EX. REMARKS ON THE EXISTENCE THEORY

**Bifurcation.** Consider the general vector system \( x' = f(t, x) \). Suppose two solutions \( u \) and \( v \) lying in a rectangular region \( R \) agree at some value \( t_1 \) but do not agree at some other value \( t_2 \). Without loss of generality, we assume \( t_2 > t_1 \). Starting at \( t_1 \) we diminish \( t \) until it first happens that the solution agree. (That a first value exists can be deduced from the completeness axiom for real numbers but is taken for granted here.) If the point so obtained is \( t_0 \) then \( u \) and \( v \) have a common value, say \( x_0 \), at \( t_0 \) but they do not agree throughout any open interval containing \( t_0 \), no matter how small. A point \((t_0, x_0) \in R \) with this property is said to be a bifurcation point. What we have shown is that, if uniqueness fails, there is always at least one bifurcation point.

This principle apply to the general vector equation and show that uniqueness is essentially a local property.

**Theorem 1.** Consider the DE
\[
x^{(n)} = f(t, x, x', x'', \ldots, x^{(n-1)})
\]
with the initial conditions
\[
x(t_0) = x_0, \quad x'(t_0) = x_1, \quad \ldots \quad x^{(n-1)}(t_0) = x_{n-1}.
\]
If every point \((t_0, x_0) \in R \) is interior to a rectangle in which the partial derivatives \( \partial f / \partial x^i \) are bounded then the initial value problem has no bifurcation point in \( R \).

**Singular solutions.** By definition, a singular solution is a solution such that every point of its graph is a bifurcation point. For the scalar case, a singular solution is essentially the same as an envelope of a family of solution, and a brief discussion is given now. We shall explain where two equations \( f_p = 0 \) and \( F_c = 0 \) for the envelope come from.

We consider the DE \( f(t, x, x') = 0 \), where \( f = f(t, x, p) \) has continuous partial derivatives with respect to a family of curves that has an envelope. Near any point \((t_0, x_0) \) at which \( \partial f / \partial p \neq 0 \) the implicit function theorem states that \( f(t, x, p) = 0 \) can be solved for \( p = g(t, x) \), where \( g \), like \( f \), has continuous partial derivatives. By Theorem 1 the resulting differential equation
\[
x' = g(t, x)
\]
can have only one solution through \((t_0, x_0) \). But through a point on the envelope, there are in general, two solutions: the envelope and the member of the family to which the envelope is tangent. This failure of uniqueness explains why \( \partial f / \partial p = 0 \) is to be expected on the envelope.

Next let a family of curves \( C_c \) be given by \( F(t, x, c) = 0 \) and suppose the family has an envelope \( \mathcal{E} \). The envelope is tangent to \( C_c \) at some point \((\tau(c), \xi(c)) \) and is expressed in the parametric form \( t = \tau(c) \), and \( x = \xi(c) \). If \( C_c \) is given as \( t = \sigma(s) \) and \( x = \eta(s) \), the defining equations are
\[
F(\tau(c), \xi(c), c) = 0, \quad F(\sigma(s), \eta(s), c) = 0.
\]
Let us now assume that \( F, \tau, \xi, \sigma, \eta \) have continuous first derivatives and that the curves \( C_c \) are smooth, in the sense that \( \sigma' \) and \( \eta' \) do no vanish simultaneously. Differentiating (1) by the chain rule yields
\[
F_t \tau' + F_x \xi' + F_c = 0, \quad F_t \sigma' + F_x \eta' = 0.
\]
Since the envelope is tangent to \( C_c \) at \((\tau(c), \xi(c)) \), the tangent vectors \((\tau', \xi')\) and \((\sigma', \eta')\) are parallel there. Thus, \( \tau' = \lambda \sigma' \) and \( \xi' = \lambda \eta' \), where \( \lambda \) is some scalar. The second equation (2) now shows that the first reduces to \( F_c = 0 \).
REFERENCES


Problems.

1. Show that the equation $x' = (x - t)^{1/3}$ does not satisfy a Lipschitz condition near the line $x = t$, yet it has a unique solution through every point of that line. (Hint: Set $y = x - t$ and separate variables.)

2. Let $f(t) = t/|t|$ for $t \neq 0$ and $f(0) = k$. It will be shown that no matter how the constant $k$ is chosen the equation $dx/dt = f(t)$ has no solution on an interval containing the origin. This may serve to explain why, as a rule, a hypothesis of continuity is necessary. Since $f(t) = -1$ for $t < 0$ and $f(t) = 1$ for $t > 0$ we must have

$$x = \begin{cases} -t + c_1 & \text{for } t < 0 \\ t + c_2 & \text{for } t > 0, \end{cases}$$

where $c_1$ and $c_2$ are constant. Existence of the derivative $x'(0)$ requires the continuity of $x$ at $t = 0$ and hence $c_1 = c_2$. But, taking $c_1 = c_2 = c$, the function $x$ is not differentiable at $t = 0$. Hence, $x$ does not satisfy the differential equation at $t = 0$ no matter how $f(0)$ is defined.

3. (a) By choosing $1/t = n\pi/2$, where $n$ is an integer, show that the function defined for $t \neq 0$ by $x = \sin(1/t)$ oscillates infinitely often between the values $-1, 0, 1$ as $t \to 0$ and hence has no limit.

(b) For $t \neq 0$, check that the function $x = \sin(1/t)$ satisfies

$$t^4 x'' + 2t^3 x' + x = 0, \quad t \neq 0.$$ 

This shows that the conclusion asserting existence of a limit at the boundary can fail even for linear equations if the coefficients are unbounded.