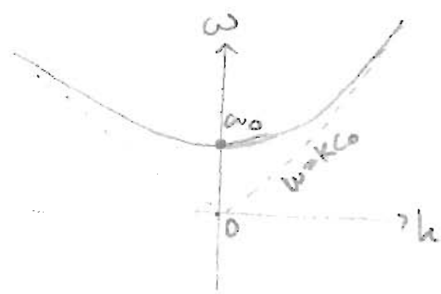
Consider dispersive waves of the form $\rho = A e^{i(kx-\omega t)}$.

Then $\frac{\partial^n}{\partial x^n} \leftrightarrow (ik)^n$ and $\frac{\partial^n}{\partial t^n} \leftrightarrow (-i\omega)^n$.

So the dispersion relation is

$$-\omega^2 = c_0^2(-k^2) - \omega_0^2$$

$$\boxed{\omega = \pm\sqrt{\omega_0^2 + c_0^2 k^2}}$$



$$u_p = \frac{\omega}{k} = \pm\sqrt{c_0^2 + \left(\frac{\omega_0}{k}\right)^2}$$

$$u_g = \frac{d\omega}{dk} = \pm\frac{1}{2}\frac{2c_0^2 k}{\sqrt{\omega_0^2 + c_0^2k^2}} = \pm\frac{c_0^2}{\sqrt{c_0^2 + \left(\frac{\omega_0}{k}\right)^2}}$$

$$\boxed{u_p u_g = c_0^2}$$

✓

5. (15 POINTS) Consider the following "spreading" problem with space-dependent diffusion:

$$\frac{\partial c}{\partial t} = a\frac{\partial}{\partial x}\left(|x|^{\epsilon}\frac{\partial c}{\partial x}\right), \quad c(x,0) = Q_0\,\delta(x).$$

(a) Use dimensional analysis to show that $c(x.t)$ has the form of a similarity solution,

$$c(x,t) = \frac{A}{(Bt)^{\alpha}}\,\Phi\left(\frac{x}{(Bt)^{\alpha}}\right).$$

What are the parameters, A, B, and α?

		units
governed param	c	C
governing params	x	L
	t	T
	a	$L^{2-\epsilon}/T$
	Q_0	CL

$n = 4$ params
$k = 3$ independent units

$$\Pi = \Phi(\Pi_1)$$

where $\Pi_1 = \dfrac{x}{(at)^{1/(2-\epsilon)}}$, $\Pi = \dfrac{c\,(at)^{\frac{1}{2-\epsilon}}}{Q_0}$

$$\boxed{A = Q_0, \quad B = a, \quad \alpha = \frac{1}{2-\epsilon}}$$

note: $\epsilon = 0 \Rightarrow \alpha = \frac{1}{2}$ as in linear diffusion with $a = D =$ constant.

(b) Substitute the similarity form into the PDE and derive the scaling function, $\Phi(z)$. [Hint: since the solution must be even, $c(-x,t) = c(x,t)$, only solve for $x > 0$.]

ODE $\boxed{-\alpha(\Phi + z\Phi') = (|z|^{\epsilon}\Phi')'}$ subject to $\int_{-\infty}^{\infty} \Phi(z)\,dz = 1$ and $\Phi(\pm\infty) = 0$

Only consider $z > 0$

$-\alpha(z\Phi)' = (z^{\epsilon}\Phi')$

$-\alpha z\Phi = z^{\epsilon}\Phi'$ using $\Phi(\infty) = 0$ (smoothly)

$-\alpha z^{1-\epsilon}\,dz \quad \frac{d\Phi}{\Phi}$ separate variables

If $\epsilon < 2$

$-\frac{\alpha z^{2-\epsilon}}{2-\epsilon} + K = \log\Phi$ Note: $\frac{\alpha}{2-\epsilon} = \frac{1}{(2-\epsilon)^2}$

$$\boxed{\Phi = K' e^{-|z|^{2-\epsilon}/(2-\epsilon)^2} \quad \text{where} \quad K' = \frac{1}{\int_{-\infty}^{\infty} e^{-|z|^{2-\epsilon}/(2-\epsilon)^2}\,dz}}$$

$\boxed{\text{for } \epsilon < 2}$

If $\epsilon > 2$ get no solution! (Spreading at large x is "too fast")

Note when $\epsilon = 0$, we get $\Phi = \frac{e^{-z^2/4}}{\sqrt{4\pi}}$ as expected

but $\epsilon = 1$ gives $\Phi = \frac{e^{-|z|}}{2}$

Note: $\epsilon < 0$ is ok too! In that case spreading near 0 is very fast compared to very slow far away ($\alpha \to 0$ as $\epsilon \to -\infty$)

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6. (10 POINTS) Consider a very viscous, Newtonian fluid, whose velocity, $\vec{u}$, and pressure, P, satisfy the Stokes equations,

$$\nabla P = \eta \nabla^2 \vec{u}, \quad \nabla \cdot \vec{u} = 0$$

which contains a spherical bubble of radius, R, and surface tension, γ. (Recall that P has units of energy/volume.)

(a) Suppose that the bubble is filled with a uniform gas at pressure, P_0, and remains in equilibrium. From the energy balance, $\gamma A = P_0 V$, where A and V are the surface area and volume of the bubble, respectively, calculate the "Laplace pressure", $P_0(\gamma, R)$.

$$4\pi R^2 \gamma = \frac{4}{3}\pi R^3 P_0$$

$$\boxed{P_0 = \frac{3\gamma}{R}}$$

Note: P_0 has units $\frac{M(L/T)^2}{L^3} = \frac{M}{LT^2}$, so γ has units $\frac{\text{energy}}{\text{area}} = \frac{M}{T^2}$.

(b) Now suppose that the bubble is empty, so that surface tension causes it to collapse in finite time, hindered only by the viscosity of the surrounding fluid. Use dimensional analysis to show how the bubble lifetime, t_c, depends on R, η, and γ.

From $\nabla P = \eta \nabla^2 \vec{u}$, we see that η has units N given by:

$$\frac{1}{L}\frac{M}{LT^2} = N \frac{1}{L^2}\frac{L}{T} \Rightarrow N = \frac{M}{LT}.$$

	units
t_c	T
R	L
γ	M/T^2
η	M/LT

$n = 3$ governing params

$k = 3$ independent units

$\Rightarrow$ Π = constant with no dimensionless governing parameters.

$$\therefore \Pi = \frac{t_c \gamma}{\eta R} \Rightarrow \boxed{t_c = \Pi \cdot \frac{\eta R}{\gamma}}$$

An exact solution of this problem due to Hopper (1990) gives the constant $\Pi = 1$. (if I recall correctly)

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7. (15 POINTS) Use the Method of Characteristics to solve

$$\rho \frac{\partial \rho}{\partial x} + 2e^x \frac{\partial \rho}{\partial y} = -\rho^2$$

subject to the boundary condition, $\rho = 1$ on the curve $y = e^x$.

$$\boxed{\frac{dx}{d\eta} = \rho, \quad \frac{dy}{d\eta} = 2e^x, \quad \frac{d\rho}{d\eta} = -\rho^2}$$

on characteristics labeled by ξ = initial x value; we parameterize the curve with BC $\rho = 1$ as $x = \xi$, $y = e^{\xi}$ at $\eta = 0$

$$\boxed{\rho = \frac{1}{1+\eta}}$$

$$\frac{dy}{d\eta} = 2(1+\eta)e^{\xi} \Rightarrow \boxed{y = e^{\xi}(1+\eta)^2}$$

$$\frac{dx}{d\eta} = \frac{1}{1+\eta} \Rightarrow \boxed{x = \xi + \ln(1+\eta)}$$

Note: characteristic curves are given by

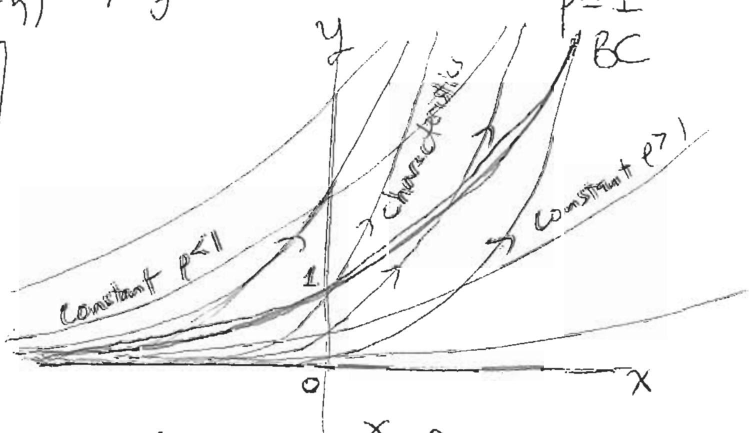
$$\ln y = \xi + 2\ln(1+\eta) = 2x - \xi$$

$$\boxed{y = (e^{-\xi})e^{2x}}$$

the solution is then given by

$$e^x = (1+\eta)e^{\xi} = (1+\eta)\frac{y}{(1+\eta)^2} = \frac{y}{(1+\eta)} = \rho y$$

or simply $\boxed{\rho = e^x/y}$



Note: solution reaches $y > e^x$ for $\eta > 0$

$0 < y < e^x$ for $\eta < 0$

and never $y \le 0$.