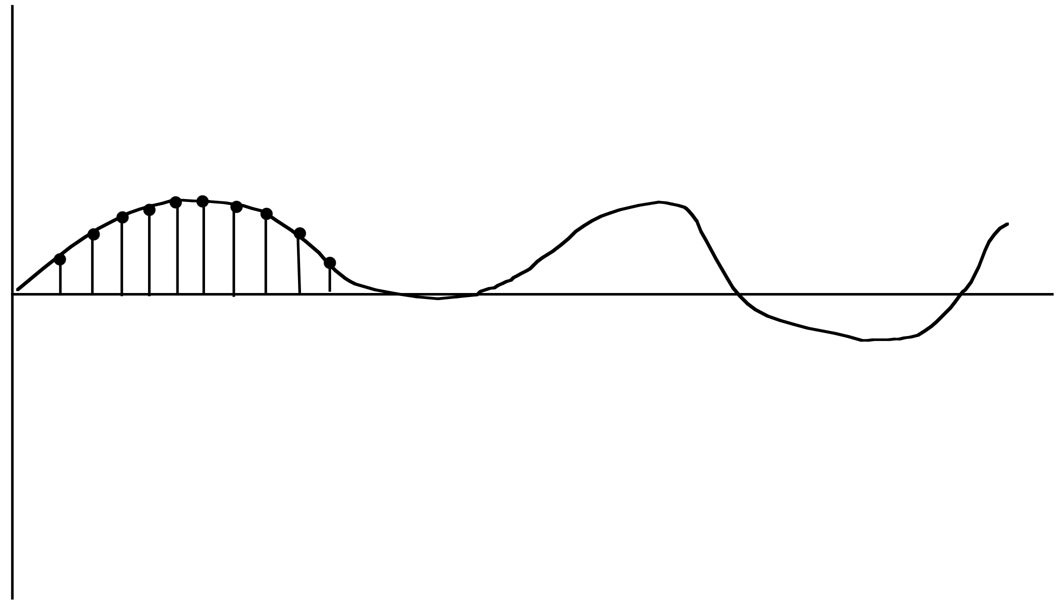
Lecture 3: The Sampling Theorem

Eytan Modiano
AA Dept.

Sampling

- **Given a continuous time waveform, can we represent it using discrete samples?**
  - **How often should we sample?**
  - **Can we reproduce the original waveform?**



The Fourier Transform

- **Frequency representation of signals**
- **Definition:** $$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- **Notation:**

X(f) = F[x(t)]
X(t) = F-1 [X(f)]
x(t) ↔ X(f)

Unit impulse δ(t)

$$\delta(t) = 0, \forall t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

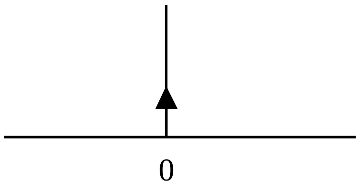
$$\int_{-\infty}^{\infty} \delta(t) x(t) = x(0)$$

$$\int_{-\infty}^{\infty} \delta(t-\tau) x(\tau) = x(t)$$

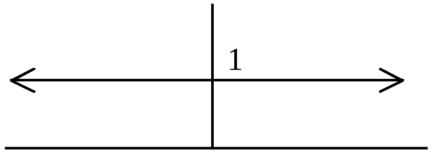
$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = e^{0} = 1$$

$$\delta(t) \Leftrightarrow 1$$

$\delta(t)$



$F[\delta(t)]$

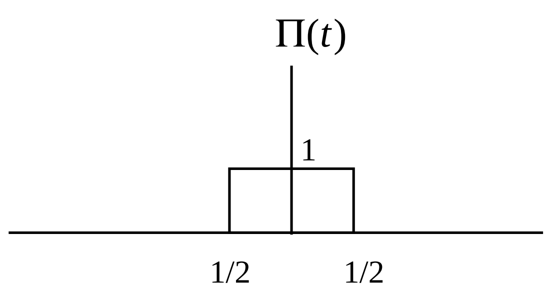


Rectangle pulse

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & otherwise \end{cases}$$

$$F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt$$

$$= \frac{e^{-j\pi t} - e^{j\pi f}}{-j2\pi f} = \frac{Sin(\pi f)}{\pi f} = Sinc(f)$$



Properties of the Fourier transform

- **Linearity**
  - **x1(t) <=> X1(f), x2(t) <=>X2(f) => αx1(t) + βx2(t) <=> αX1(f) + βX2(f)**
- **Duality**
  - **X(f) <=> x(t) => x(f) <=> X(-t) and x(-f)<=> X(t)**
- **Time-shifting: x(t-τ) <=> X(f)$e^{-j2\pi f\tau}$**
- **Scaling: F[(x(at)] = 1/|a| X(f/a)**
- **Convolution: x(t) <=> X(f), y(t) <=> Y(f) then,**
  - **F[x(t)*y(t)] = X(f)Y(f)**
  - **Convolution in time corresponds to multiplication in frequency and visa versa**

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau$$

Fourier transform properties (Modulation)

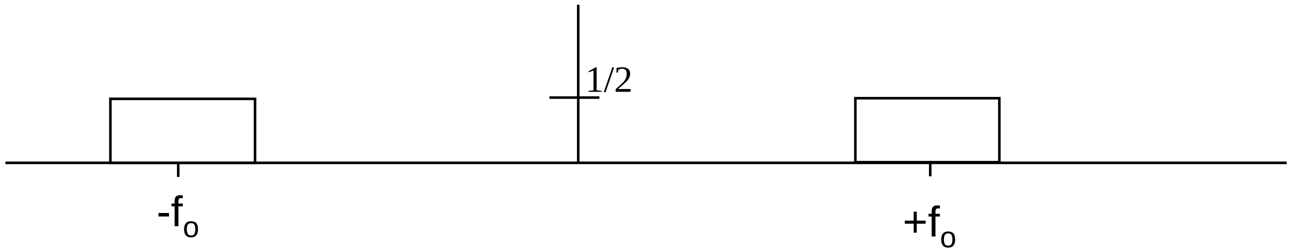
$$x(t)e^{j2\pi f_o t} \Leftrightarrow X(f - f_o)$$

$$Now,\ \cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$x(t)\cos(2\pi f_o t) = \frac{x(t)e^{j2\pi f_o t} + x(t)e^{-j2\pi f_o t}}{2}$$

$$Hence,\ x(t)\cos(2\pi f_o t) \Leftrightarrow \frac{X(f - f_o) + X(f + f_o)}{2}$$

- **Example: x(t)= sinc(t), F[sinc(t)] = Π(f)**
- **Y(t) = sinc(t)cos(2$\pi$$f_o$t) <=> ($\Pi$(f-$f_o$)+$\Pi$(f+$f_o$))/2**



More properties

- **Power content of signal**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- **Autocorrelation**

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

$$R_x(\tau) \Leftrightarrow |X(f)|^2$$

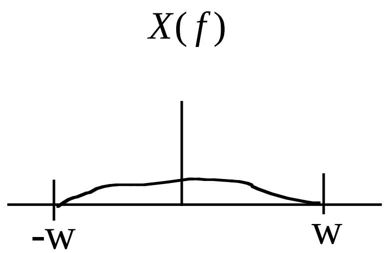
- **Sampling**

$$x(t_o) = x(t)\delta(t-t_o)$$

$$x(t)\sum_{n=-\infty}^{\infty}\delta(t-nt_o) = \text{sampled version of x(t)}$$

$$F[\sum_{n=-\infty}^{\infty}\delta(t-nt_o)] = \frac{1}{t_o}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{t_o})]$$

The Sampling Theorem



- **Band-limited signal** $X(f) = 0, \; for\ all\ f, |f| \geq W$
  - **Bandwidth < W**

Sampling Theorem: If we sample the signal at intervals Ts where Ts <= 1/ 2W then signal can be completely reconstructed from its samples using the formula

$$x(t) = \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \operatorname{sinc}[2W'(t - nT_s)]$$

$$Where, \qquad W \leq W' \leq \frac{1}{T_s} - W$$

$$With\ T_s = \frac{1}{2W} \Rightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}[(\frac{t}{T_s} - n)]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(\frac{n}{2W}) \operatorname{sinc}[2W(t - \frac{n}{2W})]$$

Proof

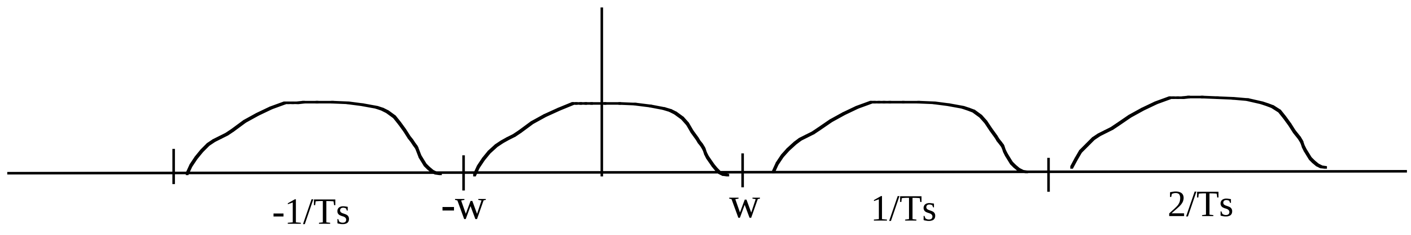
$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

$$X_\delta(f) = X(f) * F[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)]$$

$$F[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})$$

$$X_\delta(f) = \frac{1}{Ts}\sum_{n=-\infty}^{\infty}X(f-\frac{n}{T_s})$$

- **The Fourier transform of the sampled signal is a replication of the Fourier transform of the original separated by 1/Ts intervals**



Proof, continued

- If 1/Ts > 2W then the replicas of X(f) will not overlap and can be recovered

- How can we reconstruct the original signal?
  - Low pass filter the sampled signal

- Ideal low pass filter is a rectangular pulse $H(f) = T_s \Pi(\frac{f}{2W})$

- Now the recovered signal after low pass filtering

$$X(f) = X_\delta(f) T_s \Pi(\frac{f}{2W})$$

$$x(t) = F^{-1}[X_\delta(f) T_s \Pi(\frac{f}{2W})]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) Sinc(\frac{t}{T_s} - n)$$

Notes about Sampling Theorem

- When sampling at rate 2W the reconstruction filter must be a rectangular pulse
  - Such a filter is not realizable
  - For perfect reconstruction must look at samples in the infinite future and past
- In practice we can sample at a rate somewhat greater than 2W which makes reconstruction filters that are easier to realize
- Given any set of arbitrary sample points that are 1/2W apart, can construct a continuous time signal band-limited to W