16.36: Communication Systems Engineering

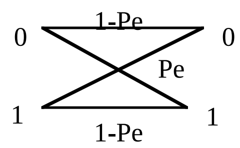
Lectures 12/13: Channel Capacity and Coding

Eytan Modiano

Channel Coding

- **When transmitting over a noisy channel, some of the bits are received with errors**

  <u>Example</u>: Binary Symmetric Channel (BSC)



Pe = Probability of error

- **Q: How can these errors be removed?**

- **A: Coding: the addition of redundant bits that help us determine what was sent with greater accuracy**

Example (Repetition code)

Repeat each bit n times (n-odd)

Input	Code
0	000……..0
1	11..……..1

Decoder:

- **If received sequence contains n/2 or more 1's decode as a 1 and 0 otherwise**
  - **Max likelihood decoding**

P (error | 1 sent) = P (error | 0 sent)
= P[more than n / 2 bit errors occur]

$$= \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} P_e^{\,i} (1-P_e)^{n-i}$$

Repetition code, cont.

- **For $P_e < 1/2$, P(error) is decreasing in n**
  - **$\Rightarrow$ for any ε, $\exists$ n large enough so that P (error) < ε**

<u>Code Rate</u>: ratio of data bits to transmitted bits

  - **For the repetition code R = 1/n**
  - **To send one data bit, must transmit n channel bits "bandwidth expansion"**

- **In general, an (n,k) code uses n channel bits to transmit k data bits**
  - **Code rate R = k / n**

- **Goal: for a desired error probability, ε, find the highest rate code that can achieve p(error) < ε**

Channel Capacity

- **The capacity of a discrete memoryless channel is given by,**

$$C = \max_{p(x)} I(X;Y)$$

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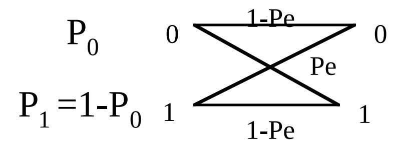
<u>Example</u>: Binary Symmetric Channel (BSC)



$I(X;Y) = H(Y) - H(Y|X) = H(X) - H(X|Y)$

$H(X|Y) = H(X|Y=0)*P(Y=0) + H(X|Y=1)*P(Y=1)$

$H(X|Y=0) = H(X|Y=1) = P_e\log(1/P_e) + (1-P_e)\log(1/ 1-P_e) = H_b(P_e)$

$H(X|Y) = H_b(P_e) \Rightarrow I(X;Y) = H(X) - H_b(P_e)$

$H(X) = P_0 \log(1/P_0) + (1-P_0)\log(1/ 1-P_0) = H_b(p_0)$

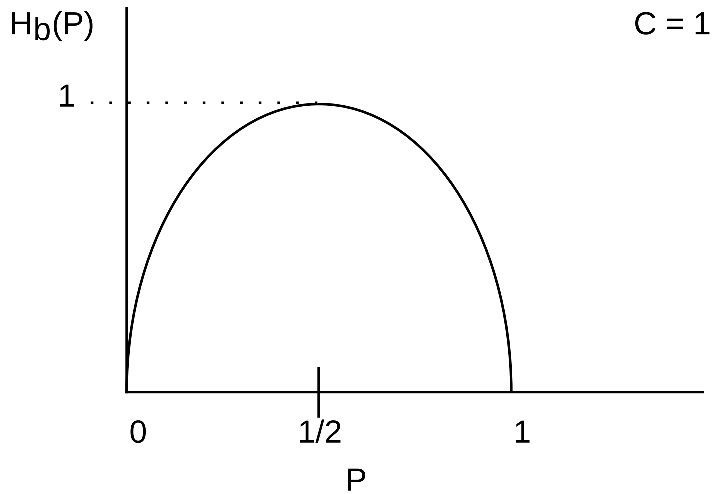
$\Rightarrow I(X;Y) = H_b(P_0) - H_b(P_e)$

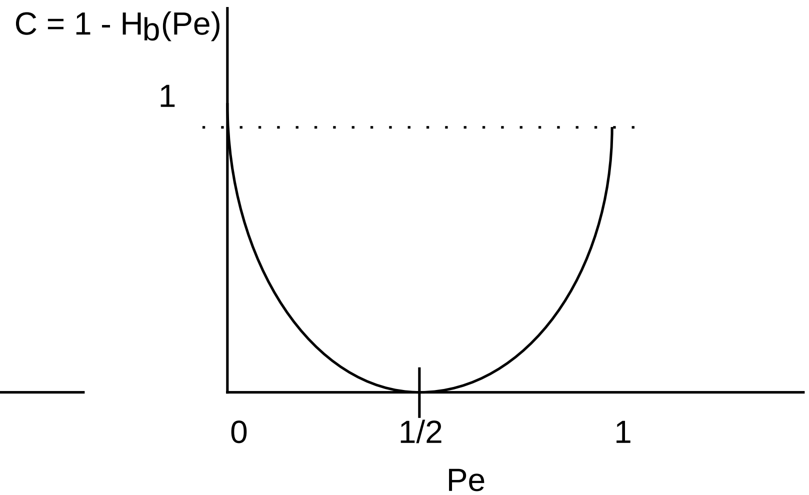
Capacity of BSC

$$I(X;Y) = H_b(P_0) - H_b(P_e)$$

- $H_b(P) = P \log(1/P) + (1-P) \log(1/\ 1-P)$
  - $H_b(P) <= 1$ with equality if $P=1/2$

$$C = \max_{P0} \{I(X;Y) = H_b(P_0) - H_b(P_e)\} = 1 - H_b(P_e)$$





$C = 0$ when $P_e = 1/2$ and $C = 1$ when $P_e = 0$ or $P_e=1$

Channel Coding Theorem (Claude Shannon)

Theorem: For all $R < C$ and $\varepsilon > o$; there exists a code of rate R whose error probability $< \varepsilon$

- **ε can be arbitrarily small**
- **Proof uses large block size n**

  as $n \rightarrow \infty$ capacity is achieved

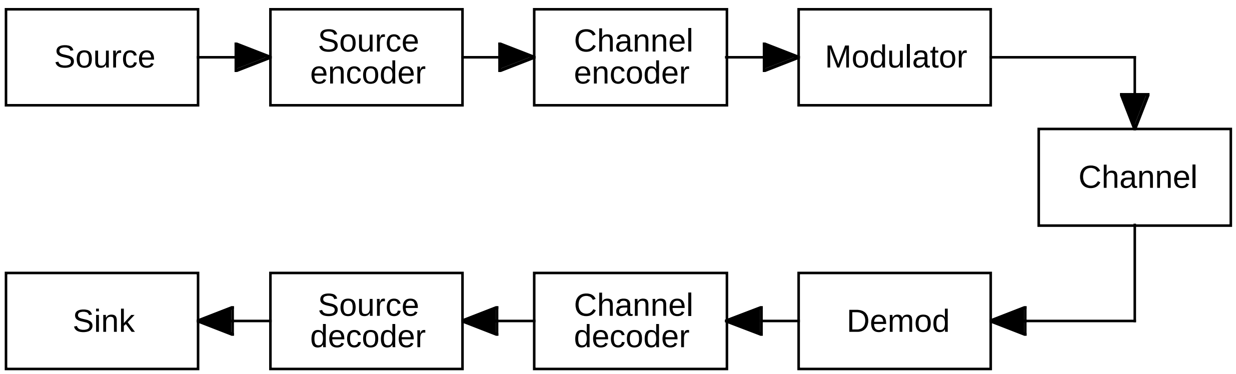
- **In practice codes that achieve capacity are difficult to find**
  - **The goal is to find a code that comes as close as possible to achieving capacity**

- **Converse of Coding Theorem:**
  - **For all codes of rate $R > C$, $\exists\, \varepsilon_0 > 0$, such that the probability of error is always greater than ε_0**

    For code rates greater than capacity, the probability of error is bounded away from 0

Channel Coding

- **Block diagram**



Approaches to coding

- **Block Codes**
  - Data is broken up into blocks of equal length
  - Each block is "mapped" onto a larger block

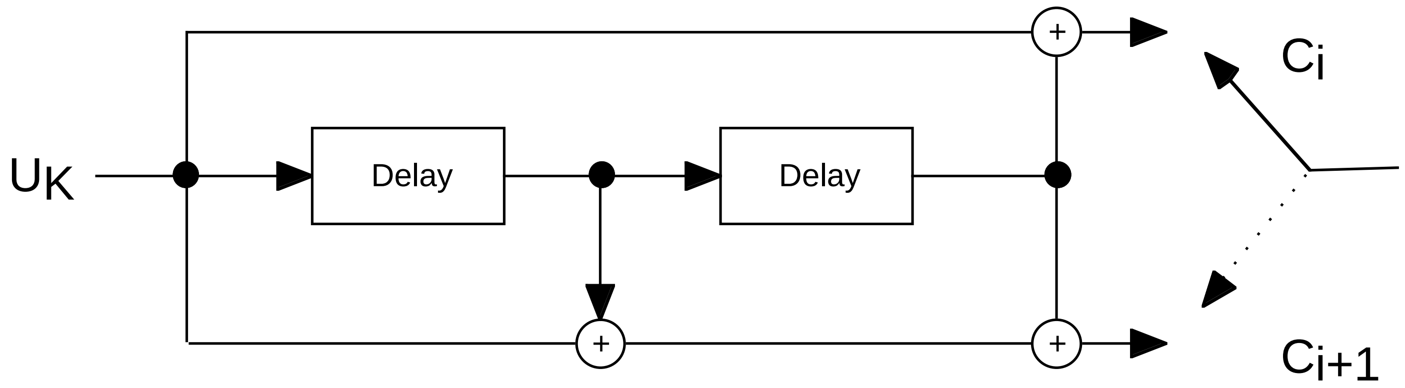
Example: (6,3) code, n = 6, k = 3, R = 1/2

$000 \rightarrow 000000$	$100 \rightarrow 100101$
$001 \rightarrow 001011$	$101 \rightarrow 101110$
$010 \rightarrow 010111$	$110 \rightarrow 110010$
$011 \rightarrow 011100$	$111 \rightarrow 111001$

- **An (n,k) binary block code is a collection of 2^k binary n-tuples (n>k)**
  - n = block length
  - k = number of data bits
  - n-k = number of checked bits
  - R = k / n = code rate

Approaches to coding

- **Convolutional Codes**
  - **The output is provided by looking at a sliding window of input**



$$C_{(2K)} = U_{(2K)} \oplus U_{(2K-2)}, \quad C_{(2K+1)} = U_{(2K+1)} \oplus U_{(2K)} \oplus U_{(2K-1)}$$

$\oplus$ mod(2) addition (1+1=0)

Block Codes

- A block code is systematic if every codeword can be broken into a data part and a redundant part
  - Previous (6,3) code was systematic

<u>Definitions</u>:

- Given $X \in \{0,1\}^n$, the <u>Hamming Weight</u> of X is the number of 1's in X

- Given X, Y in $\{0,1\}^n$, the <u>Hamming Distance</u> between X & Y is the number of places in which they differ,

$$d_H(X,Y) = \sum_{i=1}^{n} X_i \oplus Y_i = Weight(X+Y)$$

$$X + Y = [x_1 \oplus y_1, x_2 \oplus y_2, \cdots, x_n \oplus y_n]$$

- The <u>minimum distance</u> of a code is the Hamming Distance between the two closest codewords:

$$d_{min} = \min_{C_1, C_2 \in C} \{d_H(C_1,C_2)\}$$

Decoding

- r may not equal to u due to transmission errors
- Given r how do we know which codeword was sent?

Maximum likelihood Decoding:

Map the received n-tuple r into the codeword C that maximizes,

P { r | C was transmitted }

Minimum Distance Decoding (nearest neighbor)

Map r to the codeword C such that the hamming distance between r and C is minimized (I.e., min d_H (r,C))

$\Rightarrow$ For most channels Min Distance Decoding is the same as Max likelihood decoding

Linear Block Codes

- **A (n,k) linear block code (LBC) is defined by 2^k codewords of length n**

$$C = \{ C_1 \ldots . C_m \}$$

- **A (n,k) LBC is a K-dimensional subspace of $\{0,1\}^n$**
  - **(0…0) is always a codeword**
  - **If $C_1, C_2 \in C$, $C_1 + C_2 \in C$**

- **Theorem: For a LBC the minimum distance is equal to the min weight (W_{min}) of the code**

$$W_{min} = \min_{(\text{over all } Ci)} \text{Weight}(C_i)$$

Proof: Suppose $d_{min} = d_H(C_i, C_j)$, where $C_1, C_2 \in C$

$d_H(C_i, C_j) = \text{Weight}(C_i + C_j)$,

but since C is a LBC then $C_i + C_j$ is also a codeword

Systematic codes

Theorem: Any (n,k) LBC can be represented in Systematic form where: data = $x_1..x_k$, codeword = $x_1..x_k\, c_{k+1}..x_n$

- Hence we will restrict our discussion to systematic codes only

- The codewords corresponding to the information sequences: e_1 = (1,0,..0), e_2=(0,1,0..0), e_k = (0,0,..,1) for a basis for the code
  - Clearly, they are linearly independent
  - K linearly independent n-tuples completely define the K dimensional subspace that forms the code

Information sequence	Codeword
$e_1 = (1,0,..0)$	$g_1 = (1,0,..,0, g_{(1,k+1)} \ldots g_{(1,n)})$
$e_2=(0,1,0..0)$	$g_2 = (0,1,..,0, g_{(2,k+1)} \ldots g_{(2,n)})$
$e_k = (0,0,..,1)$	$g_k = (0,0,..,k, g_{(k,k+1)} \ldots g_{(k,n)})$

- $g_1, g_2, \ldots, g_k$ form a basis for the code

The Generator Matrix

$$G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & & & g_{2n} \\ \vdots & & & \\ g_{k1} & & & g_{kn} \end{bmatrix}$$

- **For input sequence x = ($x_1,\ldots,x_k$): C_x = xG**
  - **Every codeword is a linear combination of the rows of G**
  - **The codeword corresponding to every input sequence can be derived from G**
  - **Since any input can be represented as a linear combination of the basis ($e_1, e_2, \ldots, e_k$), every corresponding codeword can be represented as a linear combination of the corresponding rows of G**
- **Note: $x_1 \longleftrightarrow C_1$, $x_2 \longleftrightarrow C_2$ => $x_1+x_2 \longleftrightarrow C_1+C_2$**

Example

- **Consider the (6,3) code from earlier:**

100 $\rightarrow$ 100101; 010 $\rightarrow$ 010111; 001 $\rightarrow$ 001011

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Codeword for (1,0,1) = (1,0,1)G = (1,0,1,1,1,0)

$$G = \left[\begin{array}{c:c} I_K & P_{Kx(n-K)} \end{array} \right]$$

$$\mathrm{I_K} = \mathrm{KxK\ identity\ matrix}$$

The parity check matrix

$$H = \left[\begin{array}{c:c} P^T & I_{(n-K)} \end{array}\right]$$

$$\mathrm{I}_{(n\text{-}K)} = (n - \mathrm{K})\mathrm{x}(\mathrm{n} - \mathrm{K}) \text{ identity matrix}$$

Example:

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Now, if c_i is a codework of C then, $c_i H^T = \vec{0}$

- "C is in the null space of H"
- Any codeword in C is orthogonal to the rows of H

Decoding

- v = transmitted codeword = $v_1 \dots v_n$
- r = received codeword = $r_1 \dots r_n$
- e = error pattern = $e_1 \dots e_n$

- $r = v + e$

- $S = rH^T$ = Syndrome of r

  $= (v+e)H^T = vH^T + eH^T = eH^T$

- S is equal to '0' if and only if $e \in C$
  - I.e., error pattern is a codeword

- $S \neq 0$ => error detected
- $S = 0$ => no errors detected (they may have occurred and not detected)

- Suppose $S \neq 0$, how can we know what was the actual transmitted codeword?

Syndrome decoding

- **Many error patterns may have created the same syndrome**

  For error pattern $e_0 \Rightarrow S_0 = e_0 H^T$

  Consider error pattern $e_0 + c_i$ ($c_i \in C$)

  $$S'_0 = (e_0 + c_i)H^T = e_0 H^T + c_i H^T = e_0 H^T = S_0$$

- **So, for a given error pattern, e_0, all other error patterns that can be expressed as $e_0 + c_i$ for some $c_i \in C$ are also error patterns with the same syndrome**

- **For a given syndrome, we can not tell which error pattern actually occurred, but the most likely is the one with minimum weight**
  - **Minimum distance decoding**

- **For a given syndrome, find the error pattern of minimum weight (e_{min}) that gives this syndrome and decode: $r' = r + e_{min}$**

Standard Array

$$M = 2^K$$

$$\begin{array}{ccccc} C_1 & C_2 & \cdots & C_M & Syndrome \\ e_1 & e_1 + C_2 & & e_1 + C_M & S_1 \\ \vdots & e_2 + C_2 & & e_2 + C_M & S_2 \\ e_{2^{(n-K)}-1} & & & & S_{2^{(n-K)}-1} \end{array}$$

- **Row 1 consists of all M codewords**
- **Row 2 e_1 = min weight n-tuple not in the array**
  - **I.e., the minimum weight error pattern**
- **Row i, e_i = min weight n-tuple not in the array**

- **All elements of any row have the same syndrome**
  - **Elements of a row are called "co-sets"**
- **The first element of each row is the minimum weight error pattern with that syndrome**
  - **Called "co-set leader"**

Decoding algorithm

- Receive vector r

1) Find $S = rH^T$ = syndrome of r

2) Find the co-set leader e, corresponding to S

3) Decode: $C = r+e$

- "Minimum distance decoding"
  - Decode into the codeword that is closest to the received sequence

Example (syndrome decoding)

- **Simple (4,2) code**

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Data	codeword
00	0000
01	0101
10	1010
11	1111

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad H^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

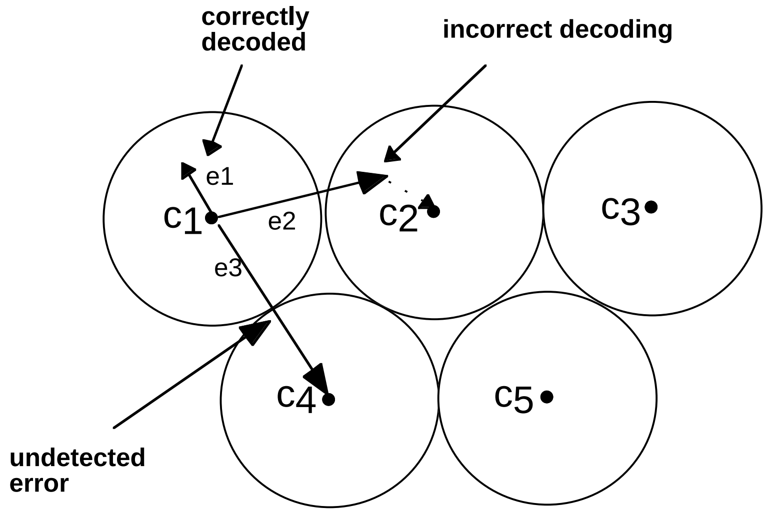
Standard array

0000	0101	1010	1111	Syndrome
1000	1101	0010	0111	10
0100	0001	1110	1011	01
1100	1001	0110	0011	11

Suppose 0111 is received, S = 10, co-set leader = 1000

Decode: C = 0111 + 1000 = 1111

Minimum distance decoding



- Minimum distance decoding maps a received sequence onto the nearest codeword
- If an error pattern maps the sent codeword onto another valid codeword, that error will be undetected (e.g., e3)
  - Any error pattern that is equal to a codeword will result in undetected errors
- If an error pattern maps the sent sequence onto the sphere of another codeword, it will be incorrectly decoded (e.g., e2)

Performance of Block Codes

- Error detection: Compute syndrome, $S \neq 0$ => error detected
  - Request retransmission
  - Used in packet networks
- A linear block code will detect all error patterns that are not codewords
- Error correction: Syndrome decoding
  - All error patterns of weight < $d_{min}/2$ will be correctly decoded
  - This is why it is important to design codes with large minimum distance (d_{min})
  - The larger the minimum distance the smaller the probability of incorrect decoding

Hamming Codes

- **Linear block code capable of correcting single errors**
  - **$n = 2^m - 1$, $k = 2^m - 1 - m$**
    (e.g., (3,1), (7,4), (15,11)…)
  - **$R = 1 - m/(2^m - 1)$ => very high rate**
  - **$d_{min} = 3$ => single error correction**

- **Construction of Hamming codes**
  - **Parity check matrix (H) consists of all non-zero binary m-tuples**

Example: (7,4) hamming code (m=3)

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$