Pick up: Solution to Hwk 4. (If you have not)
Start on Hwk 5!

Rev. Session: FRI, 4:15 pm

Review Linear Translation Invariant PDE
(uniform PDE)

\[ u = e^{ikx - iw^t} \Rightarrow w = W(k) \text{ dispersion relation} \]

\[ \mathcal{F} \text{ in } x \xrightarrow{x \to \infty} \mathcal{F} \text{ t to } 0(1) \]

\[ \mathcal{F} \text{ t to } -\infty, \text{ group velocity at } k=0 \]

\[ R \text{ translation } k=0 \text{ at } r \]

\[ \text{formula breaks down when } k = k^* \text{, } k^* : W''(k^*) = 0 \]

Inside transition region

\[ I(x,t) \sim \frac{F(k^*)}{\sqrt{2\pi |t|W''(k^*)}} \]

\[ x = \frac{W'(k^*)t}{t} \]

\[ \gamma = \frac{1}{3} W''(k^*) \]

\[ A_{i r} \text{ function} \]

\[ A_{i r} = \frac{1}{i!} \sqrt{\frac{3}{2}} K_{1/3} \left( \frac{2}{3} z^{3/2} \right) \]

Hankel Modified Bessel function
Last time we proved:
\[
\begin{align*}
\bar{R} = \Theta_k \in \mathbb{R} & : \text{local wave number} \\
W(\bar{R}) = \Theta \in \mathbb{R} & : \text{local frequency}
\end{align*}
\]

- proved last time.

Also:
\[
\left| \frac{(R)_{x}}{R} \right| : \text{"small" means } \ll \left( \text{"characteristic length of the system"} \right)^{-1} = \ell_c
\]
and any function of this is "small".

\[
\left| \frac{(R)_{t}}{R} \right| : \text{"small" means } \ll \left( \text{"characteristic time of the system"} \right)^{-1} = \tau_c
\]
and any function of this is "small".

This means that the wave of the form:
\[
I(x,t) \sim A(x,t) e^{i \Theta(x,t)}
\]
has:
\[
\left| \frac{Ax}{A} \right| : \text{"small"}, \quad \left| \frac{At}{A} \right| : \text{"small" etc.}
\]

because $\bar{R}$ has this property and enters in the definition of $A, \Theta$.

$A, \Theta$ are defined as a function of $x,t$ only through $\bar{R}(x,t)$ which is slowly varying.

Such waves are called SLOWLY VARYING NON UNIFORM WAVES.

Generalize to Non-uniform PDE's:

Assume solution $u(x,t) \sim e^{i \Theta(x,t)} \sum_{n=0}^{N} A_n(x,t)$

(we have to know that the PDE has a wave-like solution)

Assumptions:

(i) $A_0, A_k, \Theta_k$ and maybe other coefficients of the PDE are $O(1) : \text{"NOT small"}$

(ii) Increasing the order of derivative in $(x,t)$ by 1, also...
increases the order of approximation by 1

e.g. 
\[ A_0 x = O(\varepsilon), \quad \Theta x, x = O(\varepsilon) \]
\[ A_0 t = O(\varepsilon) \quad \Theta t, t = O(\varepsilon) \]
(anything you differentiate in \(x, t\) once gives you an extra \(\varepsilon\))

\[ \frac{|A_n|}{A_0} = O(\varepsilon^n) \]

(if you increase the order of approximation by 1, you add an extra \(\varepsilon\))

Physical foundation for these assumptions: we are trying to find a slowly varying solution.
(we need this \(\varepsilon\) to describe systematically a silver sol.)

Example: Non-uniform Non-linear Klein-Gordon equation

\[ u_{tt} - \left[ \alpha(x,t)^2 \, u_x \right]_x + \beta(x,t)^2 \, u = 0 \quad \{ -\infty < x < \infty \}
\quad \{ t > 0 \}
\]

Assume \( u \sim e^{i\Theta(x,t)} \sum_{n=0}^N A_n(x,t) \) : "slowly varying"

\[ A_0 x = O(\varepsilon) \]

\[ A_0 t = O(\varepsilon) \]

\[ A_0 x = O(\varepsilon^2) \]

\[ A_1 x = O(\varepsilon^2) \]

We cannot use Fourier transform because \(\alpha(x,t), \beta(x,t)\) depend on \(x\)!

Assumptions:

1. \( A_0, \Theta x, \Theta t, \alpha, \beta : O(1) \)
2. \( A_0 x, A_0 t, \ldots : O(\varepsilon) \)
   \( \alpha x, \beta x, \ldots : O(\varepsilon) \) etc.
3. \( \frac{|A_n|}{A_0} = O(\varepsilon^n) \)

Start \( u(x,t) \sim e^{i\Theta(x,t)} (A_0 + A_1) \)

\[ u_x \sim e^{i\Theta} \left[ A_0 \alpha x + A_0 \beta x + i \Theta x (A_0 + A_1) \right] \varepsilon \]

\[ u_{xx} \sim e^{i\Theta} \left[ A_0 \alpha x x + A_0 \beta x x + 2i \Theta x (A_0 + A_1) \right] \varepsilon \]

\[ u_{xxx} \sim e^{i\Theta} \left[ A_0 \alpha x x x + A_0 \beta x x + i \Theta x x (A_0 + A_1) \right] \varepsilon \]
\[ u_+ \sim \left[ A_0, h + A_1, h + i \Theta_+ (A_0 + A_1) + 2i \Theta_+ (A_0, h + A_1, h) - \Theta_+^2 (A_0 + A_1) \right] e^{i \theta} \]

\[ (\partial^2 u_+) \sim 2 \left[ 2 \partial_x \partial_y A_0, x + \partial_x \partial_y A_1, x + i \partial_x \Theta (A_0 + A_1) \right] \]

\[ + d^2 \left[ A_0, xx + A_1, xx + i \Theta_{xx} (A_0 + A_1) + 2i \Theta_x (A_0, x + A_1, x) - \Theta_x (A_0 + A_1) \right] e^{i \theta} \]

**Substitution in the PDE:**

\[ [A_0, h + A_1, h + i \Theta_+ (A_0 + A_1) + 2i \Theta_+ (A_0, h + A_1, h) - \Theta_+^2 (A_0 + A_1)] \]

\[ - 2 \partial_x \partial_y [A_0, x + A_1, x + i \Theta_x (A_0 + A_1)] \]

\[ - \partial^2 [A_0, xx + A_1, xx + i \Theta_{xx} (A_0 + A_1) + 2i \Theta_x (A_0, x + A_1, x) - \Theta_x (A_0 + A_1)] \]

\[ + \beta^2 (A_0 + A_1) = 0 \]

**Collect the terms with equal powers of \( \varepsilon \):**

**Terms of order \( \varepsilon^0 \):**

\[ - \Theta_+^2 A_0 + \Theta_+^2 d^2 A_0 + \beta^2 A_0 = 0 \]

\[ [ - \Theta_+^2 + d^2 \Theta_+^2 + \beta^2 ] A_0 = 0 \]

\[ \Theta_+^2 \text{ "Dispersion relation" but local } \quad (\omega = -\Theta_+, \kappa = \Theta_x) \]

1st order PDE

(any 1st order PDE is in principle solvable with the method of CHAR PDE \( \Rightarrow \) ODE)

**Terms of order \( \varepsilon^1 \):**

\[ (\partial^2 \Theta_x + \beta^2 - \Theta_+^2) A_1 + i (\Theta_+ A_0 + 2 \Theta_+ A_0, h - 2 \partial_x \Theta_+ A_0 - d^2 \Theta_{xx} A_0 - 2 d^2 \Theta_x A_0, x) = 0 \]

\[ (-2 \Theta_+ A_0 + 2 d^2 \Theta_x A_0, x) = (\Theta_+ - 2 \partial_x \Theta_x - d^2 \Theta_{xx}) A_0 \]

1st order PDE = CHAR = solvable = ODE's

(coupled system of ODE's)
Try to solve (Eq. 3) by CHAR:

\[ \begin{align*}
\frac{d\theta}{dx} + \frac{d\theta}{dA} &= \frac{\partial A}{\partial x} \\
\partial \theta &= 2 \frac{d\theta}{dx} \quad \text{(Eq. 3)} \\
\text{slope of CHAR:} \\
\frac{dx}{dt} &= \frac{d^2 \theta}{dx^2} \\
\end{align*} \]

From Eq. (3) \[ \frac{d^2 \theta}{dx^2} + \frac{\partial^2}{\partial t^2} = \theta_t \]

By def \( \omega = -\theta_t \), \( \kappa = \theta_x \)

\[ \frac{d^2 \kappa}{dx^2} + \frac{\partial^2}{\partial t^2} = \omega^2 \]

Preserve the order of the dispersion relation:

\[ (\kappa - \kappa + \partial \kappa, \omega - \omega + \partial \omega) \quad \text{\( \partial, \partial \), not varying, local} \]

\[ \frac{\partial^2 \partial \omega}{\partial t \partial x} = \frac{\partial \omega}{\partial x} \Rightarrow \frac{\partial \omega}{\partial t} = \frac{d^2 \kappa}{dx^2} \]

Thus, slope of CHAR is:

\[ \frac{dx}{dt} = \frac{d^2 \theta}{dx^2} = \frac{d^2 \kappa}{dx^2} = \omega \frac{d\omega}{dk} \rightarrow \text{group velocity} \]

"Values of \( A \) propagate on curves of slope the group velocity."

We want to find connections between hyperbolic equations and dispersive systems.

ii) Recall:

A conservation law is a PDE of the form:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = 0 \]

\[ \text{Density} \]

\[ \text{Flux} \]

**Dispersive systems:**

**Local wave number and local frequency satisfy**

\[ \mathbf{local} : \begin{cases} \omega = -\theta_t \Rightarrow w_x = -\theta_t \\ \kappa = \theta_x \Rightarrow k_t = \theta_t x \end{cases} \]

\[ \frac{\partial \kappa}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \]

Wave number: density of wave crests

Frequency: flux of wave crests

Conservation law
what is conserved in a dispersive system, is the number of wave crests.

Recall: Constitutive relation \( q = Q(S) \) \( \Rightarrow \) PDE for \( q \)

**Dispersive systems:**

the constitutive relation is the dispersive relation \( w = W(K) \)

\[
\frac{\partial K}{\partial t} + \frac{\partial w}{\partial x} = 0 \Rightarrow \frac{\partial K}{\partial t} + W'(K)\frac{\partial K}{\partial x} = 0
\]

The \( C_g \) is the speed of propagation of constant values of \( K \).

\( \gamma \) is the slope of the CHAR.

The speed of propagation of const \( K = C_0 \) is equal to \( W'(K_0) \)

**Shocks:** In traffic flow (based on density) shocks are signs of a pathology. Here, multiple valued solutions are allowed (not excluded a priori).