March 9, 2014
Lecture 14

Grat. 1 Take-home Mon. March 15 4hrs
To be picked 4-5 pm

Review session: Fri 4:15-5:45 pm

Nonlinear PDE => Linear PDE

\[ u_t - a \Delta u + b|u|^2 = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n \]

\[ u(t, t=0) = g(x) \quad \text{on} \Omega \times \{ t=0 \} \]

Steps.

1. Take \( w = \phi(u) \); find \( \phi \) s.t. PDE for \( w \) is linear

   \[ a \phi''(u) + b \phi'(u) = 0 \quad \text{ODE} \]

   \[ \phi(u) = C_1 e^{-\frac{bu}{a}} + C_2 = w \]

   \( C_2 \) is arbitrary

2. Choose \( C_1 \) and \( C_2 \) to accommodate boundary conditions

Example: Suppose \( u = 0 \), \( |\xi| = \infty \) and want \( w = 0 \), \( |\xi| = \infty \)

Then you choose \( C_1 = C_2 \)

3. Solve PDE for \( w \):

\[ \begin{cases}
   w_t = a \phi' \phi \\
   w(x, \theta) = \phi(u(x, \theta)) = \phi(g(x)) \\
   w = 0, \quad |\xi| = \infty
\end{cases} \]

Let \( \eta = 1 - C_2 \), \( \phi(u) = 1 - e^{-\frac{bu}{a}} = w \)

In this way, we obtain homogeneous conditions. We can apply directly our IT method.

\[ w(\xi, t) = \int \hat{w}(\xi) \frac{e^{-\frac{b|\xi|^2}{4at}}}{(4\pi at)^{\frac{n}{4}}} d\xi \]

\[ w(\xi, t) = -\frac{a}{b} \ln(1 - w) \]

(for fixed problem in \( w \), no matter how you play with the transformation and/or \( \text{const} \, \gamma = C_1, C_2 \), you must have the same result)
Laplace equation \( \nabla^2 u = 0 \) \{ electro-magneto-statics, steady state diffusion, rotational fluid flow \}

Diffusion equation by setting \( \frac{\partial u}{\partial t} = 0 \)
we expect some similarities.
Laplace = steady state of diffusion

"Natural" BVP: which means well-posed problems
\[
a u + b \frac{\partial u}{\partial n} = c \quad a, b, c \text{ given} \\
a \neq 0 \quad (\text{for uniqueness})
\]

Example:

\[
\begin{aligned}
\begin{cases}
\nabla^2 u = 0, \quad 0 \leq r < 1, \quad 0 \leq \theta < \pi \\
u(1, \theta) = f(\theta)
\end{cases}
\end{aligned}
\]
(this is a well-posed problem)

Assumptions:
- \( u \) single valued
- \( u(0, \theta) \) finite

Step 1: Identify geometry, independent variables \((r, \theta)\)

Laplace in polar coordinates: \( \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \)

Step 2: Apply PDE: \[
\left( r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} \right) u = 0
\]

\( z_1(r), z_2(r) \)
depends only on \( r \)
depends only on \( \theta \)

⇒ Apply separation of variables: Try \( u(r, \theta) = R(r) \Theta(\theta) \)

L(multiplicative / additive)
$$\Rightarrow \frac{1}{r^2} \left[ r^2 R''(r) + r R'(r) \right] + \overline{\Theta}(\theta) = 0$$

\[ \text{depends only on } r \quad \text{depends only on } \theta \]

only way to have the sum equals 0 is each one to be const.

\[ \Rightarrow \begin{cases} \Theta''(r) + \frac{\Theta'(r)}{r} = \text{const} & \text{I} \\
\frac{r^2 R'' + r R' - d R}{r} = 0 & \text{II} \\
\text{two ODEs} \quad \text{PDE} \Rightarrow \text{ODE} \end{cases} \]

\[ \Theta''(r) + \frac{\Theta'(r)}{r} = 0 \quad 0 < \theta < 2\pi \]

+ BC: usual assumption is that the solution $u$ is single-valued $\Rightarrow$ therefore the BC are periodic

\[ \Theta(0) = \Theta(2\pi) \quad \text{means periodicity} \]

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**physical**

the solution is $\Theta(r) = A \pm e^{\pm i \theta}$ periodicity

\[ d = m^2, \ m \text{ integer} \]

\[ r^2 R'' + r R' - m^2 R = 0, \quad 0 < r < 1 \]

equidimensional

Try $R = r^p$ find $p$ from ODE $p(p-1) + p - m^2 = 0$

\[ \begin{cases} p = m, \ m \neq 0 & \Rightarrow R = r^m \\
p = 0, \ u = 0 \quad \text{(double)} & \Rightarrow R = 1, \ \ln r \end{cases} \]

suppose $u > 0$, some of the solutions blow up at zero this is not a problem for the $\Theta u = 0$

$r^2$ can be defined even if a function blows up at 0

we have a new "physical" assumption $u(0, \theta)$ finite

\[ R(0) \text{ finite } \Rightarrow R(r) = r^m, \ m \]

Steps Apply BC:

Condition at $r=1$: $u(1, \theta) = f(\theta)$

$u(r, \theta) = r^m e^{\pm i m \theta}$
How to satisfy the BC? Take a linear superposition of all possible $u_i$'s:

$$u(\mathbf{r}, \theta) = \sum_{m=0}^{\infty} A_m r^m e^{im\theta} = A_0 + \sum_{m=1}^{\infty} \left[ A_m \cos(m\theta) + B_m \sin(m\theta) \right] r^m$$

at $r=1$

$$u(1, \theta) = A_0 + \sum_{m=1}^{\infty} \left[ A_m \cos(m\theta) + B_m \sin(m\theta) \right] = f(\theta) \Rightarrow \begin{cases} A_0 = \text{can be found} \\ B_m \end{cases}$$

Example of Nonlinear PDE => Laplace eqn.

**Potential functions:** help converts systems of PDE's to a Linear PDE

Example: Inviscid, incompressible fluid

$$\ddot{\mathbf{u}} = (\ddot{u}_1, \ddot{u}_2, \ddot{u}_3), \quad \ddot{\mathbf{r}} = (\dddot{x}_1, \dddot{x}_2, \dddot{x}_3)$$

PDE's

$$\begin{cases}
\nabla \cdot \nabla \ddot{u} = -\nabla p + \mathbf{F} \to \text{external force (given)} \\
\n\nabla \cdot \ddot{u} = 0 \quad \text{pressure} \\
\n\nabla \times \ddot{u} = 0
\end{cases}$$

$$\ddot{\mathbf{u}} = \begin{pmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3 \\
\end{pmatrix}, \quad \ddot{\mathbf{u}} \cdot \ddot{\mathbf{u}} = (\dddot{u}_1^2 + \dddot{u}_2^2 + \dddot{u}_3^2)$$

Recall: In electricity, the electric field $\mathbf{E}$ satisfies

$$\begin{cases}
\nabla \cdot \mathbf{E} = 0 \quad \text{in source free region} \\
\n\nabla \times \mathbf{E} = 0
\end{cases}$$

$\mathbf{E}$ satisfies 2 equations but it is not necessarily zero.

we solve this by introducing a potential function

$$\dddot{u} = -\nabla \phi$$

from $\nabla \cdot \dddot{u} = 0 \Rightarrow \nabla^2 \phi = 0$ (additive const often present)

$$\begin{array}{c}
\nabla \phi \nabla \phi = -\nabla p + \mathbf{F} \\
\text{(suppose $\mathbf{F}$ is conservative)} \\
\downarrow \nabla \cdot \mathbf{F} = -\nabla^2 \phi \\
\Rightarrow \nabla p = \nabla \phi_t - \nabla \phi \nabla \phi - \nabla f = \nabla \left( \phi_t + \frac{1}{2} (\nabla \phi)^2 - f \right) = p = \phi_t - \frac{1}{2} (\nabla \phi)^2 - f + k(\psi)
\end{array}$$