Name SOLUTIONS

18.311, Principles of Applied Mathematics, Spring 2005, Bazant

Midterm Exam – Thursday, March 31, 2005

Instructions: Please write your name on every page. This closed-book exam will last 85 minutes. Point totals for each problem are given out of 100.

1. (50 POINTS TOTAL) Consider the traffic-flow PDE,

$$\frac{\partial \rho}{\partial t} + u_m \left(1 - \frac{2\rho}{\rho_j}\right) \frac{\partial \rho}{\partial x} = 0. \tag{1}$$

with the initial traffic density,

$$\rho(x,0) = \begin{cases} \frac{\rho_j}{2} & \text{if } x \le 0 \\ \frac{\rho_j}{2}\left(1 + \frac{x}{d}\right) & \text{if } 0 < x < d \qquad \text{fan} \\ \rho_j & \text{if } x \ge d \end{cases} \tag{2}$$

(a) (25 POINTS) Solve for $t > 0$ until a shock forms, and sketch the time evolution of $\rho(x,t)$.

Characteristics $\frac{dx}{dt} = c(\rho) = \text{constant} \quad \left(\frac{d\rho}{dt} = 0\right)$

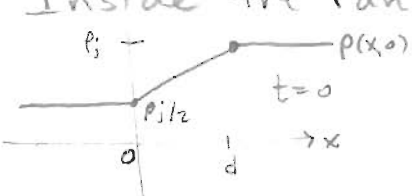
$c(\rho) = u_m(1 - 2\rho/\rho_j)$, $c(\rho_j/2) = 0$, $c(\rho_j) = -u_m$

So,

$$\boxed{\rho(x,t) = \begin{cases} \rho_j/2 & x \le 0 \\ \rho_F(x,t) & 0 < x < d - u_m t \\ \rho_j & x \ge d - u_m t \end{cases}}$$

until a shock forms at time $\boxed{t_s = d/u_m}$

Inside the fan,

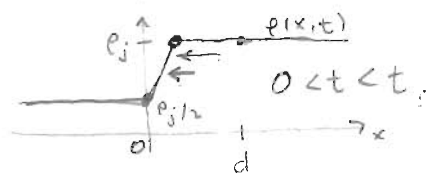


$$x = c(\rho_F)t + x_0 \quad \text{and} \quad \rho_F = \frac{\rho_j}{2}\left(1 + \frac{x_0}{d}\right) \text{ at } t=0$$

$$= u_m\left(1 - \frac{2\rho_F}{\rho_j}\right)t + x_0 \qquad x_0 = d\left(\frac{2\rho_F}{\rho_j} - 1\right)$$

= characteristic label.

$$= \left(1 - \frac{2\rho_F}{\rho_j}\right)(u_m t - d)$$



$$\boxed{\rho_F(x,t) = \frac{\rho_j}{2}\left(1 + \frac{x}{d - u_m t}\right)}$$

linear interpolation between $\rho(0,t) = \rho_j/2$ and $\rho(d - u_m t, t) = \rho_j$

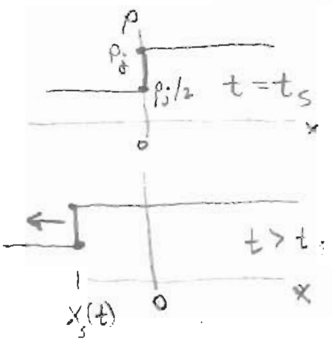
(b) (10 POINTS) Solve for $\rho(x,t)$ after the shock forms.

$$c(\rho) = u_m\left(1 - \frac{2\rho}{\rho_j}\right) = q'(\rho)$$

$$q(\rho) = u_m \rho \left(1 - \rho/\rho_j\right)$$

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]} = \frac{\cancel{q(\rho_j)} - q(\rho_j/2)}{\rho - \rho_j/2}$$

$$= \frac{-u_m\,(\cancel{\rho_j/2})(1 - \rho_j/2/\rho_j)}{\cancel{\rho_j/2}} = -\frac{u_m}{2}$$

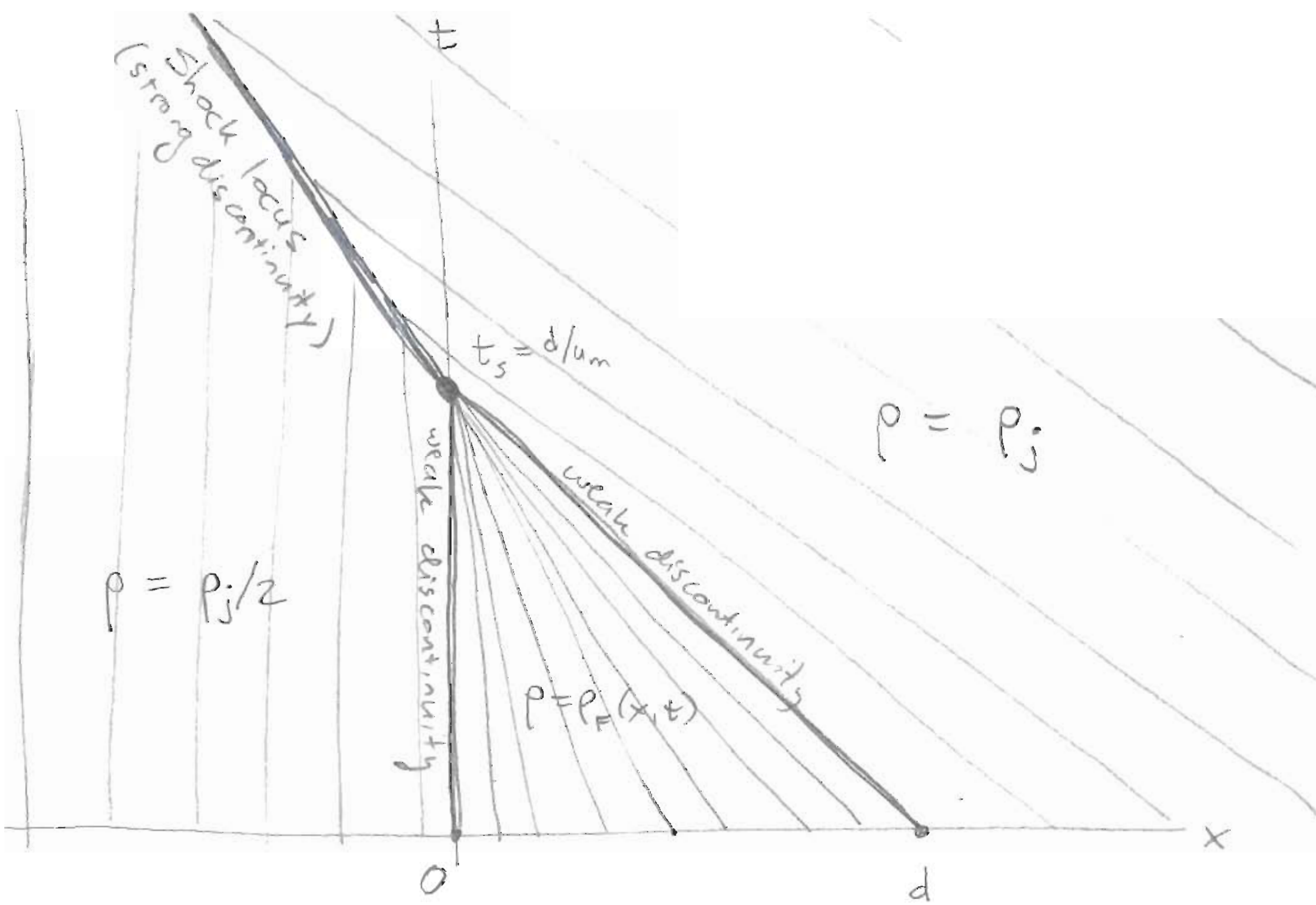


Shock starts at $x = 0$ at $t = t_s = d/u_m \Rightarrow X_s(t) = -\frac{u_m}{2}\left(t - \frac{d}{u_m}\right)$

so

$$\boxed{\rho(x,t) = \begin{cases} \rho_j/2, & x < \dfrac{d - u_m t}{2} \\ \rho_j, & x > \dfrac{d - u_m t}{2} \end{cases}}$$

(c) (15 POINTS) Carefully draw a space-time diagram showing characteristics leaving the x axis for $t > 0$. Label the trajectories of weak discontinuities (starting at $x = 0$ and $x = d$), regions of constant density ($\rho = \rho_j/2$ and $\rho = \rho_j$), and the shock locus.



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2. (25 POINTS) Solve the first-order quasi-linear PDE,

$$\psi_t + \psi\psi_x = 1$$

for $t > 0$ subject to the initial condition $\psi(x, 0) = x$

$$\frac{d\psi}{dt} = 1 \quad \text{on} \quad \frac{dx}{dt} = \psi$$

haracteristics

$$\psi = t + \xi$$

where

$\xi = \psi(t=0) = x(t=0)$

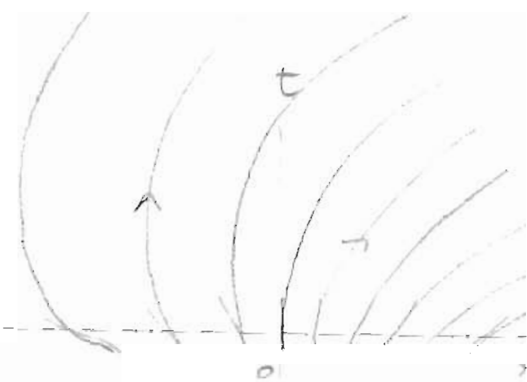
= characteristic label

$$\frac{dx}{dt} = t + \xi$$

$$x = \frac{1}{2}t^2 + \xi t + \xi$$

$$= \xi(t+1) + \frac{1}{2}t^2$$

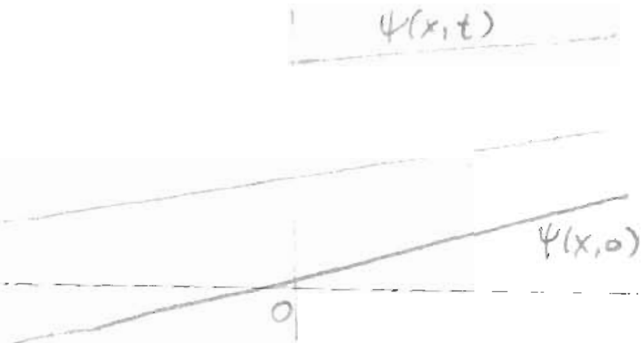
$$\Rightarrow \quad \xi = \frac{x - \frac{1}{2}t^2}{t+1}$$



characteristics are parabolae

So,

$$\boxed{\psi = t + \frac{x - \frac{1}{2}t^2}{t+1} = \frac{t^2 + 2(t + x)}{2(t+1)}}$$



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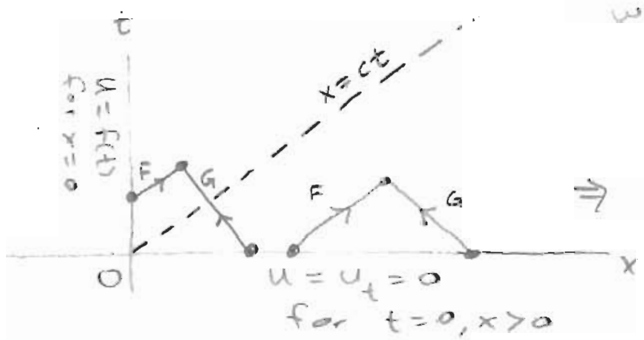
3. (25 POINTS) Solve the linear wave equation,

$$u_{tt} = c^2 u_{xx},$$

for $x > 0$ and $t > 0$ for a semi-infinite string initially at rest, $u(x,0) = u_t(x,0) = 0$, displaced at one end according to $u(0,t) = f(t)$.

d'Alembert : $u(x,t) = \underbrace{F(x-ct)}_{\text{right-moving wave}} + \underbrace{G(x+ct)}_{\text{left moving wave}}$



$\Rightarrow$ $u = 0$ for $x > ct$

For $x < ct$, $G = 0$, but $F \neq 0$

$$u = F(x - ct)$$

$$u(x,0) = F(-ct) = f(t)$$

$$\Rightarrow \quad F(z) = f(z/(-c))$$

$$F(x - ct) = f\left(t - \frac{x}{c}\right)$$

$$\boxed{u(x,t) = \begin{cases} f\left(t - \frac{x}{c}\right), & 0 < x < ct \\ 0, & x > ct \end{cases}}$$