Name ________________________________

18.311, Principles of Applied Mathematics, Spring 2005, Bazant

Final Exam – Monday, May 16, 2005

Instructions: Please write your name on every page. This closed-book exam will last three hours. Point totals are given for each problem (out of 100). Graded exams, solutions, and final grades will be available after May 18. I hope you enjoyed the class. –MZB

1. (20 POINTS) Consider the initial traffic density for a red light turning green,

$$\rho(x,0) = \begin{cases} \rho_j & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

and assume a *parabolic* velocity-density relationship,

$$u(\rho) = u_{max}\left(1 - \left(\frac{\rho}{\rho_j}\right)^2\right)$$

in the Lighthill-Whitham theory of traffic flow.

(a) Derive a PDE for $\rho(x,t)$ expressing the conservation of cars.

(b) What is the density cars at the traffic light, $\rho(0,t)$, after it turns green ($t > 0$)?

(c) Determine the boundaries of an expansion fan, $x_-(t) < x < x_+(t)$, such that $\rho = \rho_j$ for $x < x_-(t)$ and $\rho = 0$ for $x > x_+(t)$.

(d) Solve for $\rho(x, t)$ inside the expansion fan, and plot the solution for $t > 0$.

2. (15 POINTS) Solve the following (dimensionless) river-flow equation for $x > 0$ and $t > 0$,

$$A_t + \sqrt{A}A_x = -A, \quad A(x,0) = x^2.$$

Plot $A(x,t)$ at some times $t > 0$, and sketch some characteristics in the (x,t) plane.

3. (20 POINTS) *Sketch* the solution, $\rho(x,t)$, to each of the following PDEs for several times $t > 0$ for the initial condition, $\rho(x,0) = e^{-x^2}$. DO NOT SOLVE ANALYTICALLY.

   (a) $\rho_t + \rho_x = 0$

   (b) $\rho_t + (1-\rho)\rho_x = 0$

   (c) $\rho_t = \rho_{xx}$

   (d) $\rho_t + \rho\rho_x = \rho_{xx}$

4. (10 POINTS) Use a Green function to solve the linear diffusion equation, $\rho_t = D\rho_{xx}$, subject to the initial condition,

$$\rho(x,0) = \begin{cases} \rho_o & \text{if } |x| < \ell \\ 0 & \text{if } |x| \geq \ell \end{cases}$$

Express your answer in terms of the error function,

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2}\, dy,$$

and sketch the solution at several times $t > 0$.

5. (10 POINTS) Consider the dispersive wave equation

$$\rho_{tt} = c_0^2(\rho_{xx} + a^2 \rho_{xxxx})$$

(a) Derive the dispersion relation, $\omega(k)$.

(b) Show that in the limit $|k| \ll a^{-1}$ there is no dispersion (phase velocity=group velocity), and $u(x,t)$ approximately satisfies d'Alembert's wave equation.

6. (10 POINTS) Consider the Klein-Gordon equation,

$$u_{tt} = c_0^2 u_{xx} + g(u)$$

for some nonlinear function $g(u)$. Show that a nontrivial solitary wave, $u(x,t) = f(x - ct)$, cannot travel at the linear wave speed ($c \neq c_0$), and derive an ODE for $f(z)$.

7. (15 POINTS) Consider the porous medium equation,

$$\rho_t = a(\rho^2)_{xx}$$

where a is a (constant) nonlinear permeability and $\rho(x,t)$ is the concentration of a fluid spreading from a point source: $\rho(x,0) = q\delta(x)$, $\rho(\pm\infty) = 0$.

(a) Use dimensional analysis to show that there exists a similarity solution of the form,

$$\rho(x,t) = \frac{A}{(Bt)^\nu} F\left(\frac{x}{(Bt)^\nu}\right)$$

What are A, B, and ν?

(b) Show that the scaling function $F(z)$ satisfies the ODE

$$3(F^2)'' + zF' + F = 0$$

subject to $F(\pm\infty) = 0$ and $\int_{-\infty}^{\infty} F(z)dz = 1$.

(c) Solve for $F(z)$.