Where is the "jump"?

Recall: conservation law

\[
\frac{d}{dt} \int_{x_1}^{x_2} g(x,t) \, dx = \frac{d}{dt} \left[ q(x_2,t) - q(x_1,t) \right]
\]

(integration form)

in order to obtain the PDE, we assumed that \( g \in C^1 \) (differentiable)

Solutions to the integration form are called WEAK solutions of PDE.

The shock wave is a weak solution – it doesn't satisfy the PDE at all points.

\[
g(x_1,t) = \begin{cases}
0 & \text{for } x < x_1, \\
1 & \text{for } x > x_2
\end{cases}
\]

Recall:

\[
\frac{d}{dt} \int_{a(t)}^{b(t)} g(x,t) \, dx = \frac{d}{dt} \left[ b(t) g(b(t)) - a(t) g(a(t)) \right] + \int_{a(t)}^{b(t)} \frac{d}{dx} \frac{dg}{dt}(x,t) \, dx
\]

Conservation law:

\[
x_2' \frac{d}{dt} g(x_2', t) - x_1' \frac{d}{dt} g(x_1', t) = \int_{x_1}^{x_2} \frac{d}{dt} g(x,t) \, dx
\]

take \( x_2' = x_1' \)

\[
\frac{d}{dt} g(x_1,t) = \frac{d}{dt} g(x_2,t)
\]

values right and left of the shock

February 23, 2019 Lecture 6

Reading: Whitham: 5.1-5.5

Review session on Transforms: Fri 4:30

Review IVP

\[
\begin{cases}
\frac{d}{dt} g(x,t) + G(x) = 0 \\
G(x) = \text{G}(x) ; \quad x = \gamma
\end{cases}
\]

\( \gamma > 0 \), assume \( G'(\gamma) < 0 \) somewhere

\( t = 0 \leq t_b \) wave breaks for \( t > t_b \)

\( t = t_B \)

Condition for breaking: \( t + F'(\gamma) t = 0 \)

the pass can't be arbitrary.

Remedies: A Plan "shock"

Restriction: "shock speed" \( \frac{dx}{dt} = \frac{G(x)}{x - x_1} \) "cut" these planes
the shock moves at speed prescribed by the conservation
of what we remove = what we add. (Conservation of mass)

B Modify the PDE

\[ q = G(\psi) - q = G(\psi, \psi_x) = \psi_x - u\psi_x, \quad u > 0 \]

New PDE: \( \frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} = 0 \)

the PDE is the interplay between diffusion and

- diffusion term
- convection term
c (non-linear)

\[ \frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} \]

smooth sol. but varies rapidly

Assume a particular solution:

\[ \psi = \phi(x-Ut), \text{ traveling wave: if you move with} \]

\[ L = \text{L} = \text{const}, \quad \text{speed} U (x-Ut=\text{const}) \]

you see \( \psi = \text{const} \)

Want to find \( U \) s.t. \( \psi \) satisfies new PDE with \( \psi = \phi_1, \phi_2 \)

(we see \( \phi = \text{const} = \phi_1 \) on left, \( \phi = \phi_2 = \text{const} \) on right)

call \( X = x-Ut \)

\[ \psi_t = \frac{\partial}{\partial t} \psi(X) = -U \phi'(X), \quad \psi_x = \phi'(X), \quad \psi_{xx} = \phi''(X) \]

\[ \Rightarrow -U \phi'(X) + \phi'(\psi) \phi'(X) = \psi''(X) \quad \text{(an ODE)} \]

\[ \frac{d}{dX} \left[ \psi(X) - U \phi \right] = \frac{d\phi}{dx} \]

\[ \psi(X) - U \phi + A = \frac{d\phi}{dx} \]

condition for \( \phi \) from the picture:

\[ \begin{cases} \psi = \phi_1, & X < +\infty \\ \psi = \phi_2, & X > -\infty \end{cases} \]

\[ \psi = \phi_1, \quad U(\phi_1) + A = 0 \]

\[ \psi = \phi_2, \quad U(\phi_2) + A = 0 \Rightarrow U = \frac{\phi_1 - \phi_2}{s_2 - s_1} \]

How narrow is this region \( \mathcal{R} \)?

\[ \Rightarrow \text{Must specify} \quad \psi(X) = \psi_1^2 + \psi_2^2, \quad d > 0 \quad \text{(we can assume this without loss of generality)} \]
Take a traveling wave solution observer that moves with the speed of the shock and has a constant density.

Take $g = g(x-Ut)$  

$Q(g) - U_0 g + A = \nu g''(x)$

which is quadratic

$Q(g) - U_0 g + A = 0$  

$Q(g) - U_0 g + A > 0$  

$\Rightarrow Q(g) - U_0 g + A = d(g-p_1)(g-p_2)$

These are the roots

$d(g-p_1)(g-p_2) = \nu g''(x) = \frac{dp}{d(g-p_1)(g-p_2)} = \frac{dX}{\nu}$

$g_2-p_1 = C \cdot E \nu$  

$g_1-p_2$

$g_1-g_2 = \frac{C}{\nu}$

$g+p_1 \Rightarrow \text{LHS} \rightarrow \infty \Rightarrow \text{RHS} \rightarrow \infty \Rightarrow X \gg \frac{\nu}{d(g-p_1)}$

$g+p_2 \Rightarrow \text{LHS} \rightarrow 0 \Rightarrow \text{RHS} \rightarrow 0$

$\Rightarrow X \ll 0$ and $|X| \gg \frac{\nu}{d(g-p_1)}$

Estimate of width of $R: O(\frac{2\nu}{d(g-p_1)})$

when $U \rightarrow 0$ (diffusivity) the width of the region goes linearly to 0 (shock) $\rightarrow$ Bl

$\Rightarrow Q(g): \text{quadratic}$  

$St + C(g) Sx = \nu Sxx$  

$c(g) = C'(g)$

$\Rightarrow Q(g) = \nu Sx$  

$\Rightarrow Q(g) = \nu Sx \\

\Rightarrow c' = \frac{C'}{C} C(g) Sx = \frac{C'}{C} \nu Sx$  

$\Rightarrow \frac{\partial}{\partial t} (c(g) \nu Sx) = \frac{1}{\nu} \frac{\partial}{\partial x} \left( \frac{c'(g)}{C} c''(g) \right)$

$\Rightarrow \frac{\partial}{\partial t} \left( c(g) C \frac{\partial}{\partial x} c(g) \right) = \frac{\partial}{\partial x} \left( \frac{c'(g)}{C} c''(g) \right)$

$\Rightarrow \nu \frac{\partial c(g)}{\partial t} + \nu \frac{\partial}{\partial x} \left( \frac{c'(g)}{C} c''(g) \right) = \frac{\partial}{\partial x} \left( \frac{c'(g)}{C} c''(g) \right)$

$\Rightarrow \nu \frac{\partial}{\partial t} c(g) + \nu \frac{\partial}{\partial x} c(g) \frac{\partial}{\partial x} = \nu \frac{\partial}{\partial x} \frac{\partial}{\partial x} c(g)$

Burger's equation

A non-linear PDE can be converted to linear PDE

Simple PDE that involves diffusion and non-linearity
System of PDE's

Ex. \( u_{tt} - y u_{xx} = 0 \) (wave equ. \( y^2 c^2 > 0 \))

Let's convert it to system of PDE's:
\[
\begin{align*}
\gamma & \equiv u_x \quad \text{(definition)} \\
\omega & \equiv u_t
\end{align*}
\]
assuming some continuity
\[
\begin{align*}
\gamma & = \omega_t \\
\omega & = \omega_x \\
\gamma_x - \gamma \omega_x & = 0
\end{align*}
\]

"second order PDE" \( \rightarrow \) "system of first order PDE" maybe we can apply the method we learned about 1st order PDE.

Recall: \( a \omega_x + b \gamma_y = 0 \) (quasi-linear).

CHAR: \[
\begin{cases}
\delta x = a \epsilon \\
\delta y = b \epsilon
\end{cases}
\]
along CHAR, \( \delta w = 0 \)

the direction of the CHAR are known.

FOR PDE system is not known a priori. The direction of the CHAR must be determined.

Take \[
\begin{cases}
\delta x = d \epsilon \\
\delta y = \beta \epsilon
\end{cases}
\]

\[
\begin{align*}
\delta v & = (\nu_x \delta x + \nu_t \delta t) = (\nu_x d + \nu_t \beta) \epsilon \\
\delta w & = -u - (w_x d + w_t \beta) \epsilon
\end{align*}
\]

Good for quasi-linear.

We require that along CHAR, \( m_1 \delta v + m_2 \delta w = 0 \) (take this as data)

impose \( \delta v = 0 \) and \( \delta w = 0 \) is too strong, we will loose solutions...

Superposition of variations zero along CHAR

\[
\begin{cases}
\nu_x \delta x + \nu_t \delta t = 0 \\
\omega_x \delta x - \omega_t \delta t = 0
\end{cases}
\]

Compare coefficients in front of same derivatives
\[
\begin{align*}
\nu_t: & \quad \epsilon_1 = m_1 \beta \\
\omega_x: & \quad -\epsilon_1 = m_2 \alpha \\
\nu_x: & \quad -\epsilon_2 \gamma = m_1 \alpha \\
\omega_t: & \quad \epsilon_2 = m_1 \beta
\end{align*}
\]
\[ (-\beta \cdot \alpha)(e_1) = (0) \quad \text{or} \quad \left( \begin{array}{c} \beta \\ \alpha \end{array} \right)(e_2) = (0) \]

Non-trivial solution: \( d^2 - \beta \gamma x = 0 \) \( \Rightarrow d = \pm \beta \sqrt{\gamma} \)

Without loss of generality, let's take \( \beta = 1 \) \( \Rightarrow d = \pm \sqrt{\gamma} \)

we find that the direction of the \text{CHAR}

on which superposition of the solutions \( \pm \text{out} \) is given by \( \pm \sqrt{\gamma} \).

February 28, 2004 Lecture 2

Pick up:  
- Handout 5  
- Homework 2  
- Solution to Homework 1  
- Practice set 2

Review session on Friday 4-5 pm.

PDE systems:  
THEM E PDE \( \Rightarrow \) ODE (3)

Ex. \( u_{xx} - \gamma u_{xx} = 0 \), \( \Rightarrow \) \[
\begin{aligned}
&\begin{cases}
v_t - w_x = 0 \\
v = w_x \\
w_t - \gamma w = 0
\end{cases}
\end{aligned}
\]

Matrix form: \[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\gamma & 0
\end{pmatrix}
\begin{pmatrix}
v_t \\
w_t \\
\gamma w
\end{pmatrix}
= \begin{pmatrix} 0 \end{pmatrix}
\]

Steps:

1. Define \text{CHAR}: \[ \delta x = d \epsilon \]

2. Combine PDE sys. into \text{ eqn. } \( e_1(v_t - w_x) + e_2(w_t - \gamma w) = 0 \) \text{ any } e_1, e_2

Read last equation as a statement about variations \( \delta v, \delta w \)

along \text{CHAR}

Eq. (II): \[ m_2 v - m_1 w = 0 = m_1(v_x d + v_t \beta) + m_2(w_t + w_x \beta) \]

Eq. (I): \[ e_1(v_t - w_x) + e_2(w_t - \gamma w) = 0 \]

Compare coeffs. of \( v_t, v_x, w_t, w_x \) to find \( m_1, m_2, e_1, e_2 \)

\[
\begin{cases}
e_1 = m_1 \beta & \text{(1) } e_2 = m_2 \\
e_1 = m_2 \beta & \text{(2) } x e_2 = m_1 \beta
\end{cases}
\]

Find \( d = \pm \beta \sqrt{\gamma} \) if \( \beta = 1 \)

\[
\begin{cases}
e_1 = m_1 \beta & \text{or} \quad d = \pm \beta \sqrt{\gamma} & \text{if } \beta = 1
\end{cases}
\]

\[
\begin{cases}
e_1 = m_2 \beta & \\
e_1 = m_2 \beta
\end{cases}
\]