ODE

Ex. 1

ODE: \( \frac{du}{dx} = 0 \), \( u = u(x) \)

solution is \( u = C \), all \( x \)

condition: \( u(x_0) = u_0 \Rightarrow C = u_0 \)

\( \Rightarrow u = u_0 \) for all \( x \)

Conclusion: ODE's involve arbitrary constants.

Ex. 2

PDE: \( \frac{\partial u}{\partial x} = 0 \), \( u = u(x, y) \)

solution is \( u(x, y) = C(y) \) "arbitrary function of \( y \)"

General solutions to PDE's involve arbitrary functions.

\[ Y \quad u = u_0 \]

\[ b \]

\[ \quad \quad \quad X \]

by knowing the solution at \( (0, b) \)

we know the solution at the horizontal line passing through \( (0, b) \)

\[ Y \quad u \text{ given} \]

\[ e \]

\[ \quad \quad \quad X \]

by knowing \( u \) on the segment \( y_1 < y < y_2 \)

we know the solution in the whole strip \( y_1 < y < y_2 \), all \( x \)

\( y = \text{const.} \) "characteristic" by def. line on which \( u \) is constant

\[ Y \quad \text{CHAR} \quad Y \]

\[ \text{given} \quad L \]

not possible if \( u \) has to be continuous
If \( u \) given on \( \mathcal{L} \) then for some \( y, y \leq y', \) I have more than one value of \( u \).

Hence, generally, along a \( \text{CHAR} \) \[ \text{PDE} \Rightarrow \text{ODE} \] theme for the course.

If a PDE reduces to ODE, it is considered solvable.

Initial Value Problem (IVP) if values of \( u \) are given on some open, smooth curve (e.g., \( \mathcal{L} \)).

Generally, PDE is an eqn of the form:

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, \ldots) = 0 \quad \text{where} \quad u = u(x, y) \]

Order of the PDE is the order of the highest derivative of \( u \).

Linear PDE: It is linear function of \( u \) and its derivatives.

\[ \text{Ex. 3 PDEs} \quad \text{– const} \]

- \( u_t - c u_x = 0 \) (kinematic eqn) traffic flow, gas dynamics
- \( u_{tt} - c^2 u_{xx} = 0 \) (wave eqn)
- \( u_{xx} + u_{yy} = 0 \) (Laplace eqn)
- \( \nabla^2 u = 0 \)

The issue of conditions for PDE

\[ \text{Ex. 4} \]
\[ \begin{cases} u'' + u = 0, & 0 \leq x \leq 1 \\ u(0) = 0 \\ u'(0) = 1 \end{cases} \quad \text{IVP for ODE} \]

\[ \text{Ex. 5} \]
\[ \begin{cases} u'' + u = 0, & 0 \leq x \leq 1 \\ u(0) = 0 \\ u'(1) = 1 \end{cases} \quad \text{BVP for ODE} \]
Ex. 6 \[ u_t - c u_x = 0 \quad -\infty < x < +\infty \quad t > 0 \]

conditions are usually

\[ t \quad \begin{array}{c} \text{IVP} \quad (\text{there is a unique solution}) \\ \begin{array}{c} \text{"data"} \\ \text{"data"} \\ x \\ x \\ \uparrow \end{array} \\ \begin{array}{c} u \quad \text{given} \\ \text{given} \\ \uparrow \end{array} \\ x \end{array} \]

Ex. 7 \[ u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < +\infty \quad t > 0 \]

\[ t \quad \begin{array}{c} \text{IVP} \quad (\text{there is a unique solution}) \\ \begin{array}{c} \text{"Cauchy data"} \\ \text{"Cauchy data"} \\ \begin{array}{c} u, u_t \quad \text{given} \\ \text{given} \\ \uparrow \end{array} \\ x \end{array} \]

Ex. 8 \[ \nabla^* u = 0 \quad \mathcal{S} \quad \mathcal{S} \]

A) \( u \) given on \( \mathcal{S} \) BVP "Dirichlet"
B) \( \frac{\partial u}{\partial n} \) given on \( \mathcal{S} \) BVP "Neumann"
C) \( a u + b \frac{\partial u}{\partial n} \) given on \( \mathcal{S} \) BVP "Robin" (mixed, bc's)

for these conditions (ex. 6.7.8) solutions not only exist but are unique.

In physical problems, solutions: exist \[ \begin{cases} \text{be unique} \end{cases} \]

Ex. 9 \[ \begin{cases} u'' = 0 \quad \text{if} \quad 0 < x < 1 \\ u(0) = 0, u'(1) = 1 \end{cases} \]

\[ u'' = 0 \rightarrow u(x) = Ax + B \quad u'(x) = A \]

* \( u'(0) = 0 \rightarrow A = 0 \) \( \Rightarrow \) must be true, together \( \Rightarrow \) no solution
* \( u'(1) = 1 \rightarrow A = 1 \) \( \Rightarrow \) impossible

Alternatively: \( 0 = \int_0^1 u''(x) \, dx = u'(1) - u'(0) = 1 - 0 = 1 \) ! contradiction
Ex. 10 \[ \nu^2 \omega = 0 \]

BVP: \( \nabla \cdot \mathbf{u} = 0 \) or Neumann bd's

Recall: Divergence theorem \[ \int_S \nabla \cdot \mathbf{A} \, dS = \int_C \mathbf{n} \cdot \mathbf{A} \, dl \]

\[ 0 = \int_S \nabla \cdot \mathbf{u} \, dS = \int_C \mathbf{n} \cdot \mathbf{u} \, dl = \int_0^{2\pi r} \mathbf{u} \, dl = 1 \times 2\pi r \]

no solution with this data

ill-posed problems = no solution exists for the given eqn with the given data

Ex. 11 \[ \nu^2 \omega = 0 \]

BVP: \( \frac{\partial \omega}{\partial t} = 0 \) no contradiction this time

let \( \omega_1 \) is a solution then \( \omega_2 = \omega_1 + \omega_3 \) is also a solution

3) solution exists but it is not unique

Well-posed problems (classical definition by Hadaward):

1) the solution must exists
2) the solution must be unique
3) the solution depends continuously on data
   
   (e.g., small perturbation of the data results in small perturbation of solution)

Concepts that apply both to linear and nonlinear PDE's

Wave

\[ u(x,t) \]

\[ \lambda \text{ wavelength} \]
\[ \kappa = \frac{2\pi}{\lambda} \text{ wave number (how many crests/length)} \]

\[ u(x,t) = A \cos(\omega t - \kappa x) \]

\[ \omega \text{ frequency (in time)} \]

satisfies the wave equation \[ u_{tt} - c^2 u_{xx} = 0 \]

\[ c = \frac{\omega}{\kappa} \text{ w/rt} \]
\[ u = A e^{i(\omega t - kx)} \] satisfies the kinematic eqn
\[ u_t + c u_x = 0 \quad c = \frac{\omega}{K} = \text{"wave speed"} \]

This is an example of solutions (waves) that satisfy some equations.

In general,
\[ u = f(x-ct) \quad \rightarrow c \]

bump/ perturbation that propagates at speed \( c \)

Generalizations of the previous equations:
\[ u_t + c(u) u_x = 0 \]
- \( c(u) = u \Rightarrow u_t + u u_x = 0 \)
- \( u_t + u u_x = v u_{xx} \) Burger's eq.

The common concept is a wave-like solution.

February 9, 2004 Lecture 2

Optional Reading: Kevorkian § 5.1, 5.2
Debnath § 3.2-3.5

Review of Lec.1:

IVP \quad \text{data on } \partial \Omega \quad \text{BVP} \quad \text{data on } \partial \Omega

\( \partial \Omega \): smooth, open
\( \partial \Omega \): closed curve

Well-posed problems:
- \( \square \) Solution exists and is unique
- \( \square \) \ndepends continuously on data \} \neq \text{ill-posed problem}

THEME: \[ \text{PDE } \Rightarrow \text{ODE} \]
both for linear and nonlinear PDE

Useful concept: Wave

Simplest PDE: \[ u_t + c u_x = 0 \quad c = \frac{\omega}{K} \]

(help understand: Modify ↓ speed = wave #
the solution before actually finding it)
\[ u_t + c(u) u_x = 0 \]