FS. FOURIER SERIES

The trigonometric series

\[ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

is a Fourier series if its coefficients are given by

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n = 0, 1, 2, \ldots \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n = 1, 2, \ldots . \]

Note that the constant term \( \frac{1}{2}a_0 \) is the mean value of \( f(x) \) over the interval \((-\pi, \pi)\).

The Fourier series represents a function defined on the interval \((-\pi, \pi)\) for \( x \) in that interval and represents a \( 2\pi \) periodic function for all values of \( x \).

The Fourier cosine series and the Fourier sine series are, respectively,

\[ f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx, \]
\[ f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx. \]

Series (2) represents an even function on the interval \((-\pi, \pi)\) if it converges when \( 0 < x < \pi \). When \( f \) is defined only on the interval \((0, \pi)\), the series is the Fourier series for the even periodic extension of \( f \) with period \( 2\pi \). The Fourier sine series represents an odd function on the interval \((-\pi, \pi)\) or an odd periodic function of period \( 2\pi \). It also represents a function \( f \) defined only on the interval \((0, \pi)\).

Example. (1) Let \( f(x) = 0 \) for \(-\pi < x \leq 0\) and \( f(x) = x \) for \( 0 < x < \pi \). Its Fourier series expansion is

\[ f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n - 1}{n^2 \pi^2} \cos nx - \frac{(-1)^2}{n} \sin nx \right). \]

Below, we will prove that this Fourier series converges to \( f(x) \) for \(-\pi < x < \pi \). Then, the series also represents the periodic extension of the function \( f \) on an open interval \((2n - 1)\pi, (2n + 1)\pi)\), \( n \in \mathbb{Z} \).

(2) The Fourier cosine series (2) for \( f(x) = \sin x \) on the interval \((0, \pi)\) is

\[ \sin x \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}, \quad 0 < x < \pi. \]

Let us assume the correspondence here is an equality for each value of \( x \) in the interval \( 0 \leq x \leq \pi \) (which is the case). Then, for all values of \( x \) outside the interval the series converges to the even periodic extension with period \( 2\pi \) of \( \sin x \), which is \( f(x) = |\sin x| \).
The Fourier sine series of \( f(x) = x^3 \) on the interval \((0, \pi)\) is
\[
x^3 \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 \pi^2 - 6}{n^3} \sin nx, \quad 0 < x < \pi.
\]
Since \( x^3 \) is an odd function, the series here is also the Fourier series for \( f(x) = x^3 \) on the interval \((-\pi, \pi)\).

In calculations involving Fourier series it is often advantageous to use complex exponentials; The complex Fourier series of \( f \) is
\[
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]
Note the convergence of the infinite sum above is identical to that of (1).

**Example.** Let \( f(x) = e^{ax} \) on \(-\pi < x < \pi\), where \( a \) is a nonzero real constant. Using Euler’s formula, we write
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]
Since \( e^{\pm in\pi} = (-1)^n \), the evaluation of the integral gives
\[
e^{ax} \sim \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(-1)^n}{a^2 + n^2} e^{inx}.
\]

A theorem that gives conditions under which a Fourier series converges to its function is called a Fourier theorem. One such theorem is now established.

**Theorem 1.** If \( f(x) \) is a piecewise continuous on the interval \((-\pi, \pi)\), its Fourier series converges at each value of \( x \) to
\[
\frac{1}{2} (f(x+) + f(x-)).
\]

Instead giving its proof, we make a few comments. In the theory of trigonometric Fourier series, it is customary to assume that the functions have period \( 2\pi \) and that they have the value given by (4) at all points \( x \) where (4) is meaningful. For instance, if \( f \) is given on \((-\pi, \pi)\) then the periodic extension is, initially, undefined \((2n+1)\pi\), where \( n \) is an integer. The missing definition at these points is now supplied by (4)

If \( f \) satisfies the hypothesis of Theorem 1 and is continuous for all \( x \), a classical theorem known as the Dini-Lipschitz test (not provided there) shows that the convergence is uniform.

If both \( f \) and \( f' \) are piecewise continuous on the interval \([-\pi, \pi]\), then the one-sided derivatives of the periodic extension of \( f \) exist at every point, so the series converges everywhere to the mean value of the limit from the right and left for the periodic extension.

**Corollary 2.** If \( f \) is a piecewise continuous on the interval \(0 < x < \pi\) and, for convergence, if the value of \( f \) is defined at each point where it is discontinuous as the mean value of its right-hand and left-hand limits there, then at each point \( x \) of that interval where the one-sided derivatives exist, \( f \) is represented by its Fourier cosine series
\[
f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx
\]
where
\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx.
\]
It is also represented by its Fourier sine series.
Uniform convergence. Our treatment of the uniform convergence of Fourier series will depend on an important inequality, and we now present a simple but indirect derivation of it.

Let \( A_n \) and \( B_n \), \( n = 1, 2, \ldots, m \), denote real numbers where at least one of the \( A_n \), say \( A_N \) is nonzero. If the quadratic equation
\[
x^2 \sum_{n=1}^{m} A_n^2 + 2x \sum_{n=1}^{m} A_n B_n + \sum_{n=1}^{m} B_n^2 = 0
\]
in the variable \( x \) has a real root \( x_0 \), we need only write that equation in the form
\[
\sum_{n=1}^{m} (A_n x + B_n)^2 = 0
\]
to see that \( A_n x_0 + B_n = 0 \) for each \( n \), in particular, when \( n = N \). Consequently, the only possible value of \( x_0 \) is \( x_0 = -B_N / A_N \) and we find that our quadratic equation cannot have two distinct real roots. Its discriminant is therefore negative or zero, that is,
\[
(\sum_{n=1}^{m} A_n B_n)^2 \leq \left( \sum_{n=1}^{m} A_n \right)^2 \left( \sum_{n=1}^{m} B_n \right)^2.
\]
This condition is known as the Cauchy-Schwarz inequality. When \( m \) is finite, it simply states that the square of the inner product of two vectors in the \( m \)-dimensional space does not exceed that products of squares of their norms.

The proof of the following theorem uses the Cauchy-Schwarz inequality.

**Theorem 3.** Let \( f \) be a continuous function on the interval \([-\pi, \pi]\) such that \( f(-\pi) = f(\pi) \) and its derivative \( f' \) be piecewise continuous on that interval. If \( a_n \) and \( b_n \) are the Fourier coefficients of \( f \), then the series
\[
\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}
\]
converges.

From the comparison test, we note that each of the series
\[
\sum_{n=1}^{\infty} |a_n|, \quad \sum_{n=1}^{\infty} |b_n|
\]
converges as a consequence of the theorem above.

The next theorem is closely related to the above theorem.

**Theorem 4.** Under the hypothesis of Theorem 3 the convergence of the Fourier series
\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
to \( f(x) \) on the interval \([-\pi, \pi]\) is absolute and uniform with respect to \( x \) on the interval.

**Sketch of the proof.** The conditions on \( f \) and \( f' \) ensure the continuity, and also the existence of one-sided derivatives, of the periodic extension of \( f \) for all \( x \). It follows from the Fourier theorem, Theorem 1, that the Fourier series converges to \( f(x) \) everywhere in the interval \([-\pi, \pi]\).

We now observe that
\[
|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n|
\]
and also that the series of constants \( |a_n| + |b_n| \) converges since each of series \( \sum_{n=1}^{\infty} |a_n| \) and \( \sum_{n=1}^{\infty} |b_n| \) converges. The comparison test and the Weierstrauss test therefore apply to show that the convergence of the Fourier series is indeed absolute and uniform.
A Fourier series cannot converge uniformly on an interval that contains a discontinuity of its sum since a uniform convergent series of continuous functions always converges to a continuous function. Hence, some continuity of \( f \) is necessary in order to ensure uniform convergence of the Fourier series.

Modification of Theorem 3 and Theorem 4 for cosine and sine series, or for Fourier series on an interval \((-l,l)\) are apparent. For instance, it follows from Theorem 4 that the Fourier cosine series for a continuous function \( f \), on the interval \([0,\pi]\) converges uniformly to \( f(x) \) on that interval provided that \( f' \) is piecewise continuous on that interval. For the sine series, however, the further condition \( f(0) = f(\pi) = 0 \) is needed.

**Gibbs’s phenomenon.** The Gibbs phenomenon, named after the American physicist J. Willard Gibbs, refers to the peculiar manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity: the \( n \)th partial sum of the Fourier series has large oscillations near the jump, which may increase the maximum of the partial sum above that of the function itself. The overshoot does not die out as the frequency increases but approaches a finite limit.

We now study the partial sums \( s_N(x) \) of the series

\[
\phi(x) = \frac{1}{2} \left( \pi - x \right) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi)
\]

in the neighborhood of \( x = 0 \). Since \( \phi(x) \) is discontinuous at \( x = 0 \) the series cannot converge uniformly at \( x = 0 \). For \( x > 0 \) we use that

\[
\frac{1}{2}x + s_N(x) = \int_0^x \left( \frac{1}{2} + \sum_{n=1}^{N} \cos nx' \right) dx' = \int_0^{Nx} \frac{\sin t}{t} \, dt + o(1)
\]

uniformly in \( 0 \leq x \leq \pi \) \([3]\). Hence, \( s_N(x) \) is uniformly bounded, and

\[
(6) \quad s_N(x) = \int_0^{Nx} \frac{\sin t}{t} \, dt + R_N(x),
\]

where \( |R_N(x)| < \epsilon \) if \( x < \epsilon \) and \( N > N_0(\epsilon) \).

Consider the integral

\[
G(n) = \int_0^{\pi} \frac{\sin t}{t} \, dt.
\]

The integrals of \((\sin t)/t\) over the intervals \((k\pi, (k+1)\pi)\) decrease in absolute value and are of alternating sign when \( k = 0, 1, 2, \ldots \). This shows that the curve \( y = G(x) \) has a wave-like shape with maxima \( M_1 > M_3 > M_5 > \cdots \) at \( \pi, 3\pi, 5\pi, \ldots \) and minima \( m_2 < m_4 < m_6 < \cdots \) at \( 2\pi, 4\pi, \ldots \). Substituting \( x = \pi/m \) in (6), we obtain

\[
s_N(\pi/N) \to G(\pi) > G(\infty) = \frac{1}{2}\pi.
\]

Thus, though \( s_N(x) \) tends to \( \phi(x) \) at every fixed \( x \) in \( 0 < x < 2\pi \), the curve \( y = s_N(x) \), which pass through the point \((0,0)\), condense to the interval \( 0 \leq y \leq \phi(0+) = \frac{1}{2}\pi \) is

\[
\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} \, dt = 1.179 \ldots
\]

Similarly, to the left of \( x = 0 \) the curve \( y = s_N(x) \) condense to the interval \(-G(\pi) \leq y \leq 0 \). This behavior is called *Gibbs’s phenomenon*, and its generalized form may be described as follows.

*J. Willard Gibbs, Fourier series, Nature 59 200 (1898) and 606 (1899).*
Suppose that a sequence \( \{f_n(x)\} \) converges for \( x_0 < x \leq x_0 + h \), say, to the limit \( f(x) \) and that \( f(x_0 + 0) \) exists. Suppose that when \( n \to \infty \) and \( x \to x_0 \) independently, we have

\[
\limsup f_n(x) > f(x_0 + 0) \quad \text{or} \quad \liminf f_n(x) < f(x_0 + 0).
\]

Then, we say that \( \{f_n(x)\} \) shows Gibbs’s phenomenon in the right-hand neighborhood of \( x = x_0 \). Similarly for the left-hand neighborhood. If \( f(x) = \lim f_n(x) \) is defined and is continuous at \( x = x_0 \) the absence of the phenomenon at the point \( x_0 \) is equivalent to the uniform convergence of \( \{f_n(x)\} \) at \( x_0 \).

REFERENCES


Problems.

1. (a) Write the Fourier series on \((-\pi, \pi)\) of the function

\[
f(x) = \begin{cases} 
0, & -\pi \leq x \leq 0 \\
\sin x, & 0 \leq x \leq \pi.
\end{cases}
\]

(Hint. Write the function in the form

\[
f(x) = \frac{1}{2}(\sin x + |\sin x|).
\]

(b) Verify directly that the Fourier series of the function \( f(x) \) in part (a) converges uniformly on the interval \([-\pi, \pi]\) except at the points \( x = 0, \pm \pi \).

2. Show that

\[
|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.
\]

By writing \( x = 0 \) and \( x = \pi/2 \) in this expansion, obtain the following sums:

\[
\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{\pi}{4}.
\]

3. Using the Fourier series of \(|x|\) on the interval \([-\pi, \pi]\), show that

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.
\]

4. Show that the function

\[
f(x) = \begin{cases} 
\sin(1/x), & x \neq 0 \\
0, & x = 0
\end{cases}
\]

is continuous at \( x = 0 \) but that neither of the one-sided derivatives exists at that point.

5. Prove that the Fourier coefficients \( a_n \) and \( b_n \) for the function \( f \) in Theorem 3 satisfy the conditions

\[
\lim_{n \to \infty} na_n = 0, \quad \lim_{n \to \infty} nb_n = 0.
\]