\[ u = A e^{i(\omega t - kx)} \] satisfies the kinetic equation

\[ u_t + c u_x = 0 \quad c = \frac{\omega}{K} \]

this is an example of solutions (waves) that satisfy some equations.

In general,

\[ u = f(x - ct) \]

bump/PERTURBATION that propagates at speed c

Generalization of the previous equation:

\[ u_t + c(u) u_x = 0 \]

- \[ c(u) = u \Rightarrow u_t + uu_x = 0 \]
- \[ u_t + uu_x = u u_{xx} \] Buder's equ.

the common concept is a wave-like solution.

February 9, 2004 Lecture 2

Optional Reading

Kevorkian § 5.1, 5.2
Debnath § 3.2-3.5

Review of Lect. 1:

IVP \quad data on \partial \Omega \quad BVP \quad data on \partial \Omega

\[ \Omega : \text{smooth, open} \]
\[ C: \text{closed curve} \]

Well-posed problem: \[ \begin{cases} 0 \text{ Solution exists} \\ \text{and} \end{cases} \begin{cases} * \text{ is unique} \\ \text{and} \end{cases} \begin{cases} \text{depends continuously on data} \end{cases} \]

\[ \begin{cases} \text{both for linear and nonlinear PDE} \end{cases} \]

THEME: \[ \text{PDE } \rightarrow \text{ODE} \]

Useful concept: Wave

Simplest PDE: \[ u_t + c(u) u_x = 0 \quad c = \frac{\omega}{K} \]

(speed - wave #)

\[ \text{(help understand)} \]

\[ \text{modify} \]

\[ \text{the solution before actually find, if) } u_{t} + c(u) u_{x} = 0 \]
PDE's describe natural phenomena, or rather, our understanding of them are based on principles (physical principles).

**Example:** Conservation law (principle) (of "mass")

Material (fluid) moves, its mass should be conserved.

Useful quantities:

- \( g(x,t) \): density [mass/length]
- \( q(x,t) \): flux [mass/time]

We can relate these two quantities using the PDE:

\[
\frac{\partial}{\partial t} \int_{x_2}^{x_1} g(x,t) \, dx + \int_{x_2}^{x_1} \frac{\partial}{\partial x} q(x,t) \, dx = 0
\]

(i) Total mass that remains in the box at instant \( t \):

\[
\int_{x_2}^{x_1} g(x,t) \, dx = \int_{x_2}^{x_1} g(x,t) \, dt
\]

(ii) Mass inside the box at instant \( t \):

\[
\int_{x_2}^{x_1} \left[ g(x,t+\Delta t) - g(x,t) \right] \, dx
\]

Change with time for \( \Delta t \):

\[
\int_{x_2}^{x_1} \frac{\partial}{\partial t} g(x,t) \, dx + \int_{x_2}^{x_1} \frac{\partial}{\partial x} q(x,t) \, dx = 0
\]

Conservation of mass:

\[
\int_{x_2}^{x_1} \left[ g(x,t+\Delta t) - g(x,t) \right] + \int_{x_2}^{x_1} \frac{q(x_1,t) - q(x_2,t)}{\Delta t} \, dx = 0
\]

Under some conditions of integrability (weak conditions):

\[
\frac{d}{dt} \int_{x_2}^{x_1} g(x,t) \, dx + q(x_1,t) - q(x_2,t) = 0
\]

Integral form of conservation law.

In order to get a PDE, \( x_2 \to x_1 \):

\[
\Delta x = x_1 - x_2 > 0, \quad \Delta x \to 0 : \int_{x_2}^{x_1} g(x,t) \, dx \approx g(\bar{x},t) \, \Delta x, \quad x_2 < \bar{x} < x_1
\]
\[ \Delta x \frac{\partial q}{\partial t} + q(x, t) - q(x, t) = 0 \quad \Rightarrow \quad \frac{\partial q}{\partial t} + q(x, t) - q(x, t) = 0 \quad (\Delta x \to 0) \]

\[ \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad \text{differential form of conservation law} \]

(suppose \( s, x, q \) are continuous)

The integral form is more general (weaker conditions on \( s \)); allows for example jumps in \( q \).

PDE for \( q \): We must relate \( q \) and \( q \). This is what models are about. If the model is not adequate, shocks may appear in the solution (non-physical phenomena).

Assumption: \( q = Q(q) \) known function (constitutive relation)

Conservative law \( \Rightarrow \) \[ \frac{\partial q}{\partial t} + Q'(q) \frac{\partial q}{\partial x} = 0 \quad \text{PDE for} \ q \]

(because of the assumption, \( q \) do not appear in the PDE; only \( \frac{\partial q}{\partial t}, \frac{\partial q}{\partial x} \) - somehow this is not a good assumption.)

\( (p \leftrightarrow u) \) \[ u_t + c(u) u_x = 0 \quad \text{c(u) } \leftrightarrow Q'(q) \]

"waves propagating with speed \( Q'(q) \), waves in mass conserv."

by the deriv. of the function rel. \( q \) and \( q \)

Another way to bring this equation:

\[ \frac{d}{dt} q = 0 \quad \Rightarrow \quad \frac{dx}{dt} = Q'(q) \quad \text{total derivative of} \ q \ \text{is zero if you move at speed} \ Q'(q) \]

\[ \frac{\partial q}{\partial t} + \frac{dx}{dt} \frac{\partial q}{\partial x} = 0 \quad \leftrightarrow \quad \frac{\partial q}{\partial t} + Q'(q) \frac{\partial q}{\partial x} = 0 \]

\[ x = x(t) : \text{characteristics (CHAR)} \]

Theory of 1st order PDE

I can always be converted to ODE \( \Rightarrow \) they are always solvable

\( \rightarrow \) what helps is the concept of CHAR

\[ a(x, t) u_x + b(x, y) u_y = 0 \quad , \quad u = u(x, y) \]

linear, homogeneous PDE

\( \equiv u \to lu \) satisfying same PDE
Suppose we vary $x$ and $y$ by $\Delta x \sim \Delta y = \{x \rightarrow x+\Delta x, y \rightarrow y+\Delta y\}$

How does the function $u$ vary?

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y) \approx u_x \Delta x + u_y \Delta y$$

($u$ is differentiable)

We want to find specific $\Delta x, \Delta y$ such that $\Delta u = 0 \iff u = \text{const}$

for these variations.

let $\{\Delta x = a \cdot \varepsilon, \Delta y = b \cdot \varepsilon\}$

$$\Delta u \approx (a u_x + b u_y) \varepsilon = 0$$

by using the PDE

What this means?

$$a \sim \frac{dx}{a(x,y)}, b \sim \frac{dy}{b(x,y)}$$

when $(x,y)$ satisfy this ODE, $u = \text{const}$ along this curve.

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}$$

the solution $h(x,y) = \text{const}$ (a curve)

this is the CHAR

Ex.1 Kinematic eq.: $u_t + cu_x = 0$ $c \cdot \text{constant}$

Find CHAR?

$a(x,t) = 1$, $b(x,t) = c$ $\Rightarrow$ $\frac{dt}{c} = \frac{dx}{c} \Rightarrow x(t) = ct + \text{const}$

$x - ct = \text{const} = k_1$

$\Rightarrow$ the CHAR are lines $x - ct = k_1$

Ex.2 Find the general solution of $u_t + cu_x = 0$

$F$: arbitrary

Along a specific CHAR, $k_1 \Rightarrow u = G_2 = \text{const}$ i.e. $G_1 = F(k_1)$

to each $k_1$, we map a const value $G_1$

Along a CHAR, $u = F(k_1) = F(x-ct)$

arbitrary function of 1 var.

$\Rightarrow$ the general solution $u = F(x-ct)$
along these lines $u$ is const (we find a general solution) but we haven't shown there are the unique curves on which $u$ is a const.

(add work -> see reading...)

How to find the arbitrary function?

- Initial data → the CHAR propagate it for further times
- How to find the solution at a point? find the CHAR that passes through follow the CHAR to $t=0$

**IVP** $u(x,0) = f(x) = F(x) \rightarrow$ we find $F$

$\Rightarrow u(x,t) = f(x-ct)$

Suppose we are given data on some $x-ct = K_0$ (on a CHAR) $\rightarrow$ this contradicts the structure of the PDE

An IVP is well-posed if data is given on curves $\neq$ CHAR.

**Ex. 3** $xu_x + yu_y = 0$, $y \geq 1$, $-\infty < x < \infty$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + K_1 \Rightarrow \frac{y}{x} = K_1$, $y = K_1 x$ [CHAR]

Along CHAR, $u = C = F(K_1)$

$u = F\left(\frac{y}{x}\right)$ general solution

$u(x,y; x^2) = e^{-x} = F(x)$

data is given on the parabola $y=x^2$ for $y \geq 1$ $\Rightarrow$ $u = e^{-y/x}$

this curve crosses each CHAR only 1 $\rightarrow$ good initial data
Suppose our equation is for $y > 0$

![Graph showing the direction of the CHAR across the origin.]

all CHAR cross at the origin

$u$: singular at $(0,0)$

(singular solution expresses

as a weird behavior of the CHAR)

example with $e^{-x} - n$ not defined at origin

(identify crisis: value at origin depends on which CHAR we take.)

Generalization:

$a(x,y,z) u_x + b(x,y,z) u_y + d(x,y,z) u_z = 0$

$u = u(x,y,z)$ linear, homogeneous

CHAR:

$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{d}$

$\Gamma(x,y,z) = K_1$

$\Gamma(x,y,z) = K_2$

surface $\Lambda$ surface = curve in 3D space

\[\text{Case 2}\]

$a(x,y) u_x + b(x,y) u_y = d(x,y)$

linear, non-homogeneous

same concept: $\Delta x, \Delta y \quad \begin{cases} \Delta x = a \cdot \varepsilon \\ \Delta y = b \cdot \varepsilon \end{cases}$

$\Rightarrow \Delta u = u_x \Delta x + u_y \Delta y = (a \cdot u_x + b \cdot u_y) \varepsilon = d \cdot \varepsilon$

$\varepsilon = \frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} = \frac{du}{d(x,y)}$

February 11, 2004  Lecture 3

Review of ODEs: Friday 5:30 pm

Review of Lecture 2 1st order PDE's

\[\text{Case 1}\]

$a(x,y) u_x + b(x,y) u_y = 0$ linear, homogeneous