Lectures 8-9: Signal Detection in Noise

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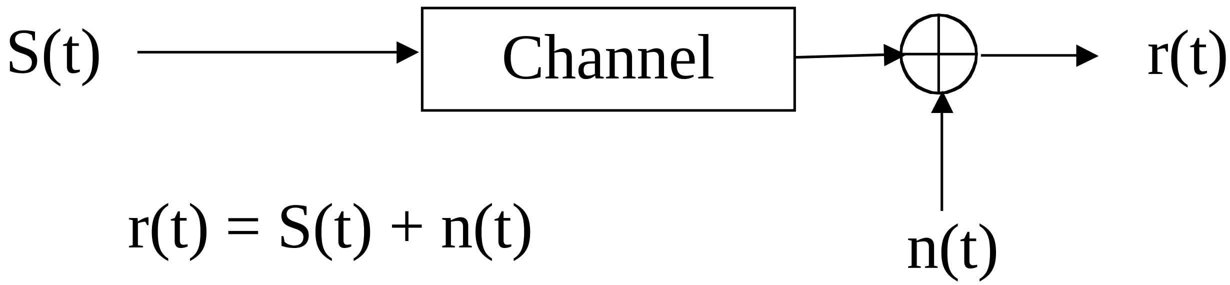
Noise in communication systems



$$r(t) = S(t) + n(t)$$

- **Noise is additional "unwanted" signal that interferes with the transmitted signal**
  - **Generated by electronic devices**

- **The noise is a random process**
  - **Each "sample" of n(t) is a random variable**

- **Typically, the noise process is modeled as "Additive White Gaussian Noise" (AWGN)**
  - **White: Flat frequency spectrum**
  - **Gaussian: noise distribution**

Random Processes

- The auto-correlation of a random process x(t) is defined as
  - $R_{xx}(t_1,t_2) = E[x(t_1)x(t_2)]$
- A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
  - $m_x(t) = E[x(t)] = m$
  - $R_{xx}(t_1,t_2) = R_x(\tau)$, where $\tau = t_1 - t_2$
- If x(t) is WSS then:
  - $R_x(\tau) = R_x(-\tau)$
  - $|R_x(\tau)| <= |R_x(0)|$ (max is achieved at $\tau = 0$)
- The power content of a WSS process is:

$$P_x = E[\lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_x(0)dt = R_x(0)$$

Power Spectrum of a random process

- **If x(t) is WSS then the power spectral density function is given by:**

$$S_x(f) = F[R_x(\tau)]$$

- **The total power in the process is also given by:**

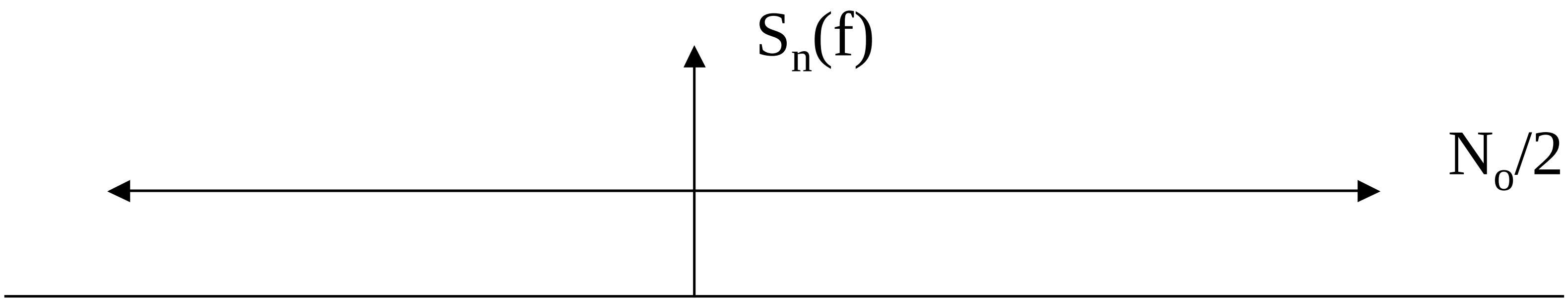
$$P_x = \int_{-\infty}^{\infty} S_x(f)df = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(t)e^{-j2\pi ft}dt\right]df$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(t)e^{-j2\pi ft}df\right]dt$$

$$= \int_{-\infty}^{\infty} R_x(t)\left[\int_{-\infty}^{\infty} e^{-j2\pi ft}df\right]dt = \int_{-\infty}^{\infty} R_x(t)\delta(t)dt = R_x(0)$$

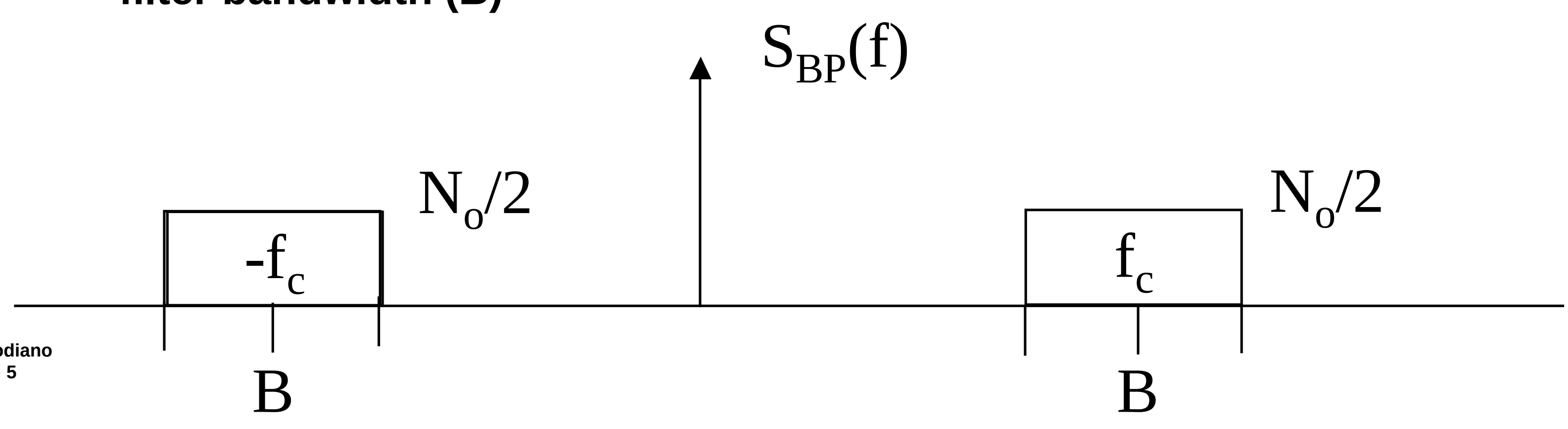
White noise

- The noise spectrum is flat over all relevant frequencies
  - White light contains all frequencies



- Notice that the total power over the entire frequency range is infinite
  - But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out

- After filtering the only remaining noise power is that contained within the filter bandwidth (B)



AWGN

- **The effective noise content of bandpass noise is BN_o**
  - **Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable**

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

  - **AKA Normal r.v., $N(0,\sigma^2)$**
  - **$\sigma^2 = P_x = BN_o$**

- **The CDF of a Gaussian R.V.,**

$$F_x(\alpha) = P[X \le \alpha] = \int_{-\infty}^{\alpha} f_x(x)dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

- **This integral requires numerical evaluation**
  - **Available in tables**

AWGN, continued

- $X(t) \sim N(0,\sigma^2)$
- $X(t_1)$, $X(t_2)$ are independent unless $t_1 = t_2$
- $$R_x(\tau) = E[X(t+\tau)X(t)] = \begin{cases} E[X(t+\tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$$

$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$$

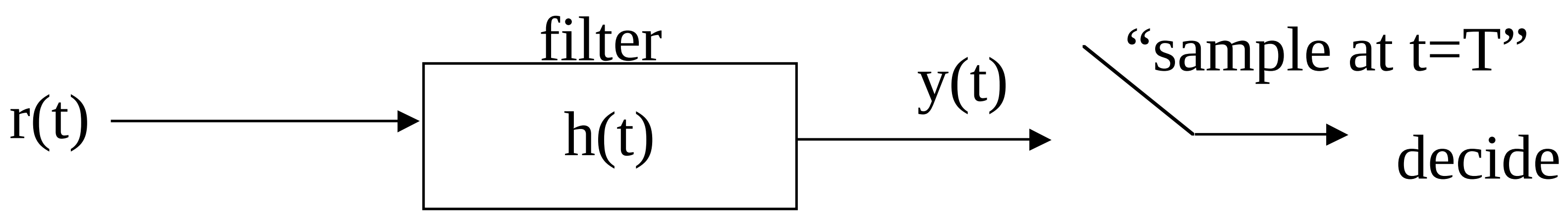
- $R_x(0) = \sigma^2 = P_x = BN_o$

Detection of signals in AWGN

Observe: $r(t) = S(t) + n(t), \; t \in [0,T]$

Decide which of $S_1, \ldots, S_m$ was sent

- **Receiver filter**
  - **Designed to maximize signal-to-noise power ratio (SNR)**



- **Goal: find h(t) that maximized SNR**

Receiver filter

$$y(t) = r(t) * h(t) = \int_0^t r(\tau)h(t-\tau)d\tau$$

Sampling at t = T $\Rightarrow$ $y(T) = \int_0^T r(\tau)h(T-\tau)d\tau$

$$r(\tau) = s(\tau) + n(\tau) \Rightarrow$$

$$y(T) = \int_0^T s(\tau)h(T-\tau)d\tau + \int_0^T n(\tau)h(T-\tau)d\tau = Y_s(T) + Y_n(T)$$

$$SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[\int_0^T s(\tau)h(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} = \frac{\left[\int_0^T h(\tau)s(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt}$$

Matched filter: maximizes SNR

Caushy - Schwartz Inequality:

$$\left[\int_{-\infty}^{\infty} g_1(t) g_2(t) dt\right]^2 \le \int_{-\infty}^{\infty} (g_1(t))^2 \int_{-\infty}^{\infty} (g_2(t))^2$$

Above holds with equality iff: $g_1(t) = c g_2(t)$ for arbitrary constant c

$$SNR = \frac{\left[\int_0^T s(\tau) h(T-\tau) d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} \le \frac{\int_0^T (s(\tau))^2 d\tau \int_0^T h^2(T-\tau)d\tau}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} = \frac{2}{N_0}\int_0^T (s(\tau))^2 d\tau = \frac{2E_s}{N_0}$$

Above maximum is obtained iff: h(T-τ) = cS(τ)

=> h(t) = cS(T-t) = S(T-t)

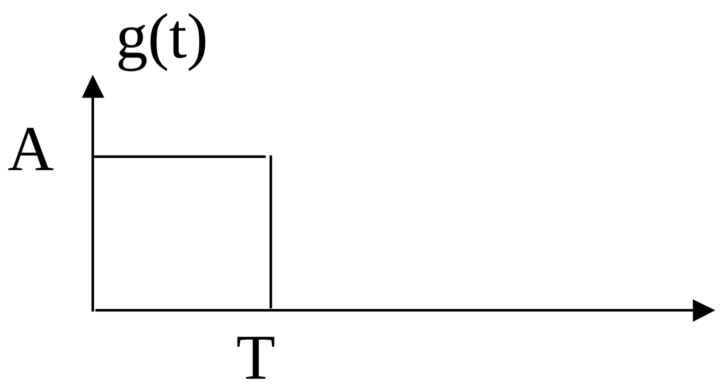
h(t) is said to be "matched" to the signal S(t)

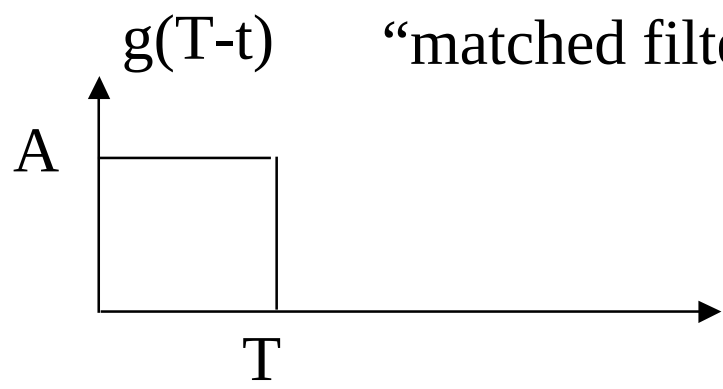
Example: PAM

$S_m(t) = A_m g(t), \; t \in [0,T]$

A_m is a constant: Binary PAM $A_m \in \{0,1\}$

Matched filter is matched to g(t)





"matched filter"

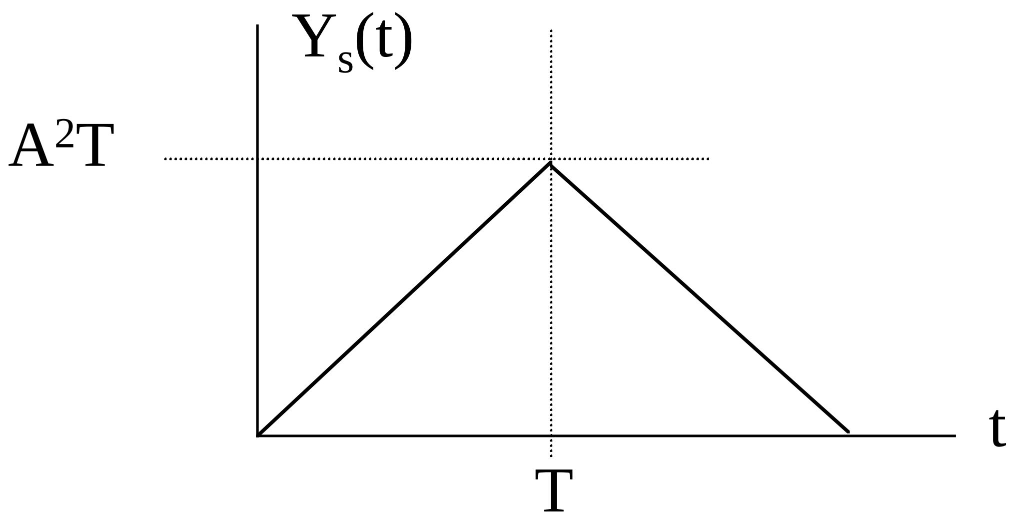
Example, continued

$$Y_s(t) = \int_0^t S(\tau)h(t-\tau)d\tau, \quad h(t') = g(T-t') \Rightarrow h(t-\tau) = g(T+\tau-t)$$

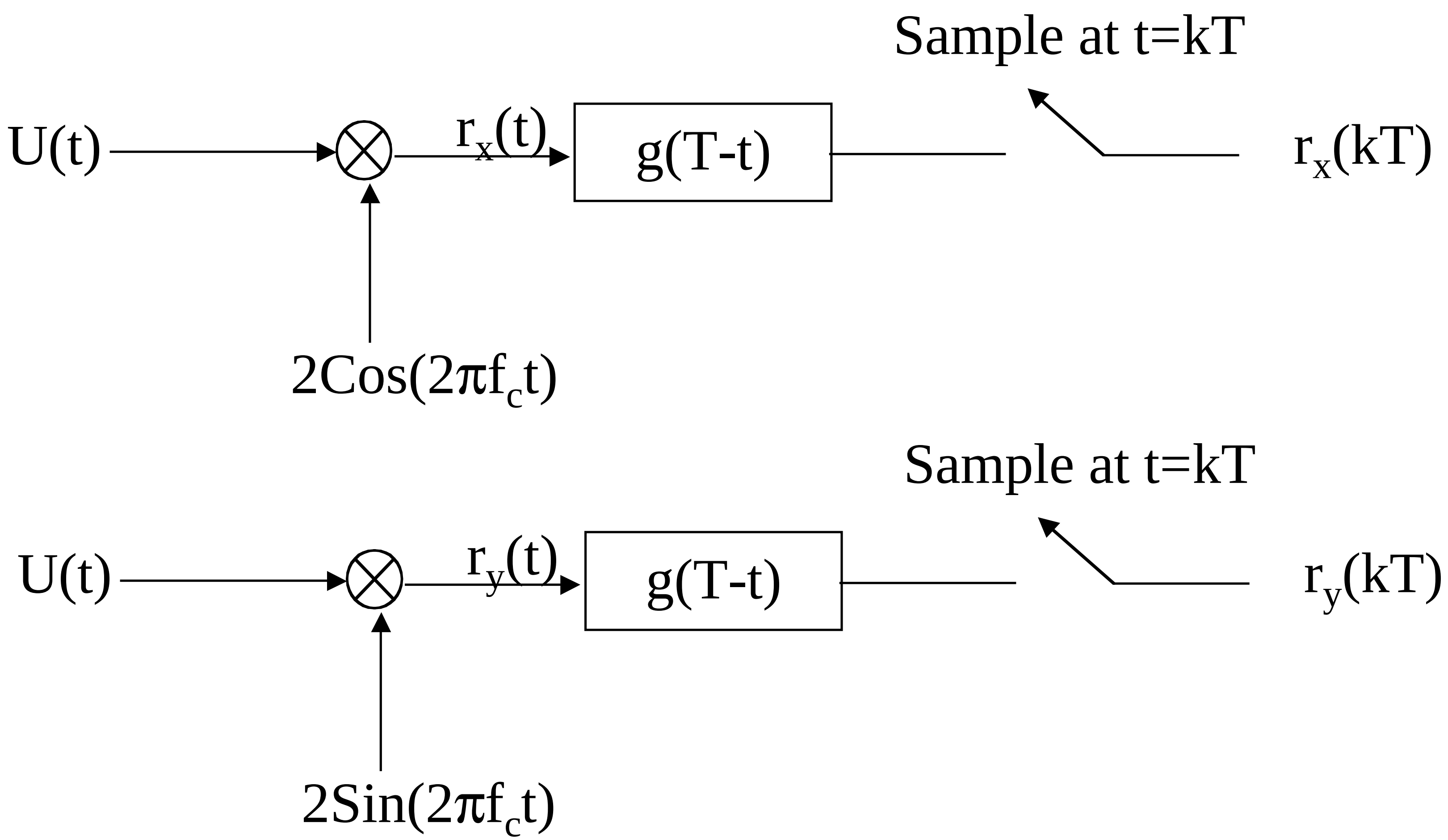
$$Y_s(t) = \int_0^t g(\tau)g(T+\tau-t)d\tau = \int_0^t g(\tau)g(T-t+\tau)d\tau$$

$$Y_s(T) = \int_0^T g^2(\tau)d\tau$$

- **Sample at t=T to obtain maximum value**



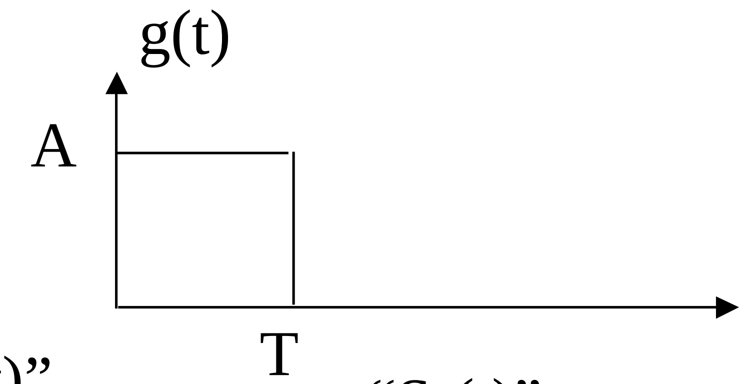
Matched filter receiver

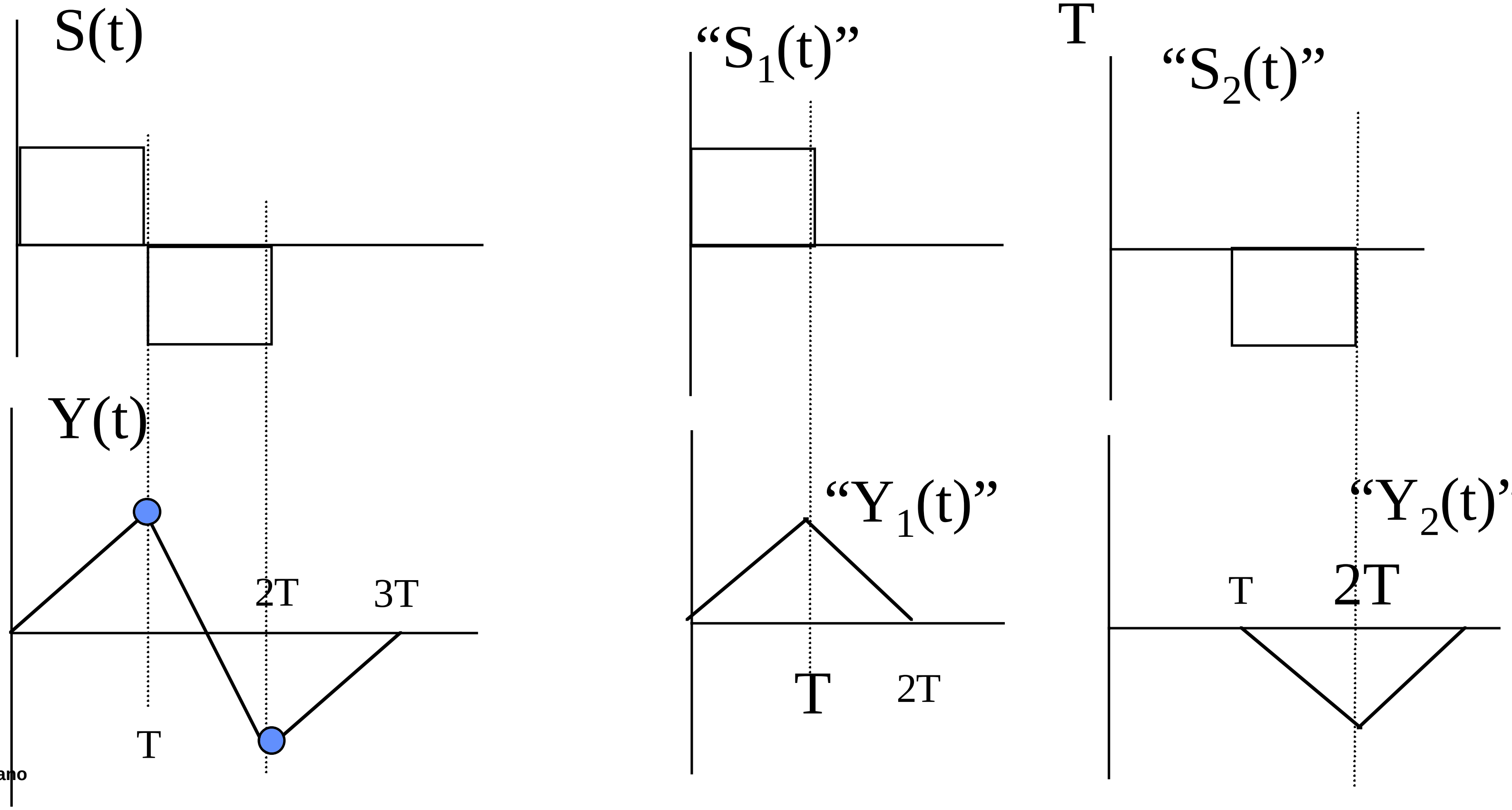


Binary PAM example, continued

$0 \Rightarrow S_1 = g(t)$

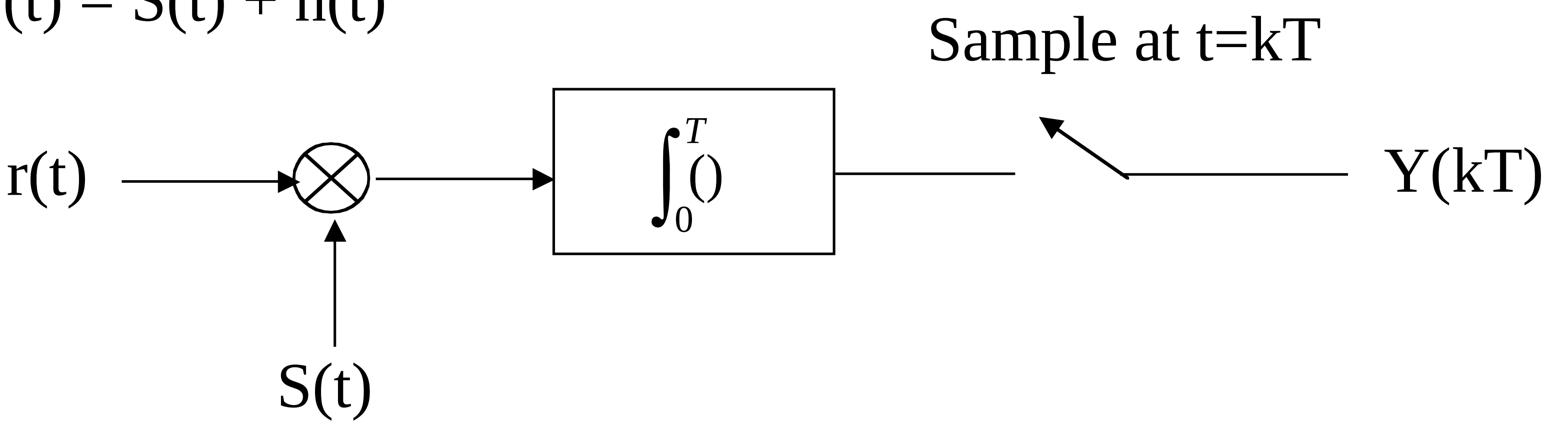
$1 \Rightarrow S_2 = -g(t)$





Alternative implementation: correlator receiver

r(t) = S(t) + n(t)



$$Y(T) = \int_0^T r(t)S(t) = \int_0^T S^2(t) + \int_0^T n(t)S(t) = Y_s(T) + Y_n(T)$$

Notice resemblance to matched filter

Signal Detection

- After matched filtering we receive $r = S_m + n$
  - $S_m \in \{S_1,..S_M\}$
- How do we determine from r which of the M possible symbols was sent?
  - Without the noise we would receive what sent, but the noise can transform one symbol into another

Hypothesis testing

- Objective: minimize the probability of a decision error
- Decision rule:
  - Choose S_m such that $P(S_m \text{ sent} \mid r \text{ received})$ is maximized
- This is known as Maximum a posteriori probability (MAP) rule
- MAP Rule: Maximize the conditional probability that S_m was sent given that r was received

MAP detector

MAP detector: $\max_{S_1 \ldots S_M} P(S_m \mid r)$

$$P(S_m \mid r) = \frac{P(S_m, r)}{P(r)} = \frac{P(r \mid S_m) P(S_m)}{P(r)}$$

$$P(S_m \mid r) = \frac{f_{r|s}(r \mid S_m) P(S_m)}{f_r(r)}$$

$$f_r(r) = \sum_{m=1}^{M} f_{r|s}(r \mid S_m) P(S_m)$$

When $P(S_m) = \frac{1}{M}$ Map rule becomes:

$\max_{S_1 \ldots S_M} f(r \mid S_m)$ (AKA Maximum Likelihood (ML) decision rule)

- **Notes:**
  - **MAP rule requires prior probabilities**
  - **MAP minimizes the probability of a decision error**
  - **ML rule assumes equally likely symbols**
  - **With equally likely symbols MAP and ML are the same**

Detection in AWGN
(Single dimensional constellations)

$$f(r \mid S_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-S_m)^2 / N_0}$$

$$\ln(f(r \mid S_m)) = -\ln(\sqrt{\pi N_0}) - \frac{(r-S_m)^2}{N_0}$$

$$d_{rS_m} = (r-S_m)^2$$

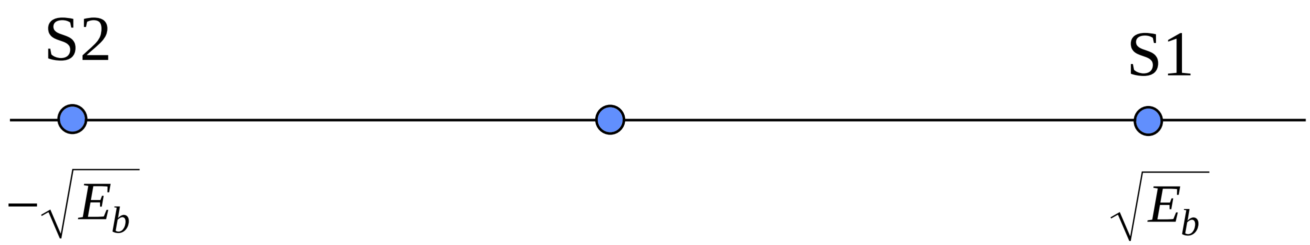
Maximum Likelihood decoding amounts to minimizing $d_{rS_m} = (r-S_m)^2$

- **Also known as minimum distance decoding**
  - **Similar expression for multidimensional constellations**

Detection of binary PAM

- **S1(t) = g(t), S2(t) = -g(t)**
  - **S1 = - S2 => "antipodal" signaling**

- **Antipodal signals with energy Eb can be represented geometrically as**

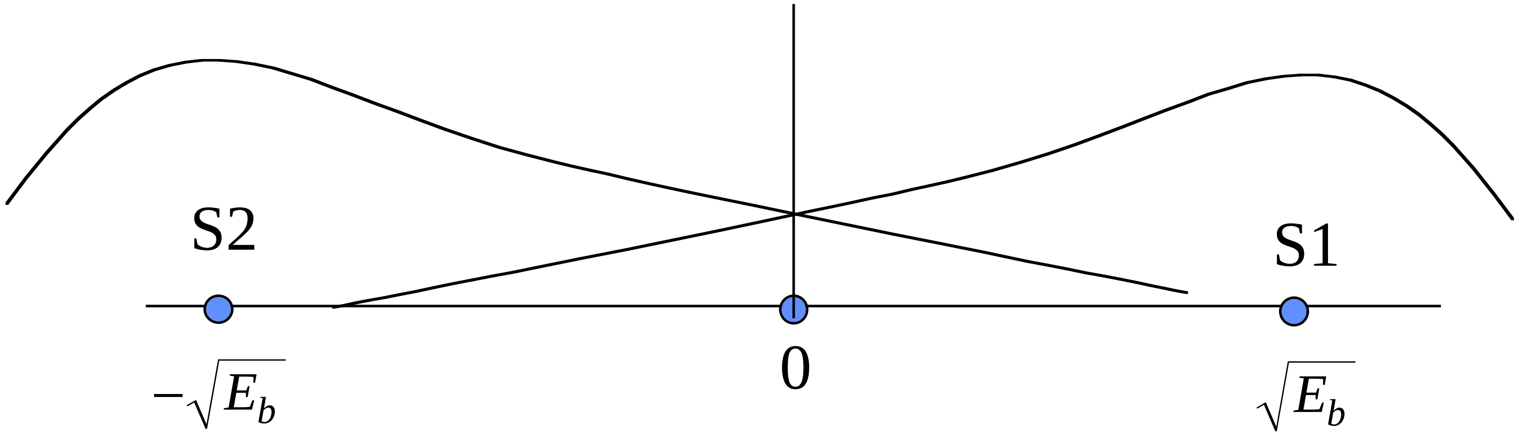


- **If S1 was sent then the received signal r = S1 + n**
- **If S2 was sent then the received signal r = S2 + n**

$$f_{r|s}(r \mid s1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2 / N_0}$$

$$f_{r|s}(r \mid s2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{E_b})^2 / N_0}$$

Detection of Binary PAM



- **Decision rule: MLE => minimum distance decoding**
  - **=> r > 0 decide S1**
  - **=> r < 0 decide S2**

- **Probability of error**
  - **When S2 was sent the probability of error is the probability that noise exceeds $(Eb)^{1/2}$ similarly when S1 was sent the probability of error is the probability that noise exceeds - $(Eb)^{1/2}$**
  - **P(e|S1) = P(e|S2) = P[r<0|S1)**

Probability of error for binary PAM

$$P_e = \int_{-\infty}^{0} f_{r|s}(r \mid s1)dr = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2 / N_0} dr$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\sqrt{E_b}} e^{-r^2 / N_0} dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2E_b / N_0}} e^{-r^2 / 2} dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E_b / N_0}}^{\infty} e^{-r^2 / 2} dr$$

$$\equiv Q(\sqrt{2E_b / N_0}) \quad where,$$

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-r^2 / 2} dr$$

- **Q(x) = P(X>x) for X Gaussian with zero mean and $\sigma^2 = 1$**
- **Q(x) requires numerical evaluation and is tabulated in many math books (Table 4.1 of text)**

More on Q function

- **Notes on Q(x)**
  - **Q(0) = 1/2**
  - **Q(-x) = 1-Q(x)**
  - **Q(∞) = 0, Q(- ∞)=1**

  - **If X is N(m,σ^2) Then P(X>x) = Q((x-m)/ σ)**

- **Example: Pe = P[r<0|S1 was sent)**

$$f_{r|s}(r\mid s1) \sim N(\sqrt{E_b}, N_0/2) => m = \sqrt{E_b}, \sigma = \sqrt{N_0/2}$$

$$P_e = 1 - P[r > 0 \mid s1] = 1 - Q(\frac{-\sqrt{E_b}}{\sqrt{N_0/2}}) = 1 - Q(-\sqrt{2E_b/N_0}) = Q(\sqrt{2E_b/N_0})$$

Error analysis continued

- In general, the probability of error between two symbols separated by a distance d is given by:

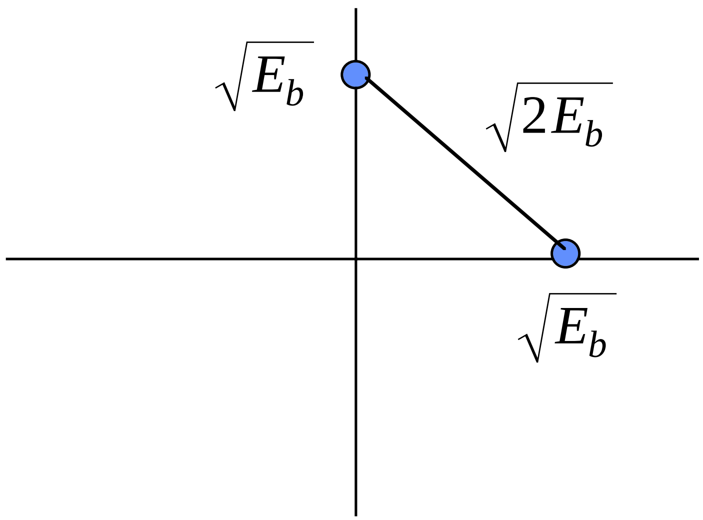
$$P_e(d) = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

- For binary PAM d = 2 $\sqrt{E_b}$ Hence,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Orthogonal signals

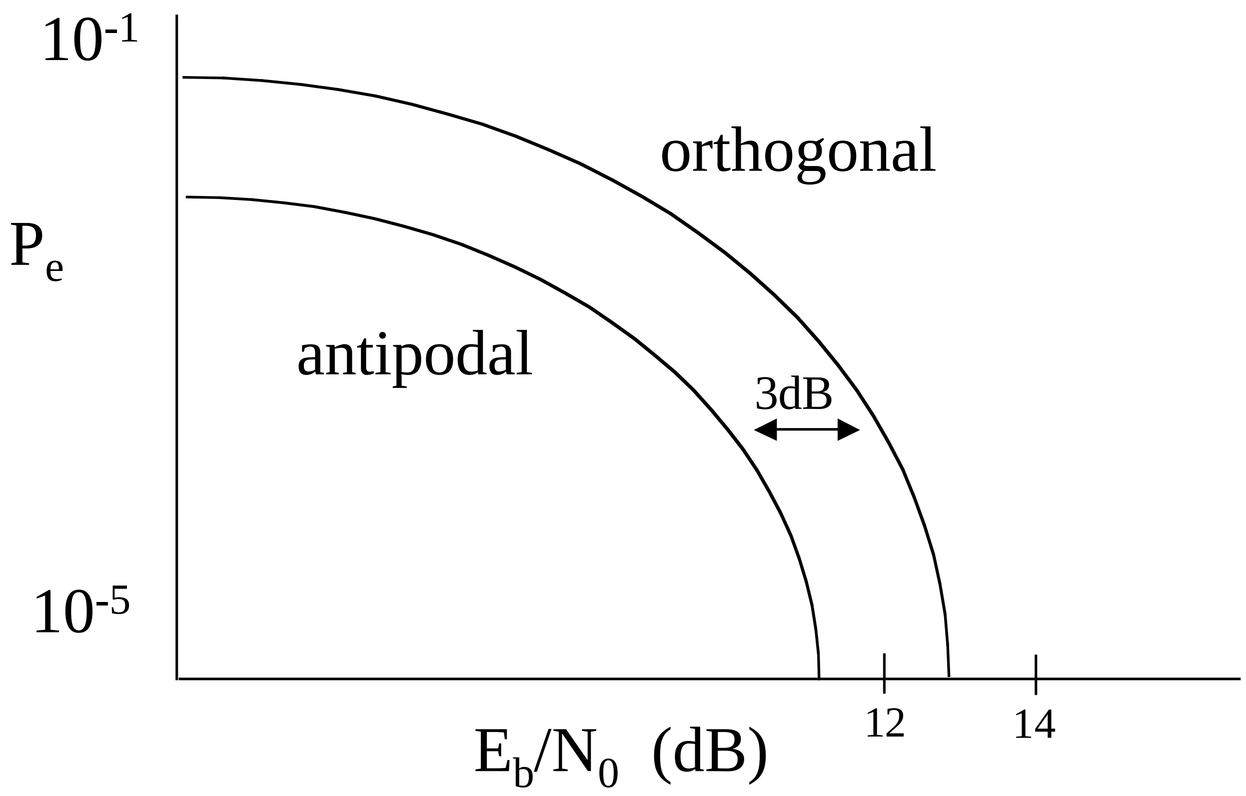
- **Orthogonal signaling scheme (2 dimensional)**



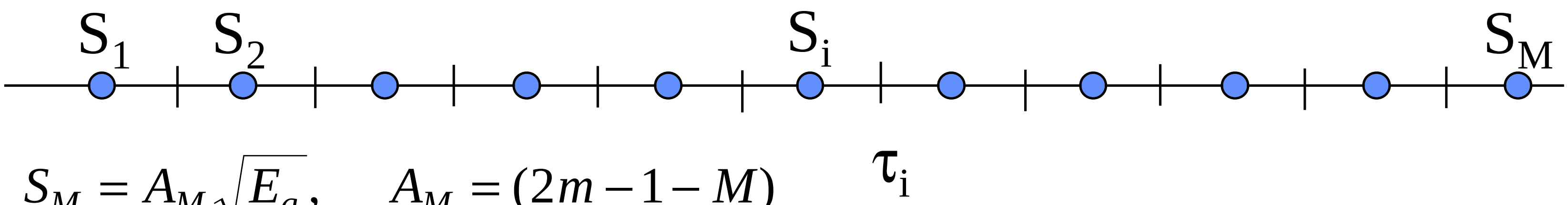
$$P_e = Q(\sqrt{\frac{d^2}{2N_0}} = Q(\sqrt{E_b / N_o})$$

Orthogonal vs. Antipodal signals

- **Notice from Q function that orthogonal signaling requires twice as much bit energy than antipodal for the same error rate**
  - **This is due to the distance between signal points**



Probability of error for M-PAM



$$S_M = A_M \sqrt{E_g}, \quad A_M = (2m-1-M)$$

$$d_{ij} = 2\sqrt{E_g} \quad for\,|i-j|=1$$

Decision rule: Choose s_i such that $d(r, s_i)$ is minimized

$$P[\text{error} \mid s_i] = P[decode\, s_{i-1} \mid s_i] + P[decode\, s_{i+1} \mid s_i] = 2P[decode\, s_{i+1} \mid s_i]$$

$$Pe = 2Q\left[\sqrt{\frac{d_{i,i+1}^2}{2N_0}}\right] = 2Q\left[\sqrt{\frac{2E_g}{N_0}}\right], \quad P_{eb} = \frac{Pe}{Log_2(M)}$$

Notes:

1) the probability of error for s_1 and s_M is lower because error only occur in one direction
2) With Gray coding the bit error rate is $P_e/\log_2(M)$

Probability of error for M-PAM

$$E_{av} = \frac{M^2 - 1}{3} E_g \Rightarrow E_{bav} = \frac{M^2 - 1}{3 Log_2(M)} E_g$$

$$E_g = \frac{3 Log_2(M)}{M^2 - 1} E_{bav}$$

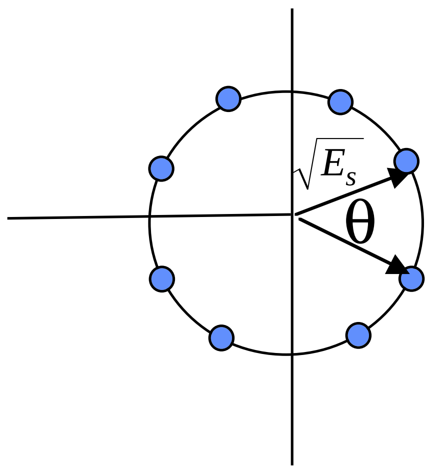
$$P_e = 2Q\left[\sqrt{\frac{6 Log_2(M)}{(M^2 - 1) N_0} E_{bav}}\right], \quad P_{eb} = \frac{Pe}{Log_2(M)}$$

accounting for effect of S_1 *and* S_M *we get :*

$$P_e = 2\left(\frac{M-1}{M}\right) Q\left[\sqrt{\frac{6 Log_2(M)}{(M^2 - 1) N_0} E_{bav}}\right],$$

Probability of error for PSK

- **Binary PSK is exactly the same as binary PAM**
- **4-PSK can be viewed as two sets of binary PAM signals**
- **For large M (e.g., M>8) a good approximation assumes that errors occur between adjacent signal points**



$$\theta = 2\pi/M$$

$$d_{ij} = 2\sqrt{E_s} Sin(\frac{\pi}{M}), \qquad |i-j| = 1$$

Error Probability for PSK

$$P[\text{error} \mid s_i] = P[decode\, s_{i-1} \mid s_i] + P[decode\, s_{i+1} \mid s_i] = 2P[decode\, s_{i+1} \mid s_i]$$

$$P_{es} = 2Q\left[\sqrt{\frac{d_{i,i+1}^2}{2N_0}}\right] = 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi / M)\right]$$

$$E_b = E_s / Log_2(M)$$

$$P_{es} = 2Q\left[\sqrt{\frac{2Log_2(M)E_b}{N_0}}\sin(\pi / M)\right], \quad P_{eb} = \frac{P_{es}}{Log_2(M)}$$