Apply the idea:

\[ u = (\frac{a}{\sqrt{\nu t}})^\beta \, h(\tau) \]

\[ \int_{-\infty}^{\infty} \, dx \, u = \int_{-\infty}^{\infty} \, dx \, f(x) = \int_{-\infty}^{\infty} \, dx \, (\frac{a}{\sqrt{\nu t}})^\beta \, h(\tau) \]

\[ = \sqrt{\nu t} \int_{-\infty}^{\infty} \, d\xi \, (\frac{a}{\sqrt{\nu t}})^\beta \, h(\xi) \]

\[ = \sqrt{\nu t} \, (\frac{a}{\sqrt{\nu t}})^\beta \int_{-\infty}^{\infty} \, d\xi \, h(\xi) \]

\[ \int_{0}^{\infty} \, dx \, f(x) = \frac{a^\beta}{(\sqrt{\nu t})^{\beta-1}} \, C \]

\[ \text{do not depend on } t \] \quad \text{should not depend on } \sigma \quad \Rightarrow \beta = 1

April 21, 2024

Lecture 23

Extra Lecture: Fri 4-5:30

Return: Hmwks (Place in envelope)

Pick up: Graded Hmwk 4

New Hmwk 6

Review: Example PDE

\[ \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\
u(x, t = 0) = f(x) \\
u = 0 \text{ "fast" on } 1x1 \rightarrow \infty
\end{array} \right. \]

From exact solution:

\[ u(x, t) = e^{-\frac{x^2}{4\nu t}} \int_{-\infty}^{\infty} \, dx \, f(x) \]

Conditions \( \frac{a}{\nu t} \ll 1 \quad \frac{x}{\nu t} \gg O(1) \)

Get \( u \) under same conditions without solving PDE exactly.
Method A: Dimensional Analysis

(i) $\frac{a}{t} \cdot \frac{x}{t} = \text{independent}$

(ii) non-dimensional parameters from $x, t$

(iii) $u = \frac{a}{t} \left( \frac{a}{t} \cdot \frac{x}{t} \right)^{\beta} = \left( \frac{a}{t} \right)^{\beta} \frac{x}{t}^{\beta}$

$\beta : \text{exponent to be found}$

$PDE = \text{ODE for } h(T)$

\[ h''(T) + \frac{T}{2} h'(T) + \frac{T}{2} h(T) = 0 \]

$\rightarrow \beta : \int dx \cdot u(x,t) = \text{constant} = \beta = 1$

$w(x,t) = \frac{a}{t} h(T)$

approximate solution

Solve for $h(T)$:

\[
\begin{cases}
\beta h''(T) + \frac{T}{2} h'(T) + \frac{T}{2} h(T) = 0 \\
h(T) \text{ has to go to 0 far from } |T| \rightarrow \infty
\end{cases}
\]

We look at the solution at later times, we don't see the fine details $\frac{a}{t} \ll 1$

\[
\text{constant} = \int_{-\infty}^{+\infty} dx \cdot u(x,t) = \frac{a}{t} \int_{-\infty}^{+\infty} dx \cdot f(x) = \frac{a}{t} \int_{-\infty}^{+\infty} dx \cdot f(x)
\]

\[
\beta = \int_{-\infty}^{+\infty} d\gamma \cdot h(\gamma) = \frac{1}{a} \int_{-\infty}^{+\infty} dx \cdot f(x)
\]

(second condition)

\[
\frac{a}{t} \int_{-\infty}^{+\infty} d\gamma \cdot h(\gamma) = 0
\]

$\Rightarrow h'(T) + \frac{1}{2} \int_{-\infty}^{T} h(\gamma) = C_1 = 0$

\[
\frac{a}{t} \int_{-\infty}^{+\infty} d\gamma \cdot h(\gamma) = 0 \Rightarrow h(T) = C_2 e^{-\frac{T}{\beta}}
\]
Apply the integral constraint:
\[
\int_{-\infty}^{\infty} d\tau \cdot h(\tau) = \int_{-\infty}^{\infty} dx \cdot f(x) = C_2 \int_{-\infty}^{\infty} dr \cdot e^{\frac{r^2}{4}}
\]
\[
\Rightarrow C_2 = \frac{1}{V_\text{vol}} \int_{-\infty}^{\infty} dx \cdot f(x)
\]
\[
\Rightarrow u(x,t) = \frac{a}{V_\text{vol}} h(\tau) = \frac{a}{V_\text{vol}} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\sigma^2}} f(x) e^{\frac{x^2}{4\sigma^2}}
\]

same as exact solution.

This becomes an exact solution only if \( f(x) = \delta(x) \).

**Method B:** Group theory based

*Invariance of PDE under "stretching transformations":*

"stretching transformation" \( \begin{align*}
x' &= \lambda^d x \\
t' &= \lambda^b t \quad b: \text{arbitrary} > 0
\end{align*} \)
\( x, \lambda, b, y \) real to be found \( \lambda^d u \) (vonst)

(i) PDE in \( (x', t', u') \); demand that the PDE is the same form as the original PDE.

\[
\frac{\partial u'}{\partial t'} - \nu \frac{\partial^2 u'}{\partial x'^2} = \lambda^{d-p} \frac{\partial u}{\partial t} - \nu \lambda^{d-2d} \frac{\partial^2 u}{\partial x^2}
\]

change notation: \( \gamma = \delta \), \( \beta = \lambda \)

\[
\gamma = \delta - 2d = 3 \quad \gamma = 2d
\]

(ii) Identify invariant quantities in the \( 2 \) systems:

\[
x' = \lambda^d x , \quad t' = \lambda^d t , \quad u' = \lambda^d u
\]
\( \lambda = \left( \frac{x'}{x} \right)^{1/4} = \left( \frac{t'}{t} \right)^{1/2} = \left( \frac{u'}{u} \right)^{1/8} \)

\( x' = \left( \frac{t'}{t} \right)^{1/2} \)

**Invariant of the problem under stretching transformation**

\( u' = \left( \frac{t'}{t} \right)^{9/2} \Rightarrow u' = \frac{u}{(t')^{9/2}} - \frac{u}{(t)^{9/2}} \)

\( \beta = \text{arbitrary} \)

\( [u'] = \frac{u}{(t')^{-\beta}} - \frac{u}{(t)^{-\beta}} \)

**Invariant under ST**

(iii) "Construct" a solution, similarity solution, that respects invariants under transformation \((x,t,u) \rightarrow (x',t',u')\)

\( u = h \left( \frac{x}{\sqrt{t}} \right) \Leftrightarrow u = \left( \frac{\alpha}{\sqrt{ut}} \right)^{\beta} h(\tau), \ \tau = \frac{x}{\sqrt{ut}} \)

\( \alpha = \frac{\alpha}{\sqrt{ut}} \ll 1 \)

Where is the condition \( t = \infty \), \( \left( \frac{\alpha}{\sqrt{ut}} \right) \ll 1 \)

The respect of the invariants by the solution \( u \) is not necessary; it happens under some assumptions: Method A: for \( t \gg 1 \)

Method B includes Method A

**Example 2: Nonlinear diffusion**

Why systems behave like scaling is related to statistical, microscopical

Recall: \( \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial x} = 0 \); simple case of diffusion \( q = -\alpha \frac{\partial u}{\partial x} \)

- \( u = \text{const} \Rightarrow \partial_t q = \alpha \frac{\partial u}{\partial x} \) (linear diffusion equation)
- \( u \neq \text{const} \); e.g. \( u = g(t) \) (thermal conduction in solids, filtration theory)

**PDE:** \( \frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} \left( g(p) + \frac{\partial u}{\partial x} \right) = 0 \); nonlinear diffusion equation

**Special case:** \( g(p) = p^n \), \( k > 0 \), \( n \) real
\( \frac{\partial^2 g}{\partial x^2} - \frac{1}{x} \frac{\partial g}{\partial x} = \frac{1}{x} \left( \frac{\partial^2 g}{\partial x^2} \right) \) NOT exactly solvable for \( n \neq 0 \)

Nice sandwich:

Try similarity solution:

PDE: \( \frac{\partial^2 g}{\partial t^2} = \frac{1}{x} \frac{\partial g}{\partial x} \) Take \( n=1 \)

\( g(x; t=0) = f(x) \)

\( g \to 0 \) fast as \( |x|+t \to \infty \) \( g \) continuous in \( x \)

**Method A**

**Quantities:** \( g \times \frac{t}{x} \frac{k}{m} \)

**Dimensions:** \[ [g] \mid L \quad T \quad \frac{L^2}{T} \quad \frac{T}{[g]} \quad L\ [g] \]

What matters for a similarity solution is the integral of the IC:

\[
\int_{-\infty}^{\infty} \frac{dx}{x} f(x) = \frac{1}{\infty} \int_{-\infty}^{\infty} f(x) \, dx = m = \text{const}
\]

You will never use the IC on such, only its integral

Identify non-dimensional parameters:

\[
[X] = \frac{[x^2]}{[t]} = \frac{x^2}{t} \quad \text{non-dimensional}
\]

\[
[m] = \frac{[x]}{[g]} = \frac{[m]}{[g]} = \frac{m^2}{[g]} \quad \text{non-dimensional}
\]

\[
[C] = \frac{[C]}{[g]} = \frac{m^2}{g^2 + sk} \quad \text{non-dimensional}
\]

\[
[g] = \frac{[g]}{[m^2/k + 1]} \quad \text{non-dimensional}
\]
Non-dimensional quantities: 
\[ x \left( \frac{k}{m} \right)^{1/3} \]

Similarity solution:
\[ g \left( \frac{k}{m^2} \right)^{1/3} = h \left( \frac{x}{(mk)^{1/3}} \right) = h\left( \frac{x}{(mk)^{1/3}} \right) \]

\[ \to \text{to be found} \]

April 30, 2014 Lecture 24

Pick up solution to HW wk 5

Similarity solution: Ex. nonlinear diffusion \( \nabla u = ku \nabla^2 u \) \( (q = -u \nabla q) \)

\[ g_{st} = k \left( g^0 g_0 \right) x \]

\[ g \to 0 \quad \text{"fast"} \to 1 \times 1 + \infty + \infty \]

\[ g(x,t=0) = f(x) = \int_{-\infty}^{\infty} g(x,t) = m = \text{count} = \int_{-\infty}^{\infty} f(x) dx \]

Method A \( n=1 \)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>( g )</th>
<th>( x )</th>
<th>( t )</th>
<th>( k )</th>
<th>( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
<td>( [g] )</td>
<td>( L )</td>
<td>( T )</td>
<td>( L^2 )</td>
<td>( \frac{[g]}{[L]} )</td>
</tr>
</tbody>
</table>

Non-dimensional parameters
\[ g \left( \frac{k}{m} \right)^{1/3} \quad \xi = \frac{x}{(mk)^{1/3}} \]

Similarity Ansatz:
\[ g \left( \frac{k}{m} \right)^{1/3} = g(\xi) \]

Why?

- before we wrote first a function of \( k \) parameters and then we took one of them to be small and factored it out on a power:
- here since the equation is non-linear, the unknown function enters in the non-dimensional parameters

\[ \frac{u}{u_0} = f \left( \frac{a}{\sqrt{vt}}, \frac{x}{\sqrt{vt}} \right) \]

Linear diffusion: \( u_t = \nabla^2 u \)

Moral: Dimensional analysis is a powerful tool if you are experienced with the problem and know what to expect.

Use stretched coordinates, otherwise.