LC. LIMIT CYCLES

Consider a curve defined by \( x = x(t), y = y(t) \) for \( 0 \leq t \leq T \) where \( T > 0 \) and \( x, y \in C \). The curve is simple if 
\[
x(t_1) = x(t_2), \quad y(t_1) = y(t_2)
\]
for \( 0 \leq t_1 < T \) hold only when \( t_1 = t_2 \). That means, it does not intersect itself. The curve is closed if the endpoint coincides with the initial point, that is, \( x(0) = x(T) \) and \( y(0) = y(T) \). A simple closed curve is called a Jordan curve. A Jordan curve is a continuous one-to-one image of a circle.

An open set \( \Omega \) is connected if every pair of points in \( \Omega \) can be joined by a simple curve that lies entirely in \( \Omega \). Intuitively, this means that the set does not consist of two or more separate pieces. An open connected set is called a domain.

The Jordan curve theorem\(^*\) states that a simple closed curve divides the plane into two domains, an interior domain which is bounded and an exterior domain.

A domain \( \Omega \) is simply connected if, whenever a Jordan curve lies in \( \Omega \) the interior of the curve is also in \( \Omega \). Intuitively, this means that \( \Omega \) has no holes.

We now study closed orbits of the system
\[
(1) \quad x' = f(x, y), \quad y' = g(x, y),
\]
where \( f, g \in C^1 \).

**Theorem 1** (Poincaré). If \( C \) is a simple closed trajectory of (1) lying in a simply connected domain \( \Omega \), then there is at least one critical point inside \( C \).

**Sketch of the proof.** Let \( C \) be the image of the circle \( x^2 + y^2 = r^2 = 1 \). It is possible to fill out \( C \) and its interior by the sets \( C_s \) where \( C_s \) is the image of the circle \( r = s, 0 \leq s \leq 1 \). Thus, \( C_s \) is a curve for \( 0 < s < 1 \), with \( C_1 = C \) and \( C_0 \) reduces to a point. The region consisting of \( C \) of the Theorem and its interior is filled out in much the same way as the circle \( r = s \) for \( 0 < s < 1 \) and the point \( r = 0 \) fill out the disk \( r \leq 1 \).

Next, if \((f, g)\) does not take the value \((0, 0)\) inside \( C \), the angle that this vector makes with the \( x \)-axis can be so defined as to be a continuous function of parameter \( t \) on \( C_s \). This function changes by a multiple of \( 2\pi \) when \( C_s \) is traversed. If the total change is \( \Delta \theta \) the quantity \( \Delta \theta/(2\pi) \) is an integer that is denoted by \( I_s \) and is called the index of \( C_s \) with respect to the vector field \((f, g)\). The index is a continuous integer-valued function of \( s \).

The differential equation shows that the vector \((f, g)\) is tangent to the closed trajectory \( C \), hence makes exactly one revolution when \( C \) is traversed. Thus the index is \(+1\) or \(-1\) when \( C \) is traversed in the positive or negative direction, respectively. If \( s \) is close to \( 0 \), however, the curve \( C_s \) is almost a point, the vector \((f, g)\) is almost constant on \( C_s \), and the index is zero.

An integer-valued function \( I_s \) continuous for \( 0 \leq s \leq 1 \) must be constant, and it cannot change from \( 1 \) to \(-1\) to \( 0 \) as described above. Hence our assumption that \((f, g)\) does not take the value \((0, 0)\) was false. In other words, there must be a point in \( C \) where \((f, g) = (0, 0)\), and this yields the existence of a critical point. \( \square \)

**Theorem 2** (The Bendixson-Dulac theorem). If \((f, g) \in C^1 \) in a simply connected domain \( \Omega \) and let \( R \) be a \( C^1 \) function such that
\[
(2) \quad (Rf)_x + (Rg)_y \neq 0 \quad \text{in} \quad (x, y) \in \Omega,
\]

\(^*\)It was first formulated by the French mathematician Marie Ennemond Camille Jordan in 1887, but the first correct proof was given by the American mathematician Oswald Veblen in 1905.
then the system (1) has no closed-trajectory in $\Omega$.

Theorem 2 for $R = 1$ was established by the Swedish mathematician Ivar Bendixson in 1901 and for general $R$ by the French mathematician Henri Dulac in 1933.

**Proof.** The proof uses knowledge of Green’s theorem. If a simple closed trajectory $C$ lies in $\Omega$, it follows

$$\int_C R(gdx - fdy) = \int_{\text{int}(C)} ((Rf)_x + (Rg)_y)dxdy,$$

where $\text{int}(C)$ denotes the interior domain of $C$. The left side is zero by virtue of the differential equations while the right side is, by hypothesis, not zero. A contradiction then proves the assertion.

**Example 3.** For the competition equations

$$x' = x(k - ax - by), \quad y' = y(m - cx - dy)$$

let us assume only that $ad \geq 0$ and that $a$ and $d$ are not both zero. By a little experiment one is lead to the choice $R = 1/(xy)$ in Theorem 2, which gives

$$Rf = \frac{k}{y} - \frac{ax}{y} - b, \quad Rg = \frac{m}{x} - c - \frac{dy}{x}.$$ 

It is obvious that $(Rf)_x + (Rg) + y \neq 0$ in the interior of the first quadrant, hence there is no closed-orbit in this region.

The following theorem was established by Poincaré in 1880 when the coefficients are polynomials, and also when they have power-series expansions. Poincaré’s method were simplified by Bendixson in 1901 and generalized to the $C^1$ case. Basically, it states that any orbit which stays in a bounded region of a plane autonomous system either approaches a fixed point or a periodic orbit (thus, chaotic behavior cannot arise).

**Theorem 4 (The Poincaré-Bendixson theorem).** If $(f, g) \in C^1$ in a domain $\Omega$ and let $\Lambda$ be a trajectory of (1) for all $t \geq 0$. Suppose $\Lambda$ is contained in a bounded closed subset of $\Omega$ that has no critical point. Then, $\Lambda^+$ is an orbit of the differential equation and is a simply closed curve.

If $\Lambda$ itself is a Jordan curve then $\Lambda = \Lambda^+$. In general, however, $\Lambda$ is not a closed curve, but spirals towards the closed curve $\Lambda^+$. When this happens, $\Lambda^+$ is called a limit cycle; a limit cycle is a nonlinear behavior. The Poincaré-Bendixson theorem is one of few known results that asserts the existence of a periodic solution for a general class of nonlinear systems. The Bendixson-Dulac theorem predicts the absence of a limit cycle.

**Example 5 (The van der Pol equation).** In 1924 the Dutch engineer Balthasar van der Pol investigated† a series of electrical circuit containing a nonlinear resistance. His analysis led to the equation

$$x'' + \mu(x^2 - 1)x' + x = B \sin \omega t,$$

where $\mu, \omega, B$ are constants with $\mu > 0$. One of his principal results is that the equation with $B = 0$ has a nonconstant periodic solution. This will now be deduced from the Poincaré-Bendixson theorem.

Setting $B = 0$ we get the equivalent system

$$x' = y, \quad y' = -x - \mu(x^2 - 1)y$$

which has \((0, 0)\) as the only critical point. To avoid the trivial solution \((x, y) = (0, 0)\) we consider a trajectory \(x = x(t)\) and \(y = y(t)\) starting at a distance \(r_0 > 0\) from the origin. The radius squared \(r^2\) satisfies

\[
rr' = xx' + yy' = -\mu(x^2 - 1)y^2.
\]

If \(|x| \leq 1\) then \(rr' \geq 0\). Hence, the trajectory does not enter the disk \(r < r_1\) where \(r_1 = \min(r_0, 1)\). If \(|x| > 1\) then \(rr' < 0\), hence the trajectory cannot cross the circle \(r = r_2 > 1\) from inside to outside on any part of the arc satisfying \(|x| > 1\).

In a similar manner, on the line \(y = y_0\) we have \(y' = -x - \mu(x^2 - 1)y_0\), which is negative if \(|x| > 1\) and \(\mu y_0 > |x|/(x^2 - 1)\). For \(|x| \geq 2\), the maximum of the expression on the right is taken on when \(|x| = 2\) and is \(2/3\). Thus, if \(\mu y_0 > 2/3\) the value of \(y\) is decreasing on the line segment \(y = y_0, x > 2\). Similarly, if \(\mu y_0 < -2/3\) the value of \(y\) is increasing on the segment \(y = y_0, x < 2\).

To deal with the strip \(|x| \leq 2\) let us consider a line \(y = y_1 + mx\) with \(y_1\) large and \(m\) positive. A downward-pointing normal vector to this line is \(\vec{n} = (m, -1)\). If the tangent to the trajectory is \(\vec{t} = (x', y')\) then

\[
\vec{t} \cdot \vec{n} = (y, -x - \mu(x^2 - 1)y) \cdot (m, -1) = my + x + \mu(x^2 - 1)y.
\]

We take \(m = \mu + 1\). Then for \(|x| \leq 2\) we have

\[
\vec{t} \cdot \vec{n} = y + x + \mu x^2 \geq y - 2 = y_1 + mx - 2 \geq y_1 = 2m = 2 > 0
\]

provided that \(y_1 > 2 + 2m\). A similar calculation applies when \(y_1\) is a large negative number.

Since \(r_2, |y_0|, |y_1|\) in the foregoing discussion can be arbitrarily large, there is no difficulty in constructing the figure so that the initial point of the trajectory is in the interior of the simple closed curve forming the boundary. Then the trajectory stays in a ring-shaped region between this curve and the circle \(r = r_1\). Since there is no critical point in this region or on its boundary, Theorem 4 applies and a periodic solution exists. The phase portraits of the unforced van der Pol oscillators for various \(\mu\) are in [1].

**Example 6 (The Liénard equation).** The van der Pol equation is an example of the Liénard equation

\[
x'' + f(x)x'(x) = 0,
\]

where \(f, g \in C^1\), \(f\) is an even function and \(g\) is an odd function. The equation is equivalent to the following autonomous system

\[
(3) \quad x' = y - F(x), \quad y' = -g(x),
\]

where \(F(x) = \int_0^x f(\xi)d\xi\).

**Theorem 7 (The Liénard theorem).** If the Liénard system \((3)\) additionally satisfies (i) \(g(x) > 0\) for all \(x > 0\), (ii) \(\lim_{x \to \infty} F(x) = \infty\), and (iii) \(F(x)\) has exactly one positive root at \(p\), \(F(x) < 0\) for \(0 < x < p\) and \(F(x) > 0\) and monotone for \(x > 0\), then the system has a unique stable limit cycle.

To decide an upper bound for the number of limit cycles of a system of differential equations of the form

\[
x' = f(x, y), \quad y' = g(x, y),
\]

where \(f\) and \(g\) are real polynomials of degree \(n\), is the main object of the second part of Hilbert’s sixteenth problem.

**REFERENCES**

Problems.

1. Show that the system
   \[ x' = x \sin xy, \quad y' = y \cos xy \]
   has no periodic solution in the first quadrant \( x, y > 0 \). Show that the system
   \[ x' = \sin xy, \quad y' = \cos xy \]
   has no periodic solution in the plane.

2. For the system
   \[ x' = -y + x(1 - x^2 - y^2), \quad y' = y(1 - x^2 - y^2), \]
   show that the orbit \( r = 1 \) is a limit cycle.

3. Show that the DE \( u'' + (u^2 + 2(u')^2 - 1)u' + u = 0 \) has a nontrivial periodic solution.

4. (Winding number). Let \( C \) be a closed curve in the \((x, y)\)-plane of class \( C^1 \) not passing through the origin and let \( \theta = \arctan(y/x) \). The quantity \( N = \Delta \theta / 2\pi \) obtained when \( C \) is traversed once is an integer called the winding number of \( C \) with respect to the origin. Show that
   \[ N = \frac{1}{2\pi} \int_C \frac{x dy - y dx}{x^2 + y^2}. \]
   If \( C \) is a Jordan curve, it can be shown that \( |N| = 1 \) or 0 when the origin is inside or outside of \( C \), respectively.

5. Repeat the discussion in Example 5 for the Liénard equation \( x'' + f(x)x' + x = 0 \) where \( f \in C^1 \) is even, \( (|x| - 1)f(x) > 0 \) for \( |x| \neq 1 \) and \( x/f(x) \) is bounded for \( |x| \geq 2 \).