18.311 Principles of Applied Mathematics, Spring 2006, Prof. Bazant

# Practice Problems on Method of Characteristics

1. Solve for $\psi(x, y)$,
$$x^2\psi_x + xy\psi_y = \psi^2$$
subject to the boundary condition $\psi = 1$ on the curve $\Gamma$ defined by $x = y^2 \neq 0$. Sketch the characteristics passing through $\Gamma$. Describe the set in the $(x, y)$ plane where the solution is defined, i.e. those points which can be reached from the curve $x = y^2$ by following characteristics.

2. Solve
$$x^2\psi_x - xy\psi_y = \psi^2$$
subject to the same boundary condition as above.

3. Consider the river equation,
$$A_t + (q(A))_x = R_o$$
where $A$ is the cross-sectional area of the river, $q(A) = Au(A)$ is the flow rate, $u(A)$ is the average water velocity given by,
$$u(A) = u_o(A/A_o)^{1/4}$$
(where $u_o$ and $A_o$ are constants), and $R_o$ is a constant rate of water entering the river (per unit length) to represent flood water spilling into the river uniformly across its length (e.g. during a heavy rain). As discussed in lecture, simple river waves "break forward" to form shock waves. Suppose we have a shock at $x = 0$ at $t = 0$,
$$A(x, 0) = \begin{cases} A_1 & \text{for} x < 0 \\ A_2 & \text{for } x > 0 \end{cases}$$
where $A_1 > A_2$.

   (a) How does the strength of the shock $[A]$ (jump in area) evolve in time?

   (b) Find the position of the shock $x_s(t)$ at all later times, and show that
$$x_s(t) \sim Ct^\nu, \text{ as } t \to \infty.$$
   What are the constants $C$ and $\nu$?

1. The boundary curve $\Gamma$ can be parameterized by $y = \xi$ and $x = \xi^2$ for all real $\xi$. The solution along the characteristic labeled by $\xi$ satisfies,

$$\frac{dx}{d\eta} = x^2, \quad \frac{dy}{d\eta} = x\,y, \quad \frac{d\psi}{d\eta} = \psi^2$$

with $x = \xi^2$, $y = \xi$, $\psi = 1$ at $\eta = 0$. Integrating two equations, we obtain,

$$x = \frac{\xi^2}{1 - \xi^2 \eta}$$

and

$$\psi = \frac{1}{1 - \eta}.$$

Substituting into the third, we obtain a separable equation, $dy/y = x\,d\eta = \xi^2 d\eta/(1 - \xi^2 \eta)$, which can be integrated to obtain,

$$y = \frac{\xi}{1 - \xi^2 \eta}.$$

(using the boundary conditions).

The solution is obtained by eliminating the characteristic variables $\eta$ and $\xi$. Since $\xi = x/y$, we have

$$x = \frac{(x/y)^2}{1 - (x/y)^2 \eta}$$

which implies

$$\eta = \left(\frac{y}{x}\right)^2 - \frac{1}{x}.$$

Substituting into $\psi = 1/(1 - \eta)$ yields the solution,

$$\psi = \frac{x^2}{x^2 + x - y^2}.$$

Note that the characteristics are straight lines passing through the origin with slope $\xi$,

$$x = \xi y.$$

The origin is actually a singular point of the PDE, because it reduces to $\psi^2 = 0$ for $x = y = 0$, which violates the condition $\psi = 1$ on $x = y^2$. Therefore, the solution is defined everywhere in the half plane $x > 0$, except on the $x$-axis $y = 0$ (which would correspond to the $\xi = \pm\infty$ characteristics).

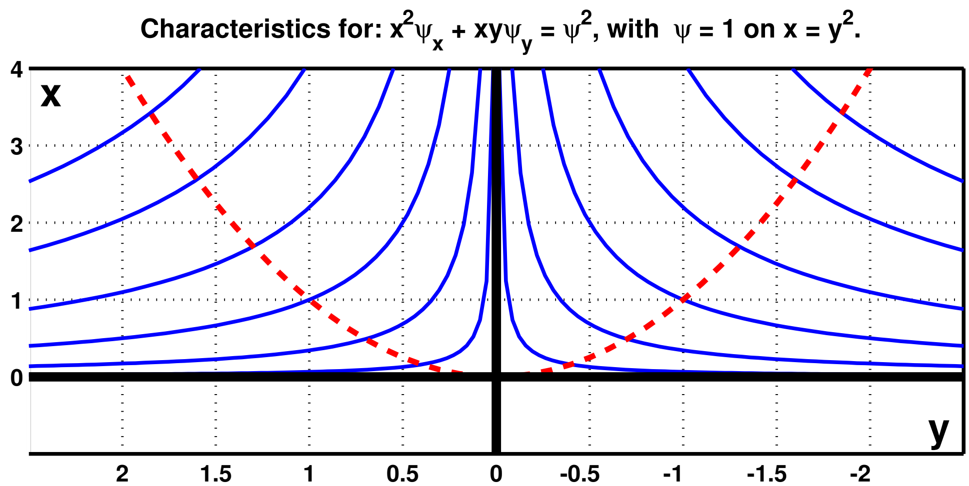


Figure 1: (Problem 2) Characteristics for the first order quasilinear P.D.E. problem given by $x^2\,\psi_x - x\,y\,\psi_y = \psi^2$, with the condition $\psi = 1$ on the curve $\Gamma$ given by $x = y^2$. The picture has been rotated by $\pi/2$ radians (rotate the page to have the $y$ axis vertical.)

2. (by R. Rosales) This is a quasilinear problem (linear in the derivatives), that can be written in terms of characteristics. Namely, if we use $s$ as a parameter along the characteristics, and use $\xi$ to parameterize the curve $\Gamma$ (i.e.: $x = \xi^2$ and $y = \xi$, for $-\infty < \xi < \infty$) we have:

$$\frac{d\psi}{d\eta} = \psi^2 \quad \text{along} \quad \frac{dx}{d\eta} = x^2 \quad \text{and} \quad \frac{dy}{d\eta} = -x\,y\,, \tag{1}$$

with the condition: $\psi = 1$, $x = \xi^2$, and $y = \xi$, for $s = 0$ and $-\infty < \xi < \infty$. The equations for $\psi$ and $x$ are separable, and have the general solutions $\psi = 1/(c_0 - \eta)$ and $x = 1/(c_1 - \eta)$, where $c_0$ and $c_1$ are constants. Once we know $x$, the equation for $y$ is easy to solve, giving the general solution $y = c_2\,(c_1 - \eta)$, where $c_2$ is a constant. Using now the conditions at $s = 0$ along the curve $\Gamma$, we obtain:

$$x = \frac{\xi^2}{1 - \xi^2\,\eta}\,, \tag{2}$$

$$y = \xi - \xi^3\,\eta\,, \tag{3}$$

$$\psi = \frac{1}{1 - \eta}\,, \tag{4}$$

where $-\infty < \eta < 1/\xi^2$.

Note that:

**1.** As $\eta \to 1/\xi^2$, $x \to \infty$ and $y \to 0$.

**2.** As $\eta \to -\infty$, $x \to 0$ and $y \to -\text{sign}(\xi)\,\infty$.

**3.** There is no characteristic curve corresponding to $\xi = 0$. The equations above just give $x \equiv y \equiv 0$. This corresponds to the fact that: For $x = 0$ equation (**??**) is singular (note that the left hand side vanishes), and only the value $\psi = 0$ is consistent with the equation. Thus for the purposes of this problem, the curve $\Gamma$ should be split into two branches: $y > 0$ and $y < 0$, with no value for $\psi$ given at $x = y = 0$.

From equations (2–4) it follows that:

**A.** The characteristics are the family of **hyperbolae** $x\,y = \xi^3$, where $-\infty < \xi \neq 0 < \infty$, and $x > 0$ (see figure 1). They cover the entire first ($x,\ y > 0$) and fourth quadrants ($x > 0 > y$), with precisely characteristic going through each point.

**B.** Substituting $\xi = \sqrt[3]{xy}$ into equation (3) we obtain $\eta = \left(\dfrac{1}{xy}\right)^{2/3} - \dfrac{1}{x}$. Finally, substituting this into equation (4) we find

$$\psi = \frac{x\,y^{2/3}}{x\,y^{2/3} + y^{2/3} - x^{1/3}}\,, \tag{5}$$

for the solution to the problem (defined in the first ($x,\ y > 0$) and fourth ($x > 0 > y$) quadrants only!) — since this is the only region that the characteristics reach.

**Note:** it is easy to check directly that (5) solves the problem. In doing the calculations, noting that $\varphi = 1/\psi$ satisfies the linear equation $x^2\,\varphi_x + x\,y\,\varphi_y = -1$, simplifies the algebra considerably.

3. (a) Since $dA/dt = R_o$ or $A = A(\xi, 0) + R_o t$ along characteristics, the shock strength is constant in time, $[A] = A_1 - A_2$. Note, however, that change in water height $[h]$ will slowly decrease if $A \propto h^2$ (e.g. for a triangular cross section).

(b) The shock velocity is given by

$$\frac{dx_s}{dt} = \frac{[q(A)]}{[A]} = \frac{u_o\left((A_1 + R_o t)^{5/4} - (A_2 + R_o t)^{5/4}\right)}{A_o^{1/4}[A]}.$$

Solving this linear ODE with the initial condition $x_s(0) = 0$ yields the shock locus,

$$x_s(t) = \frac{4u_o\left((A_1 + R_o t)^{9/4} - (A_2 + R_o t)^{9/4}\right)}{9R_o A_o^{1/4}[A]}.$$

For long times ($t \gg R_o/A_2$, to be precise), we obtain the asymptotic result,

$$\begin{aligned} x_s(t) &= \frac{4u_o(R_o t)^{9/4}\left((1+\frac{A_1}{R_o t})^{9/4} - (1+\frac{A_2}{R_o t})^{9/4}\right)}{9R_o A_o^{1/4}[A]} \\ &\sim \frac{u_o(R_o t)^{5/4}(A_1 - A_2)}{R_o A_o^{1/4}[A]} \\ &= Ct^{\nu} \end{aligned}$$

by Taylor expansion, where

$$C = \frac{u_o R_o^{1/4}}{A_o^{1/4}}$$

and $\nu = 5/4$. As the water level rises (and area increases) due to the arrival of rainwater down stream, the flood wave accelerates.