QD. SOLUTION BY QUADRATURE

When the solution of a differential equation is expressed by a formula involving one or more integrations, it is said that the equation is solvable by quadrature, and the formula is called a closed-form (or exact) solution. The term “quadrature” has its historical origin in the connection of integration with area; in plane geometry, the quadrature of the circle, is the problem of constructing a square and a circle of equal area.

Several methods of solution by quadrature were explained and illustrated in Chapter 1 of Birkhoff & Rota [1]. For example, the following theorem on the existence and the uniqueness of the first order-linear equation may be recognized as an application of the method by quadrature.

Theorem. If \( p(x) \) and \( r(x) \) are continuous on an open interval \( I \) containing the point \( x_0 \), then the initial value problem

\[
\begin{align*}
  y' + p(x)y &= r(x), \\
  y(x_0) &= y_0
\end{align*}
\]

has one and only one solution given by

\[
y(x) = e^{-P(x)}y_0 + e^{-P(x)} \int_{x_0}^{x} e^{P(t)} f(t) dt,
\]

where \( P(x) = \int_{x_0}^{x} p(t) dt \).

Other examples include homogeneous equations, exact differentials and solution by integrating factors [1].

Not all differential equations can be solved by quadrature. For example, the second-order equation

\[ y'' + p(x)y' + q(x)y = r(x) \]

cannot be solved in general by quadrature. The Riccati equation

\[ y' = A(x) + B(x)y + C(x)y^2 \]

also cannot be solved by quadrature except in special cases.

When we do have a closed-form solution, the problems of existence and uniqueness are greatly simplified. If the formula produces a solution, it provides an existence proof; and if every solution is given by the formula, it is unique.

A second-order equation of the form

(1) \( y'' = f(x, y') \)

is simplified by the substitution \( v = y' \) and becomes the first-order equation \( v' = f(x, v) \).

A second-order equation of form

(2) \( y'' = g(y, y') \)

can be simplified by the substitution \( v = y' \). By the chain rule,

\[ y'' \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = \frac{dv}{dy}. \]

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\*Proposed by ancient geometers, it is the challenge to construct a square with the same area as a given circle by using only a finitely many steps with compass and straightedge. The task was proven to be impossible, as a consequence of the fact that \( \pi \) is transcendental.
Using this in (2) we obtain the first-order equation
\[ \frac{dv}{v} = g(y, v). \]
Once \( v \) is known, \( y \) is solved by quadrature.

**Example.** When a second-order equation is reduced to an equation of the first order, a solution of the latter is often called a *first integral* of the original differential equation. Obtain a first integral for the equation
\[ 2y'' = 3y^2. \]
Setting \( v = y' \) and \( y'' = v dv/dy \) yields that
\[ 2v \frac{dv}{dy} = 3y^2, \]
which, by integration, yields that \( v^2 = y^3 + C \). Since \( v = dy/dx \), we have reduced the original second-order equation to a separable first-order equation
\[ \left( \frac{dy}{dx} \right)^2 = y^3 + C. \]

**REFERENCES**


**Problems.**

1. (a) Show that \( Pdx + Qdy \) has an integrating factor \( r = x^a y^b \) if and only if
\[ xy(P_y - Q_x) = ayQ - bxP. \]
(b) Show that \( r(x, y) = r(x) \) and \( r(x, y) = r(y) \) lead respectively to
\[ \frac{r'}{r} = \frac{P_y - Q_x}{Q}, \quad \frac{r'}{r} = \frac{Q_x - P_y}{P}. \]
If the right side is a function of \( x \) alone in the first case, or of \( y \) alone in the second, integration gives \( \ln r \) and exponentiation yields \( y \).
(c) If the equation
\[ \frac{P_y - Q_x}{Qy - Px} = h(xy) \]
holds then obtain \( r = e^{H(xy)} \), where \( H' = h \).
(d) The Bernoulli equation is
\[ y' + f(x)y = g(x)y^n \]
with \( n \neq 1 \). Try \( r(x, y) = s(x)y^{-n} \) to obtain that
\[ (y^{1-n}s)' = (1 - n)s g, \]
where \( F' = f \) and \( s = e^{(1-n)F(x)} \).
2. (Clairaut’s equation) Show that by the substitution $v = y'$, the DE
\[ y = xy' + g(y') \]
reduces to \((dv/dx)(x + g'(v)) = 0\). Taking $v = c$ or $g'(v) = -x$ obtain

\[ y = cx + g(c) \]

or the parametric solution
\[ x = -g'(v), \quad y = g(v) - vg'(v). \]
The formula (3) represents a family of straight lines. Show that its envelope, if there is one, leads to the parametric solution with parameter $c$ instead of $v$.

3. Let us consider a second-order differential operator $Ty = py'' + qy' + ry$, where $p, q, r$ are continuous functions of $x$. The adjoint operator is defined as $T^*y = (py)' - (qy)' + ry$.

Show that if there is a nonvanishing solution of $T^*u = 0$ then $Tv = f$ can be solved by quadrature. Similarly, if there is a nonvanishing solution of $Tu = 0$ then $T^*v = f$ can be solved by quadrature. (Hint. Use the Lagrange identity
\[ vTu - vT^*v = (uqv - upv' - up'v + u'pv'). \]