\[ c(s) = G'(s) \quad \text{propagation speed} \]
\[ \text{i.e. this is the slope } \tan \beta \]

Since \( U \) goes down, the slope has to go down.

\[ c(s) < 0 \quad \text{decreases} \]

As the density increases, the drivers tend to decrease their speed.

PDE for \( s \):
\[ \frac{\partial s}{\partial t} + \frac{G'(s)}{c(s)} \frac{\partial p}{\partial x} = 0 \quad c(s) < 0 \]

February 18, 2004 Lecture 5

Review Session: FRI 4-5pm
Pick up: Handout 4

Review Lecture 4:
Eikonal eqn. \[ S_x^2 + S_y^2 - 1 = 0 \]
\[ S(0,s) = u_0(s), \quad y = 0 = p \]
\[ \left( q_0(s) = u_0(s), \quad a = p^o - 1 \right) \]

CHAR:
\[ \tan \beta = \frac{q_0(s) v}{p(s)} \]
\[ q_0(s) x - p(s) y = K_s \]

- for each point \((0,s)\) a line passes through the slope is done by \( q_0/p_0 \)

Eliminate \( q_0, p_0 \) but keep \( s \) fixed = consider all possible slope of CHAR

- all points lying on the disk are affected by the data \((0,s), \text{ all } r\)

Monge cone:
\[ s(x) = \text{constant} \]

(all ways the CHAR can affect the solution by the data at point \((0,s) \to 3D \text{ cone})
Solve the IVP:
\[
\begin{align*}
&g_t + c(s) \; s_x = 0 \\
&g(x, 0) = f(x) \; \text{known} \; \; t > 0
\end{align*}
\]

Along \text{CHAR}: \quad \frac{dt}{1} = \frac{dx}{c(s)} = \frac{ds}{0} = \Rightarrow \; \begin{cases} \; s = K_1 = \text{const} \; \quad \text{along \text{CHAR}} \\
\; x = c(s) \; t + K_2 \; \quad \text{(CHAR: linear)}
\end{cases}

\text{The slope of the CHAR depends on the solution.}

\text{Initial data:}
\begin{align*}
&x = \bar{x}, \; t = 0 : \; \; s = f(\bar{x}) \\
\Rightarrow \; &\begin{cases} \; s = f(\bar{x}) \\
\; x = c(f(\bar{x})) \; t + \bar{x}
\end{cases}
\end{align*}

\text{composite function}
\begin{align*}
F(\bar{x}) = \\
\text{Scenario 1:} \quad F'(\bar{x}) > 0 \; \text{in particular} \quad \begin{cases} \; c(s) < 0 \\
\quad F'(\bar{x}) = c(s) \; f'(\bar{x}) \geq 0
\end{cases}
\end{align*}

\text{(that's the variable we choose to parametrize the data)}

\text{const because } g = \text{const}
\text{const because } g = \text{const} = 0

\text{What's the solution on the CHAR emanating from } \bar{x}_2
\begin{align*}
&g = \bar{s}_2 = f(\bar{x}_2)
\end{align*}
Scenario 2 \( F'(\tau) \leq 0 \) in particular \[ \begin{cases} c'(\xi) > 0, & \xi < \xi, \\ f'(\tau) \leq 0 \end{cases} \]

We can obtain the same situation if \[ \begin{cases} c'(\xi) < 0, \\ f'(\tau) > 0 \end{cases} \]

The problem is that we have a density that \( \tau \) is multi-defined. The condition to happen is \( F'(\tau) \leq 0 \) \( q \) is multi-valued.

What goes wrong with our model?
\[ g = c(\xi) \]

\[ \begin{array}{ccc} \mbox{move slow} & \mbox{move fast} & \mbox{move fast} \\ \uparrow & \uparrow & \uparrow \\ \mbox{only} & \mbox{only} & \mbox{only} \\ \end{array} \]

Systematic treatment:
\[ \begin{cases} s_t + c(s) s_x = 0, \\ s(x,0) = f(x), \\ -\infty < x < \infty, \ t > 0 \end{cases} \]

Define: \( F(\tau) = c(f(\tau)) \)

\( F'(\tau) < 0 \)

Correspondence that shows the influence of the data on the solution:
\[ (x,t) \rightarrow \tilde{f}(x,t) \]
\[ g = f(c) \Rightarrow \begin{cases} \dot{s}_x = f'(c) \dot{x} \\ \dot{s}_t = f'(c) \dot{t} \end{cases} \]

Along the CHAR \( x = F(s) t + \tilde{c} \) we have:
\[ \begin{align*}
1 + t F'(s) \dot{t} + \dot{s} x &= 0 \\
F'(s) \dot{t} &= -F(s) \left[ 1 + t F'(s) \right]^{-1} \\
\end{align*} \]

From eqn. (A) and (B):
\[ \frac{\dot{s}_x}{1 + F'(s) \dot{t}} = \frac{\dot{s}_t}{1 + F'(s) \dot{t}} \]

Is it possible that \( \dot{s}_x, \dot{s}_t \to \infty \)?
\[ 1 + t F'(s) = 0 \Rightarrow t = \frac{-1}{F'(s)} \text{ possible if } F'(s) < 0 \text{ (since } t > 0 \text{)} \]

What happens to \( s \)?
\[ \begin{align*}
\dot{s}_x > 0 & \Rightarrow \text{ high density propagates faster} \\
\dot{s}_x < 0 & \Rightarrow \text{ first-order PDE given thus} \\
\dot{s}_x = 0 & \Rightarrow \text{ remedy is second-order PDE, shock} \\
\end{align*} \]

What is the earliest time \( t_0 \) for all possible \( s \) that this happens?
\[ t_0 = \min \left( \frac{1}{|F'(s)|} \right) = \frac{1}{|F'|_{\text{max}}} \text{ breaking time} \]

**Remedies:**

A.) Allow for a discontinuous \( s \):
\[ g(x,t+s) \rightarrow g(x,t+s) \rightarrow \text{ SHOCK WAVE} \]

the solution is not multi-valued
but rather changes abruptly from \( s_2 \) to \( s_1 \)
Where is the "jump"? \( x_s: x_s(t) \to x_1 \)

Recall: conservation law: \[
\frac{d}{dt} \int g(x,t) \, dx = q(x,t) - q(x_1,t)
\]

(integration form)

in order to obtain the PDE, we assumed that \( g \in C^1 \)

(differentiable)

Solutions to the integration form are called weak solutions of PDE.

The shockwave is a weak solution - it doesn't satisfy the PDE at all points.

\[
\begin{align*}
& x_1 \quad x_2 \quad x_3 \quad x_4 \\
& g(x_1) \quad g(x_2) \quad g(x_3) \quad g(x_4)
\end{align*}
\]

- Recall: \[
\frac{d}{dt} \left[ \int g(x,t) \, dx \right] = b(x) g(x,t) - \frac{d}{dt} \left[ \int g(x,t) \, dx \right]
\]

Conservation law:

\[
\frac{d}{dt} \left[ \int g(x,t) \, dx \right] = q(x_2,t) - q(x_1,t)
\]

take \( x_2 = x_1 \)

\[
\begin{align*}
& x_1(t) \quad g(x_1,t) \quad g(x_1,t) \\
& x_2(t) \quad g(x_2,t) \quad g(x_2,t) \\
& x_3(t) \quad g(x_3,t) \quad g(x_3,t)
\end{align*}
\]

values right and left of the shock

February 23, 2023 Lecture 6

Reading: Whitham: 5.1-5.4

Review session on Transform: Fri 4:30pm

Review IVP \[
\begin{align*}
\frac{d}{dt} g(t) + a(t) g_x(t) &= 0 \\
G(t) &= g(0,t) \quad x = 0 \\
G(t) &= g(t) \quad x = T \quad T > 0
\end{align*}
\]

Assume \( F(t) < 0 \) somewhere

Condition for breaking: \( t + F(t) = 0 \)

the prescribed 

Remedies: A place "shock" (be arbitrary)

Restriction: "shock speed" \( \frac{dx}{dt} = \frac{G(t) - G(t)}{t - t_2} \)

wave breaks for \( t > t_b \), \( t_b = \sqrt{\frac{1}{2|F(t)|_{max}}} \)