Pick up: Handouts: 19, 10 + paper
Solution to practice Set 5
Solution to Hmwrk G

Pick up Guest in the end of the class
Place Hmwrk in box

Review:
Ex. \{ Eu(x, y) + u(y) = 1 ; 0 < x < 1 \; ; \; |y| < l
\quad u(0) = u(l) = 0

Exact solution:
\[ u(x, \varepsilon) = x - 1 + \frac{e^{-x/\varepsilon} - e^{-y/\varepsilon}}{1 - e^{-\varepsilon}} \quad \varepsilon \ll 1 \]

\[ x \ll \varepsilon \ll 1: \quad u(x, \varepsilon) \approx x - 1 \quad \text{OUTER SOLUTION} \quad \frac{x - 0^+}{x - \infty} \quad \text{OVERLAP}
\]

\[ x = 0(\varepsilon): \quad u(x, \varepsilon) \approx -1 + e^{x/\varepsilon} \quad \text{INNER SOLUTION} \quad \frac{x - 0^-}{x - \infty} \quad \text{OVERLAP}
\]

\[ u(x, \varepsilon) = \frac{(\text{OUTER}) + (\text{INNER}) - (\text{OVERLAP})}{x - 1 + e^{-x/\varepsilon} - 1} \quad \text{COMPOSITE FORMULA}
\]

Example: \[ \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + 2u = 1 \]
\[ \varepsilon = 0 \rightarrow u(x, y) \approx h(x, y) = \frac{1}{2} [A(y) e^2 x + 1] \quad \text{cannot satisfy all conditions}
\] = BOUNDARY LAYERS

Remark: \varepsilon multiplies the highest derivatives in both x & y \rightarrow we have to look for BL in x & y, all 4 boundaries.
No need for BL if the highest derivative in x or y is not multiplied by \varepsilon
Look for BL:

\[
(i) \quad x = 0: \quad w(x, y; \varepsilon) = h(x, y) + p(x, y; \varepsilon) \quad \Gamma
\]

PDE for \( p \):

\[
\nabla^2 p - \varepsilon^{-1} p_x + \varepsilon^{-1} 2p = -\Delta^2 h
\]

\( p(x, y; \varepsilon) \propto p(\xi, \eta) \quad \Rightarrow \text{to be found} \)

\[
\xi = \frac{x}{\varepsilon^u}, \quad \eta \rightarrow 0
\]

PDE in \( \varepsilon \):

\[
\nabla^2 p = p_{xx} + p_{yy} = \varepsilon^{-2u} p_{\xi\xi} + p_{\eta\eta}
\]

PDE is:

\[
(\varepsilon^{-2u}) p_{\xi\xi} + \varepsilon p_{\eta\eta} - \varepsilon^{-u} p_{\xi} + 2p = -\varepsilon^{-2u} \Delta^2 h
\]

\[
\text{set coeff. in highest derivative to 1}
\]

\[
p_{\xi\xi} + \varepsilon^{2u} p_{\eta\eta} - \varepsilon^{-u} p_{\xi} + 2 \varepsilon^{2u} p = -\varepsilon^{2u} \Delta^2 h
\]

Considerations:

1) Inside the BL \( \varepsilon = O(1) \), \( p_{\xi\xi}, p_{\eta\eta}, p, \Delta^2 h, p_{\eta\eta} = O(1) \)

2) \( w \) inside the BL must reduce to \( h(x, y) \) as \( \xi \rightarrow \infty \)

\( \Rightarrow \) "fast" usually means exponentially

Why exponentially?

\( \Rightarrow \) because of stability reason:

any small perturbation should decay

\[
p_{\xi\xi} + \varepsilon^{2u} p_{\eta\eta} - \varepsilon^{-u} p_{\xi} + 2 \varepsilon^{2u} p = -\varepsilon^{2u} \Delta^2 h
\]

\[
\left[ \begin{array}{c}
\varepsilon^{u-1 - 2u} \\
\varepsilon^{-u} - \varepsilon^{2u}
\end{array} \right]
\]

\[
\Rightarrow p_{\xi\xi} - \varepsilon^{-u} p_{\xi} = 0 \Rightarrow \frac{u = 1}{v = 1} \quad O(1) \text{ eqn. for } p
\]

The highest derivative has to be kept in the BL.

solution \( \Rightarrow p(\xi, \eta) = B_1(\xi) e + O(1) \eta \)

not acceptable because grows as \( \eta \rightarrow \infty \)
no BL at $x=0$.

Since there is no BL at $x=0$, $h(x,y)$ has to satisfy the BC

$$w(x=0, y) = 0 \Rightarrow A(y) = -1 \quad \Rightarrow \quad w(x,y) = \frac{1}{2} (1 - e^{2x})$$

\[ (ii) \quad \eta = 0 \]

$$u(x, y, \varepsilon) = h(x,y) + w(x, \eta)$$

$\eta = \frac{y}{\epsilon \mu} \to be \ found$ for $\mu \to 0$.

PDE for $w$: \[ \varepsilon^2 w = (h_{xx} + w_{xx}) + (h_{yy} + w_{yy}) = -2e^{2x} + w_{xx} \]

PDE in $\eta$ for $w$: \[ -2e^{2x} + \varepsilon^{-2\mu} w_{\eta\eta} \]

$$\varepsilon \left( -2e^{2x} + w_{xx} + \varepsilon^{-2\mu} w_{\eta\eta} \right) - \left( h_{x} + w_{x} \right) + 2 \left( h + w \right) = 0$$

Considerations:

1. Inside BL, $\eta = o(1)$, $w_{\eta\eta}, w_{xx}, h_{x}, w = o(1)$
2. $w$ as $\eta \to \infty$ goes to zero exponentially

PDE: \[ (e^{1-2\mu}) \eta \eta + \varepsilon w_{xx} - 2\varepsilon e^{2x} + w_{xx} + 2w = 0 \]

Set this coeff to 1

we want an order one equation for $w \to \eta \mu = \frac{1}{2}$

PDE: \[ w_{\eta\eta} - w_{xx} + 2w = 0 \]

! BL does not lead always to ODE's

\[ \left( e^{-2x} w \right)_{\eta\eta} - \left( e^{-2x} w \right)_{x} = 0 \]

diffusion equation for $v = e^{-2x} w$

Conditions:

- $u(x, y=0) = 0$ \quad $u(x, \eta=0) = -h(x) = -\frac{1}{2} (1-e^{2x})$
- $w(x, \eta + \omega) = 0$ \quad so $\mu = \omega$
- Since there is no BL at $x=0$, $u(x=0, y) = 0 \Rightarrow w(x=0, \eta) = 0$
\[ \text{Bl at } y=0, \quad q = \frac{y}{e^{-y^2}}; \quad u = h(x) + w(x,y) \]

\[ \begin{align*} 
\frac{\partial^2 w(x,y)}{\partial x^2} - (e^{-y^2} w) \frac{\partial w}{\partial x} - (e^{-y^2} w)_x = 0 \quad &\text{Handout 10} \\
w(x, y=0) = -\frac{1}{2} (1 - e^{-x}) \\
w(x, y=\infty) = 0 \\
w(x, 0) = 0 \\
\end{align*} \]

\[ \text{boundary of} \]
\[ \text{the solution} \]

\[ w(x,y) = \int \frac{x}{\sqrt{2\pi}} \text{erfc} \left( \frac{y}{\sqrt{2x}} \right) \, dx \]

\[ \text{complementary error function} \]

\[ w(x,y) = \frac{1}{2} (1 - e^{-x}) + w(x,y) \]

satisfies PDE to some order, and conditions at \( x=0, y=0 \)

but NOT \( x=1 \).

\[ \text{(iii)} \quad x=1 \]

\[ \begin{array}{ccc}
\text{y} & \text{outer} & \text{inner} \\
& & \text{h} \\
0 & 1 & \rightarrow \text{to be found} \\
\end{array} \]

\[ w(x,y; z) \approx h(x,y) + w(x,y) + V(x^1, y) \]

PDE for \( u \): (no \( w \) because \( w \) satisfies the PDE from step 2)

\[ \begin{align*} 
\varepsilon^{1/2} \nabla \sigma + \varepsilon \sigma \nabla y + \varepsilon^{d-1} \sigma \nabla \sigma + 2 \varepsilon = 2 \varepsilon e^{2x} \\
V \sigma + \varepsilon^{d-1} V \sigma + 2 V = 2 \varepsilon e^{2x} \\
\varepsilon^{1/2} \nabla \sigma + \varepsilon \sigma \nabla y + \varepsilon^{d-1} \sigma \nabla \sigma + 2 \varepsilon = 2 \varepsilon e^{2x} \\
\end{align*} \]

\[ \Rightarrow \varepsilon \sigma + \varepsilon^{d-1} \sigma = 0 \quad \Rightarrow \quad d=1 \]

\[ \varepsilon \sigma + V \sigma = 0 \quad \Rightarrow \quad V(x^1, y) = D(y) e^{-\sigma} + E(y) \]
\[ v(\xi, \eta) \rightarrow 0 \text{ as } \xi \rightarrow \infty \Rightarrow \xi(\eta) = 0 \]

Find \( D(\eta) \):

\[ d(\eta) = 0 \Rightarrow h(x=1, \eta) + w(x=1, \eta) + v(\xi=0, \eta) = 0 \]

\[ \Rightarrow D(\eta) = -\frac{1}{\alpha}(1 - e^2) - w(1, \frac{\eta}{\alpha}) \]

Combine all pieces together:

\[ w(x, \eta; \alpha) \approx \frac{1}{\alpha}(1 - e^{2x}) + \int_0^x e^{2x'} \frac{x'}{\alpha} \text{erfc} \left( \frac{y}{\alpha} \right) dx' \]

\[ -\left[ \frac{1}{\alpha}(1 - e^2) + \int_0^x e^{2x'} \frac{x'}{\alpha} \text{erfc} \left( \frac{y}{\alpha} \right) dx' \right] e^{-\frac{x}{\alpha}} \]

Beautiful solution because it is not trivial!

This \( w \) satisfies the PDE up to order \( \alpha \) (if you admit that highest derivatives are of order \( 1 \))

This \( w \) satisfies also conditions at \( x = 0, 1, \eta = 0 \).

Question:
1) Condition at \( \eta = 0 \) ? Satisfied or not?
2) If not why that does not matter?
3) Find a solution that satisfies \( \eta = \infty \)?

Example: (Prob. 98)

\[ \frac{\partial^2}{\partial \eta^2} - (1 - \eta^2) \frac{\partial}{\partial \eta} + \eta \frac{\partial^2}{\partial x^2} = 0 \]

Boundary condition only in \( \eta, \eta = \pm 1 \)

Let \( \eta \rightarrow -\eta \) PDE remains the same, same conditions, same symmetric interval.

\( w(x, \eta) = w(x, -\eta) \) even in \( \eta \)

We can only consider \( \eta \in [-1, 0] \)
Set $\varepsilon = 0: \quad u_x = \frac{4y^2}{1-y^2} \Rightarrow u = \frac{4xy^2}{1-y^2} + c(y)$  

no BL at $x = 0 : \Rightarrow w(x=0, y) = 0 \Rightarrow c(y) = 0$  

$h(x, y) \rightarrow \infty$ for not finite value  

May 12, 2004 Lecture 28  

Opt. Redding on Solitons:  
- Debnath 9.2-9.6, 11.7, 11.8  
- Whitham 13.10-13.12  
- Dratik & Johnson, Chs. 1.2  

Pick up: Graded Hw wk 5  

Boundary layer theory  

Example: $\varepsilon u_{yy} - (1-y^2) u_x + y^2 = 0 \quad u = u(x, y)$  

Symmetry: $u(x, y) = u(x-y)$  

$\varepsilon = 0:$  

$u_x = \frac{4y^2}{1-y^2} \Rightarrow u(x, y) = h(x, y) = \frac{4xy^2}{1-y^2} + c(y)$  

no boundary layer at $x = 0 \Rightarrow h(x=0, y) = 0 \Rightarrow c(y) = 0$  

$\Rightarrow h(x, y) = \frac{4xy^2}{1-y^2}$ when $y \to \pm 1 h(x, y) \to \infty$ outer solution blows up!  

Does not satisfy conditions at $y = \pm 1$: boundary layers?  

(i) $y = -1: \quad \gamma = 1 + y, \quad \delta > 0$ to be found  

$\gamma = O(1)$ in the boundary layer: $u(x, y; \varepsilon) = h(x, y) + p(x, y)$