Optional reading: Hinch 1.1-1.4 8.1-8.3 5.1-5.7

Review of similarity

Stretching transformations (ST):

\((x, t, u) \rightarrow (\hat{x}, \hat{t}, \hat{u})\):

\[
\hat{x} = \lambda^4 x, \quad \hat{t} = \lambda^3 t, \quad \hat{u} = \lambda^6 u
\]

\(\lambda > \text{arb} > 0, \ d, \beta, \gamma \text{ real}\)

PDE is invariant under ST if PDE - form in \((x, t, u)\)

same as PDE - form in \((\hat{x}, \hat{t}, \hat{u})\)

Idea:

(i) Find relations among \(d, \beta, \gamma\) that leave PDE invariant.

(ii) Construct desired solutions for those \(d, \beta, \gamma\)

that are themselves invariant.

Similarity solutions: Particular solutions to PDE that respects invariance under ST

Particular solutions: Not valid everywhere, at every time.

Remark: Original conditions to PDE not always invariant under ST. Choose conditions that are also invariant.
Example: Incompressible Fluid flow past an infinite plate

\[ \begin{align*}
&\begin{cases}
  u: \text{fluid velocity in } x \\
  v: \text{fluid velocity in } y \\
  \nu: \text{viscosity (const)}
\end{cases} \quad \Rightarrow \quad \begin{cases}
  u, v = 0 \\
  \text{thickness } = 0
\end{cases}
\end{align*} \]

\[ \begin{align*}
PDE \text{ sys.} &:\quad \begin{cases}
  \nu u_x + \nu u_y = \nu u_{yy} \quad \text{momentum conservation} \\
  u_x + v_y = 0 \quad \text{mass conservation}
\end{cases}
\end{align*} \]

**Condition:**

\[ \begin{align*}
&\begin{cases}
  u = U \quad y = \infty, \ x > 0 \\
  u = U_0 \quad x = 0, \ y > 0
\end{cases}
\end{align*} \]

Reduce to a single equation by introducing a streamline function \( \psi \):

1. PDE
   \[ \begin{cases}
   u = \psi_y \\
   v = -\psi_x
\end{cases} \]

2. Satisfied
   \[ \psi_x = 0 \quad \psi_y = 0 - \psi_x \]

Try similarity solutions

ODE is the Blasius equation

**Perturbation theory for PDE's:**

To help obtain approximate solutions when PDE's (or conds) contain small (non-dimensional) parameters

\[ \nabla^2 u + \epsilon \left( \frac{\partial u}{\partial y} \right)^2 = 0, \quad u = u(x, y) \]

Only method that can work on this problem is Green's function, but we want to use perturbation theory.
The PDE is not linear.  Can NOT be solved exactly.

\[ \nabla^2 u = 0 \]

For perturbation theory, everything has to be non-dimensional. \( \epsilon \) has to be non-dimensional.

**Step 1:** Set \( \epsilon = 0 \) and apply all given conditions.

If solution exists in this case, then we have regular perturbation theory.

**Regular Perturbation**

\[ \nabla^2 u = 0 \]

**Asymptotic**

\[ u \to 0 \]

separation of variables: obvious choice

(FT wants \( x \in (0, \infty) \))

Try a solution of the form:

Comment: additive separation of variables will not work

\[ \text{an over-determined problem.} \]

**Multiplicative:**

\[ u = X(x) \cdot Y(y) \]

+ homog. bc's

\[ \frac{X''}{X} = \frac{Y''}{Y} = \text{const} > 0 \quad \text{(oscillatory behavior in y)} \]

\[ \frac{X}{Y} = k^2 > 0 \quad (k > 0) \]

\[ X(x) = B e^{kx} + C e^{-kx} \]

\[ Y(y) = D \sin(ky) + E \cos(ky) \]

**Homog. condns:**

\[ u = 0 \text{ at } y = 0 \implies E = 0 \]

\[ u = 0 \text{ at } y = \pi \implies D \sin(TK) = 0 \quad k \text{ integer } k, n = 1, 2, \ldots \]

\[ u \to 0 \text{ at } x \to \infty \implies B = 0 \]
\[ u = X(x)Y(y) = e^{-\alpha x} \sin(ny) \quad \alpha = \sqrt{n^2 + \xi^2} \quad B = \delta(x - a) \]

Linear superposition:
\[ u^{(0)} = \sum_{n=0}^{+\infty} B_n e^{-\alpha x} \sin(ny) \]

\[ u^{(0)} : u \approx u^{(0)} = u(x, y, \epsilon = 0) \]

Apply condition at \( x = 0 \):
\[ w_{x=0} = A \sin y = \sum_{n=0}^{+\infty} B_n \sin(ny) \]

\[ B_n = \begin{cases} A, & n = 1 \\ 0, & \text{else} \end{cases} \]

Final solution:
\[ u = u^{(0)} = Ae^{-\alpha x} \sin y \]

Step 2: Consider \( \epsilon \neq 0 \) to leading order.

Assume \( u(x, y, \epsilon) = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \ldots \)

(for a regular perturbation, dependence on \( \epsilon \) is in power series)

\( \Rightarrow u \) has a "regular" expansion around \( \epsilon = 0 \).

Questions:
- \( \Rightarrow \) is it really a convergent expansion?
- \( \Rightarrow \) can we use it even if it is not convergent?

PDE for \( u^{(1)} \):
\[ \nabla^2 u = -\epsilon \left( \frac{\partial u^{(1)}}{\partial y} \right)^2 = \nabla^2 u^{(1)} = -\epsilon \left( \frac{\partial u^{(0)}}{\partial y} \right)^2 \]

\[ \nabla^2 u^{(1)} = -\left( \frac{\partial u^{(0)}}{\partial y} \right)^2 \quad \text{(set equal coefficients of same powers of \( \epsilon \))} \]

\[ \nabla^2 u^{(1)} = -A^2 e^{-2y} \cos^2 y \quad u^{(1)} \to 0 \quad \text{zero conditions everywhere but the PDE is not homogeneous} \]

Poisson equation

All higher order terms satisfy the Poisson equation.

Thus, we get:
a non-linear PDE with \( \epsilon \) \Rightarrow sequence of linear PDE's.

Digression: To solve a non-homogeneous, linear PDE with homogeneous boundary, we expand the solution to function that
satisfy homogeneous corresponding PDE with homogeneous be's

suitable hom. PDE: Laplace
hom. be's in {4}

Expand \( u^{(1)} \) in Fourier series: (from step 1)

\[
u^{(1)} = \sum_{n=1}^{\infty} a_n(x) \sin(ny)
\]

PDE:

\[
\sum_{n=1}^{\infty} \left( a_n^{(x)} - n^2 a_n \right) \sin(ny) = -A^2 e^{-2x} \cos^2 y
\]

RHS:

\[
-\frac{A^2 e^{-2x}}{\pi} \sum_{n=1}^{\infty} b_n \sin(ny)
\]

\[
= \cos^2 y
\]

\[
= \sin^2 y
\]

\[
= \frac{1}{2} - \cos 2y
\]

\[
\Rightarrow b_n = \frac{2}{\pi} \int_0^\pi \left( \sin^2(y) \right) \sin(ny) dy
\]

\[
= \begin{cases} 
0, & n \text{ even} \\
\frac{n^2 - 2}{\pi n^2 (n^2 + 1)}, & n \text{ odd}
\end{cases}
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \left[ a_n^{(x)} - n^2 a_n \right] \sin(ny) = -A^2 e^{-2x} \sum_{n=1}^{\infty} b_n \sin(ny)
\]

ODE for \( a_n(x) \):

\[
\begin{align*}
&\text{n: even} & a_n^{(x)} - n^2 a_n &= -A^2 e^{-2x} & b_n = 0 \\
&\text{n: odd} & a_n^{(x)} - n^2 a_n &= -A^2 e^{-2x} & \frac{\sin 2y}{\pi n(n^2 + 1)}
\end{align*}
\]

\[
x \in (0, +\infty) \quad \{ a_n(x=0) = 0 \\
a_n(x=+\infty) = 0
\]

\[
a_n^{(x)} - n^2 a_n = -A^2 e^{-2x} b_n
\]

\[
a_n(x) = F e^{-nx} + G e^{nx} + Me^{-2x}
\]

- particular

\[
G = 0 \quad \text{will fail for } n=2
\]

\[
F + M = 0 \quad \text{will fail for } n=2
\]

but does not matter here
Mis found by substitution in PDE: $(4-n^2)H = -A^2b_n \rightarrow M = \frac{A^2b_n}{a^2 \cdot 4}$.

$$n \Rightarrow F = \frac{A^2b_n}{n^2 \cdot 4}$$

$$u_n(x) = \frac{A^2b_n}{n^2 \cdot 4} \left( e^{-2x} - e^{-n^2x} \right)$$

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{1}{\pi} \frac{(-1)^{n/2}}{(n^2 - 4)^{1/2}} & n \text{ odd} \end{cases}$$

$$u(x,y) = \sum_{n=1}^{\infty} u^{(n)}(x,y) \sin(n \pi y)$$

$$w^{(n)}(x,y) = \frac{4A^2}{\pi} \sum_{n=1 \text{ odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{(n^2 - 4)^{1/2}} \sin(n \pi y)$$

$$u(x,y) = u^{(0)}(x,y) + \sum_{n=1}^{\infty} u^{(n)}(x,y) + \ldots$$

$$1\varepsilon_i \ll 1$$

**Issues about convergence:**

- Does the series $u = u^{(0)} + \varepsilon u^{(1)} + \ldots + \varepsilon^n u^{(n)} + \ldots$ converge?

  **Answer:** Yes, by theory of integral equations.

  (For sufficiently small $\varepsilon$, $1\varepsilon_i < 1\varepsilon_0$)

- More generally, do series of such form converge?

  **Answer:** No, in principle; They may be divergent, yet asymptotic.

  (Asymptotic series: you've n and you look if your finite sum is close to the true $u$ if $\varepsilon \ll 1$)

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May 5, 2004: Lecture 26

- Quiz 2: Mon, May 10, pick up in class.
  - to be returned Wed, May 12 before class
  - Review Session: PRL 4-7:30 pm

Review regular perturbations for PDE

(Advice: have eqns. in non-dimensional form)

**General problem:** PDE: $\left\{ F(x_1, x_2, \ldots x_n, \text{deriv of } u; \varepsilon) = 0, \right\}$

+ conditions