1 Overview

The class’s goals, requirements, and policies were introduced, and an overview of the topics in the class were described. Everything mentioned in the overview should be in the course syllabus, so consult that for a complete description.

2 Linear Algebra Review

This course requires a good understanding of linear algebra, so here is a quick review of the facts we will use frequently.

Definition 1 Let $M$ be an $n \times n$ matrix. Suppose that

$$Mx = \lambda x$$

for $x \neq 0$. Then we call $x$ an eigenvector and $\lambda$ an eigenvalue.

Proposition 2 If $M$ is a symmetric $n \times n$ matrix, then

- If $v$ and $w$ are eigenvectors of $M$ with different eigenvalues, then $v$ and $w$ are orthogonal ($v \cdot w = 0$).
- If $v$ and $w$ are eigenvectors of $M$ with the same eigenvalue, then so is $q = av + bw$, so eigenvectors with the same eigenvalue need not be orthogonal.
- $M$ has a full orthonormal basis of eigenvectors $v_1, \ldots, v_n$. All eigenvalues and eigenvectors are real.
- $M$ is diagonalizable:

$$M = V \Lambda V^{-1}$$

where $V$ is orthogonal ($VV^T = I_n$), with columns equal to the eigenvectors of $M$, and $\Lambda$ is diagonal, with the eigenvalues of $M$ as its diagonal entries.

In Proposition 2, it was important that $M$ was symmetric. None of the results stated there are necessarily true in the case that $M$ is not symmetric.

Definition 3 We call the span of the vectors with the same eigenvalue an eigenspace.

3 Matrices for Graphs

During this course we will study the following matrices that are naturally associated with a graph:

- The Adjacency Matrix
- The Random Walk Matrix
- The Laplacian Matrix
- The Normalized Laplacian Matrix
Let $G = (V, E)$ be a graph, where $|V| = n$ and $|E| = m$. We will for this lecture assume that $G$ is unweighted, undirected, and has no multiple edges of self loops.

**Remark** It will not be difficult to remove the assumptions that $G$ is unweighted and has no self loops; those assumptions we are making only for convenience. However, because the matrices we study are symmetric only if $G$ is undirected, we will find it difficult to generalize these results to directed graphs $G$.

**Definition 4** For a graph $G$, the **adjacency matrix** $A = A_G$ is the $n \times n$ matrix given by

$$A_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For an unweighted graph $G$, $A_G$ is clearly symmetric. We will often find the Laplacian matrix a more useful object to work with:

**Definition 5** Given an unweighted graph $G$, the **Laplacian matrix** $L = L_G$ is the $n \times n$ matrix given by

$$L_{i,j} = \begin{cases} -1 & \text{if } (i, j) \in E \\ d_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $d_i$ is the degree of the $i$th vertex.

For unweighted $G$, the Laplacian matrix is clearly symmetric. An equivalent definition for the Laplacian matrix that we will sometimes find useful is

$$L_G = D_G - A_G,$$

where $D_G$ is the diagonal matrix with $i$th diagonal entry equal to the degree of $v_i$, and $A_G$ is the adjacency matrix.

**4 Example Laplacian**

Consider the graph with adjacency matrix

$$A_G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

This graph has Laplacian

$$L_G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$L_G$ is a matrix, and thus a linear transformation. We would like to understand how $L_G$ acts on a vector $v$. To do this, it will help to think of a vector $v \in \mathbb{R}^3$ as a map $f_v : G \to \mathbb{R}$. We can thus write $v$ as

$$v = \begin{pmatrix} X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

The action of $L_G$ on $v$ is then

$$L_Gv = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} X(1) - X(2) \\ 2X(2) - X(1) - X(3) \\ X(3) - X(2) \end{pmatrix} = \begin{pmatrix} \frac{X(1) - X(2)}{2} \\ \frac{X(2) - \frac{X(1) + X(3)}{2}}{X(3) - X(2)} \end{pmatrix}$$
It is easy to see that for a general Laplacian, we will have

\[ L_G v_i = [d_i \ast (X(i) - \text{average of X on neighbors of } i)] \]

**Remark**  For any \( G, 1 = (1, \ldots, 1) \) is an eigenvector of \( L_G \) with eigenvalue 0, since for this vector \( X(i) \) always equals the average of its neighbors’ values.

**Proposition 6** We will see later the following results about the eigenvalues \( \lambda_i \) and corresponding eigenvectors \( v_i \) of \( L_G \):

- Order the eigenvalues so \( \lambda_1 \leq \ldots \leq \lambda_n \), with corresponding eigenvectors \( v_1, \ldots, v_n \). Then \( v_1 = 1 \) and \( \lambda_1 = 0 \)
- For all \( i \), \( \lambda_i \geq 0 \). If \( G \) has \( k \) connected components, then \( \lambda_1 \ldots \lambda_k \) are all 0, while \( \lambda_{k+1} > 0 \).
- One can get a lot of information about the graph \( G \) from just the first few nontrivial eigenvectors.

**5 Matlab Demonstration**

As we remarked before, vectors \( v \in \mathbb{R}^n \) can be thought of as maps from \( f_v : G \rightarrow \mathbb{R}^n \). Thus, each eigenvector assigns a real number to each vertex in \( G \). Since a point in the plane is a pair of real numbers, we can embed a connected graph into the plane using \((f_{v_2}, f_{v_3}) : G \rightarrow \mathbb{R}^2\). The following examples generated in Matlab show that this embedding provides a reasonable nearly-planar representation of these planar graphs.

![Figure 1: Plots of the first two nontrivial eigenvectors for a ring graph and a grid graph](image-url)
Figure 2: Handmade graph embedding (left) and plot of the first two nontrivial eigenvectors (right) for an interesting graph due to Dan Spielman

Figure 3: Handmade graph embedding (left) and plot of first two nontrivial eigenvectors (right) for a graph used to model an airfoil