Set $\varepsilon = 0 : \quad u_x = \frac{4y^2}{1-y^2} \quad \Rightarrow \quad u = \frac{4xy^2}{1-y^2} + c(y)$

no BL at $x = 0$ so $w(x=0, y) = 0 \quad \Rightarrow \quad c(y) = 0$

$h(x, y) \rightarrow \infty$ not finite value

May 12, 2004 Lecture 28

Opt. Redding on Solitons:
- Debnath 9.2-9.6, 11.7, 11.8
- Whitham 13.10-13.12
- Dratın & Johnson, Chs. 1.2

Pick up: Graded Hmwk 5

Boundary layer theory

Example: $\varepsilon u_{yy} - (1-y^2)u_x + y^2 = 0 \quad u = u(x, y)$

Symmetry: $u(x, y) = u(x, -y)$

I $\varepsilon = 0$:

$u_x = \frac{4y^2}{1-y^2}$ \quad $u(x, y) = h(x, y) = \frac{4xy^2}{1-y^2} + c(y)$

no boundary layer at $x = 0$ \quad $h(x=0, y) = 0 \quad \Rightarrow \quad c(y) = 0$

$\Rightarrow \quad h(x, y) = \frac{4xy^2}{1-y^2}$ ! when $y \to \pm 1$ \quad $h(x, y) \to \infty$

outer solution blows up!

Does not satisfy conditions at $y = \pm 1 = \text{boundary layers}$?

ii) $y = -1: \quad y = \frac{1+y}{\varepsilon^2}$, $\eta_0$ to be found

$y = O(1)$ in the boundary layer: $u(x, y; \varepsilon) = h(x, y) + p(x, y)$
PDE for $p$:

$$e^{2d} p_{yy} + e^{2d} h_{yy} - (1-y^2) p_x = 0 + (1-y^2) h_x = 4y^2$$

Inside the boundary layer: $\gamma = O(1)$, $p_{yy} = O(1)$, $h_{yy} = O(1)$, $p_x = O(1)$

How it blows up:

$$h(x,y) = \frac{4x(-1 + \gamma e^d)^2}{1 - (-1 + \gamma e^d)^2} \approx \frac{4x e^{-d}}{2\gamma}$$

as $y \to 1$ inside the layer

$$h(x,y) = O(e^{-d})$$

=) $eh_{yy}$ can NOT be neglected

$$p_{yy} + e^{2d} h_{yy} - e^{2d - 1} (1-y^2) p_x = 0$$

usually we neglect $e^{2d} h_{yy}$ and find $d = \frac{1}{2}$

=) can not do this here because $h_{yy} \neq O(1)$

Difficult point: outer solution scales as a negative power of $e$.

Outer solution:

$$h(x,y) = \frac{4xy^2}{1-y^2}$$

$$-1 < y < 1 \quad \gamma > O(1)$$

Inner solution: $u_{in}(x,y)$

Overlap: we let $y \to 1$ in the outer solution

$$h(x,y) \underset{y \to 1}{\approx} \frac{4x}{2\gamma} e^{-d}$$

=) set $u_{in}(x,y) = e^{-d} \Psi(x,\gamma)$ where $\Psi$ and derivatives $O(1)$
For boundary layer problems, it is advisable to separate inner and outer solution and then match them afterwards.

Outer approach \( u = h + p \)

works only if \( h \) & \( p \) are of the same order

Boundary layer \( \gamma = o(1) \), find PDE for \( \Psi \)

\[
\varepsilon \ln y + (1-y) u_{\text{in},x} + y^2 = 0
\]

\[
\Rightarrow \quad \frac{\varepsilon}{\varepsilon^{3d-1}} \psi_{xy} - 2\gamma \psi_x + y = 0
\]

Simplification:

\[
y \approx \frac{1}{1 + y - \varepsilon^d y^2}
\]

\[
\psi_{xy} - 2d \varepsilon^{3d-1} \gamma \psi_x + 4 \varepsilon^{3d-1} = 0
\]

assuming \( \psi_{xx}, \psi_{x}, \gamma = o(1) \)

in order to get an eq. of order 1 \( \Rightarrow 3d = 1 \)

\[
d = \frac{1}{3}
\]

PDE for \( \Psi \):

\[
\psi_{xy} - 2d \varepsilon^3 \psi_x + y = 0
\]

Condition:

\[
u_{\text{in}}(x,y) = \varepsilon^{-d} \psi(x,\gamma) \Rightarrow \psi(x,\gamma) = \varepsilon^{\frac{d}{3}} u_{\text{in}}(x,y)
\]

At \( y = -1 \):

\[
\psi(x,\gamma) = \varepsilon^{\frac{d}{3}} u_{\text{in}} \bigg|_{y=-1} = \varepsilon^{\frac{d}{3}}. I \approx 0 \quad \psi = o(1)
\]

\[
\Rightarrow \psi(x,\gamma = 0) = 0 \quad \text{(1)}
\]

At \( x = 0 \):

\[
\varepsilon u_{\text{in}}(x,1) = 0 \quad \Rightarrow \psi(x=0, \gamma) = 0 \quad \text{(2)}
\]

because we won't have an \( e \)-dependence...
The condition comes from matching: (outer and inner)

\[ u_{\text{out}} = h(x,y) \quad u_{\text{in}}(x,y) = e^{-d} \Psi(x,y) \]

\[ \downarrow y \rightarrow -1 \quad \downarrow y \rightarrow +\infty \]

\[
\begin{align*}
\frac{2x}{l} & = e^{-d} \\
\Psi(x,y,\pm \infty) & \approx e^{-d}
\end{align*}
\]

\[ \Rightarrow \quad \Psi(x,y,\pm \infty) \approx \frac{2x}{l} \]

Leading order condition

asymptotically equal

Safe procedure: for singular perturbation theory treat inner and outer solutions separately.

Solitons:

Introduction: Historical origin: John Scott Russell (1834)
observed solitary wave: propagates without changing shape
in Glasgow River Canal

\[
\begin{align*}
v^2 & = g(h + a) \\
\text{amplitude depends on speed} & \quad \text{does not change with time (long time)}
\end{align*}
\]

John Scott Russell was naturalist = did not know any math.

Korteweg and de Vries (1895): wrote down a PDE for \( \eta(x,t) \)

KdV eqn. \( \eta_t + c \left( 1 + \frac{a}{2h} \right) \eta_x + \left( \frac{1}{6} \chi h^2 \right) \eta_{xxx} = 0 \)

\( h, c = \text{const} \)

L = char. speed
Non-dimensional PDE: \[ u_t + 6uu_x + u_{xxx} = 0 \]

Standard KdV

Assume travelling wave:
\[ u(x,t) = f(x-\sqrt{V}t) \implies \text{find ODE for } f(\tau) \]

KdV eqn.: \[ -V f' + 6ff' + f''' = 0 \]

\[ (3f^2)' \]

\[ \implies -Vf + 3f^2 + f'' = \text{const} = A \quad | f' \]

\[ -\frac{1}{2} V f^2 + f^3 + \frac{1}{2} f'^2 = Af + B \]

Take \( f, f' \to 0 \) when \( |\tau| \to \infty \)

\[ \implies B = 0 \]

\[ \text{divide by } f, \quad -\frac{1}{2} Vf + f^2 + \frac{1}{2} \frac{f'^2}{f} = A \]

extra assumption \( \frac{f'^2}{f} \to 0 \implies A = 0 \)

First order ODE for \( f \):
\[ f' = \pm f (-2f + V)^{1/2} \]

\[ \implies \int \frac{df}{f(-2f+V)^{1/2}} = \pm \int d\tau + K \]

\[ \psi \text{ set } f = \frac{V}{2} \cosh^2 \theta \quad \to \text{new variable} \]

\[ \implies -\frac{2}{V} \int d\theta = \pm \tau + K \]

solitary wave

\[ f(\tau) = \frac{V}{2} \cosh^2 \left[ \frac{\sqrt{V}}{2} (x-\sqrt{V}t-x_0) \right] \]
Other example: Sine-Gordon PDE

\[ u_{xx} - u_{tt} = \sin u \]

- currents in Josephson-junction transmission lines (\( u \): voltage)
- dislocation of crystals (\( \sin u \): from periodic structure)
- waves in ferromagnets
- fields of laser pulses in non-linear media

Traveling wave: \( u(x,t) = \frac{\phi(x-Vt)}{F} \)

\[ F \frac{d}{dt}(\phi')^2 + \frac{\cos \phi}{1-V^2} = B = \text{const} \]

\[ \frac{d}{d\theta} \frac{2}{\sqrt{A-\cos \Psi}} = \pm \frac{2}{\sqrt{1-V^2}} (\Phi - \Phi_0) \quad \text{where} \quad \phi = \phi_0 \quad \text{at} \quad \Phi = \Phi_0, \]

A arbitrary constant

The value of \( A \) depends on \( V \)

If \( A=1 \) \( \rightarrow \) wave exists if \( 0 < V < 1 \)

integral can be find exactly

solution with \( t \): \( u = \phi(\tau) = 4 \arctan \left( x - Vt \right) \)

\[ \phi_0 = \pi \quad \text{at} \quad \tau_0 = 0 \]

---

"kink" solution

"anti-kink" solution

solution with \(-\): \( u(x,t) = 4 \arctan \left( x + Vt \right) \)

\( \phi_0 = \pi \quad \text{at} \quad \tau = 0 \)

"Inverse Scattering transform": Method to obtain "many solutions" solutions from certain PDE's