XI. SPEECH COMMUNICATION*

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A. SOME NETWORK PROPERTIES OF DIGITAL FILTERS

Several network properties for digital filters are presented in this report. Two sets of issues are discussed, those pertaining to a general matrix representation of arbitrary digital filter structures and those pertaining to the coefficient sensitivity properties of arbitrary digital filter structures.

1. Signal-Flow Graph Representation of Digital Networks

Digital networks can be placed in the category of directed graph theory or signal-flow graphs.[1,2] In this context, the regular branches of the signal-flow graph can be restricted to be either simple gains or gains with unit delays. Input signals to the network can be introduced via source branches.

For each node in the graph an equation can be written relating its node signal value to other node signal values in the network and the source branch signal entering it. For a graph with N nodes there are N of these equations and they provide a total description of the digital network. They can be written compactly in matrix form as

$$\underline{Y}(z) = z^{-1}\underline{H}_d^t\underline{Y}(z) + \underline{H}_c^t\underline{Y}(z) + \underline{X}_s(z), \tag{1}$$

where

$\underline{Y}(z)$ = the column vector of node signal values

$\underline{X}_s(z)$ = the column vector of source branch values

$\underline{H}_c$ = the matrix of coefficients for branches with simple coefficients

$\underline{H}_d$ = the matrix of coefficients for branches with coefficients and delays.

In this representation the $(m,n)^{th}$ element of $\underline{H}_c$ or $\underline{H}_d$ represents the coefficient of a

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branch directed from node m to node n.

Because $\underline{H}_c$ and $\underline{H}_d$ are matrices of constant coefficients, the inverse z transform of (1) can be taken to obtain a matrix equation corresponding to N first-order difference equations,

$$\underline{y}(k) = \underline{H}_d^t \underline{y}(k-1) + \underline{H}_c^t \underline{y}(k) + \underline{x}_s(k). \tag{2}$$

It is this set of difference equations that are actually iterated by computer or with special-purpose hardware in the operation of the digital filter.

The transfer functions of the digital network can be determined by performing a suitable matrix inverse of (1).

$$\underline{Y}(z) = \underline{T}(z)^t\, \underline{X}_s(z), \tag{3}$$

where

$$\underline{T}(z)^t = (\underline{I}-\underline{H}(z)^t)^{-1} \tag{4}$$

$$\underline{H}(z)^t = \underline{H}_c^t + \underline{H}_d^t z^{-1}. \tag{5}$$

The matrix $\underline{T}(z)$ is referred to as the transfer function matrix and $T_{mn}(z)$, the $(m,n)^{th}$ element of $\underline{T}(z)$, is the transfer function from node m to node n. For the case of a network or a graph in which a specific source $X_{si}(z)$ is considered as the input and a specific node value $Y_j(z)$ is considered as the output, the particular transfer function $T_{ij}(z)$ is referred to as the system function of the graph or network.

2. Properties of the Matrix Representation

a. Order of Computation and Node Renumbering

Thus far, no stipulation has been made about the order in which the equations in (2) are iterated. We shall assume that this order is computed from top to bottom. That is, the node values $y_i(k)$ will be computed from $i = 1$ to $i = N$ in consecutive order in the actual implementation of the network. The input can be any one of the $x_{si}(k)$ sources and the output can be any one of the node values $y_i(k)$.

The order in which the nodes are computed can be changed by renumbering the nodes or shuffling the equations. To interchange the order of computation of the i^{th} and j^{th} nodes, it is necessary to interchange the i^{th} and j^{th} elements in each of the vectors $y(k)$, $y(k-1)$ and $x_s(k)$. It is also necessary to interchange the i^{th} and j^{th} rows and the i^{th} and j^{th} columns in the matrices $\underline{H}_c$ and $\underline{H}_d$.

b. Implementation Realizability

In order to have a digital network which is physically realizable, it is necessary to impose the condition that the filter structure contain no inner loops without delay. This condition can be imposed on the set of equations in (2) by requiring that the matrix $\underline{H}_c^t$ be lower triangular and zero along the main diagonal, as shown in (6).

$$\underline{H}_c^t = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ h_{12} & 0 & 0 & & 0 \\ h_{13} & h_{23} & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ h_{1N} & \cdot & \cdot & \cdots & 0 \end{bmatrix} \tag{6}$$

If (2) is not of the form such that $\underline{H}_c^t$ satisfies (6), and the nodes cannot be renumbered in an order to achieve this form, then the network has at least one inner loop without delay, and is not realizable. A form of (2) for a given network in which $\underline{H}_c^t$ satisfies (6) will be referred to as a realizable form.

The reason for this requirement is straightforward. If the equations in (2) are iterated from top to bottom, no value of $y_i(k)$ can be computed from values of $y_j(k)$ for $j \geq i$ because these node values are not yet available for that value of k.

c. Requirement for Nonrecursive Graphs

For a nonrecursive graph, all of the signal paths must be feed-forward paths; that is, no feedback paths are allowed in the structure. This suggests that for a nonrecursive structure it is possible to number the nodes in such a way that in (2) both $\underline{H}_d^t$ and $\underline{H}_c^t$ will be lower triangular and zero along the main diagonal. If this condition cannot be met, then there is at least one feedback path in the structure. The form of (2) for a given nonrecursive network in which both $\underline{H}_c^t$ and $\underline{H}_d^t$ are lower triangular and zero on the main diagonal will be referred to as a nonrecursive form. Obviously if (2) is in nonrecursive form, then it is also in realizable form.

Another property of nonrecursive graphs is that with (2) in nonrecursive form the expression $\underline{I} - \underline{H}(z)^t$ is lower triangular. From matrix theory it can be shown that the inverse of a lower triangular matrix is also lower triangular. Therefore, with the aid of (4) it can be seen that the transpose of the transfer function matrix, $\underline{T}(z)^t$, will be lower triangular for (2) in nonrecursive form.

If a simple nonrecursive graph has N nodes, then the maximum duration impulse that can be obtained from this graph is N. This can be observed from the fact that the maximum number of feed-forward delay branches that can exist in cascade is N-1. (The first node for a nonrecursive graph cannot have a regular branch entering it, only a source branch.) Thus the system function can have a maximum order of N-1 and therefore a maximum duration impulse response of N.

d. Requirement for "In-Place" Computation

In general, it is necessary to have a maximum of 2N storage registers to store the internal variables $\underline{y}(k)$ and $\underline{y}(k-1)$ in the filter. If it is possible to do an "in-place" computation, then it is only necessary to have a maximum of N storage registers. By "in-place" computation we mean that once a node value $y_i(k)$ is computed, the value $y_i(k-1)$ is no longer needed, and the value $y_i(k)$ can be stored in its place. In order to have a structure with "in-place" computation, (2) must be of realizable form and $\underline{H}_d^t$ must be an upper triangular matrix.

It should not be implied from this that a structure with N nodes that can be realized with in-place computation will always require fewer storage registers than some other structure with N nodes which cannot be realized with in-place computation. The reason for this is that for a particular structure not all of the node signal values $\underline{y}(k-1)$ may be needed to compute the values of $\underline{y}(k)$. Those values of $\underline{y}(k-1)$ that are not needed do not have to be stored from one iteration to the next. They are intermediate variables and therefore only need to be stored temporarily within a portion of one iteration or cycle of the filter. Typically a storage register may be used for temporary storage of several intermediate variables in nonoverlapping portions of an iteration.

3. Network Sensitivity Properties

a. First-Order Sensitivity

A convenient expression has been derived by Fettweis[3] which relates the first-order sensitivity of a transfer function $T_{k\ell}(z)$ from some node k to some node ℓ in a network to the branch transmittance function $H_{pq}(z)$ of a branch from some node p to some node q in that network. This expression is

$$\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)} = T_{kp}(z)\, T_{q\ell}(z). \tag{7}$$

The desirable feature of (7) is that it relates the sensitivity of the system function with respect to the branch transmittance functions in terms of other transfer functions in the structure.

For the case of digital networks, the branches are restricted to be either coefficients or coefficients and delays. Relation (7) can be used to express the sensitivity of the system function to these coefficients. For a branch from node p to node q composed of a single coefficient c, (7) can be applied directly to obtain the coefficient sensitivity.

$$\frac{\partial T_{k\ell}(z)}{\partial c} = T_{kp}(z)\, T_{q\ell}(z). \tag{8}$$

For a branch from node p to node q, composed of a coefficient c_d and a delay, the chain rule must be used in conjunction with (7) to determine the coefficient sensitivity.

$$H_{pq}(z) = c_d z^{-1}$$

$$\frac{\partial T_{k\ell}(z)}{\partial c_d} = \frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)} \cdot \frac{\partial H_{pq}(z)}{\partial c_d} = T_{kp}(z)\, T_{q\ell}(z)\, z^{-1}. \tag{9}$$

b. Second-Order Sensitivity

The second-order sensitivity can be obtained by taking the partial derivative of the first-order sensitivity,

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = \frac{\partial}{\partial H_{pq}(z)} \left[\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)}\right]$$

$$= \frac{\partial}{\partial H_{pq}(z)} [T_{kp}(z)\, T_{q\ell}(z)]. \tag{10}$$

Applying the chain rule, we obtain

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = T_{kp}(z) \frac{\partial T_{q\ell}(z)}{\partial H_{pq}(z)} + T_{q\ell}(z) \frac{\partial T_{kp}(z)}{\partial H_{pq}(z)}.$$

Applying relation (7) to each of the partial derivatives on the left side of the equation yields

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = T_{kp}(z)\, T_{qp}(z)\, T_{q\ell}(z) + T_{q\ell}(z)\, T_{kp}(z)\, T_{qp}(z).$$

Then, by combining terms, the second-order sensitivities can be expressed in the form

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = 2T_{qp}(z)\, T_{kp}(z)\, T_{q\ell}(z). \tag{11}$$

An alternative form for this expression can be determined with the aid of (7).

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = 2T_{qp}(z)\,\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)}. \tag{12}$$

c. Third-Order Sensitivity

The third-order sensitivity can be computed in a manner similar to the procedure for determining the second-order sensitivity.

$$\begin{aligned}\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} &= \frac{\partial}{\partial H_{pq}(z)}\left[\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2}\right] \\ &= 2\,\frac{\partial}{\partial H_{pq}(z)}\left[T_{qp}(z)\,\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)}\right] \\ &= 2\left[T_{qp}(z)\,\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} + \frac{\partial T_{qp}(z)}{\partial H_{pq}(z)}\cdot\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)}\right].\end{aligned}$$

This expression can be reduced to either of two forms. If the partial derivatives are evaluated in terms of the transfer functions with the aid of (7) and (11), then the second-order sensitivity can be expressed in the form

$$\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} = 2\left[2T_{qp}(z)^2\,T_{kp}(z)\,T_{q\ell}(z) + T_{qp}(z)^2\,T_{kp}(z)\,T_{q\ell}(z)\right],$$

or

$$\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} = 6T_{qp}(z)^2\,T_{kp}(z)\,T_{q\ell}(z). \tag{13}$$

Alternatively, this relation can be expressed in terms of the second-order sensitivity.

$$\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} = 2T_{qp}(z)\left[\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2}\right] + T_{qp}(z)\left[2T_{qp}(z)\,\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)}\right].$$

With the aid of relation (12) and by combining terms, we get the form

$$\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} = 3T_{qp}(z)\,\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2}. \tag{14}$$

d. N^{th}-order Sensitivity

If the expressions for the sensitivities are tabulated, a pattern can be observed and, by induction, an expression for the n^{th}-order sensitivity can be determined.

$$\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)} = T_{kp}(z)\,T_{q\ell}(z) \tag{15}$$

$$\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} = 2T_{qp}(z)\,T_{kp}(z)\,T_{q\ell}(z) = 2T_{qp}(z)\,\frac{\partial T_{k\ell}(z)}{\partial H_{pq}(z)} \tag{16}$$

$$\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} = 6T_{qp}(z)^2\,T_{kp}(z)\,T_{q\ell}(z) = 3T_{qp}(z)\,\frac{\partial^2 T_{k\ell}(z)}{\partial H_{pq}(z)^2} \tag{17}$$

$$\frac{\partial^4 T_{k\ell}(z)}{\partial H_{pq}(z)^4} = 24\,T_{qp}(z)^3\,T_{kp}(z)\,T_{q\ell}(z) = 4T_{qp}(z)\,\frac{\partial^3 T_{k\ell}(z)}{\partial H_{pq}(z)^3} \tag{18}$$

$$\frac{\partial^5 T_{k\ell}(z)}{\partial H_{pq}(z)^5} = 120\,T_{qp}(z)^4\,T_{kp}(z)\,T_{q\ell}(z) = 5T_{qp}(z)\,\frac{\partial^4 T_{k\ell}(z)}{\partial H_{pq}(z)^4} \tag{19}$$

$$\vdots$$

It can be observed that the higher order derivatives are related by the recursive relation

$$\frac{\partial^n T_{k\ell}(z)}{\partial H_{pq}(z)^n} = n\,T_{qp}(z)\,\frac{\partial^{n-1} T_{k\ell}(z)}{\partial H_{pq}(z)^{n-1}} \quad \text{for } n > 1. \tag{20}$$

This relation can be iterated to obtain a second form that expresses the higher order derivatives in terms of three transfer functions in the network.

$$\frac{\partial^n T_{k\ell}(z)}{\partial H_{pq}(z)^n} = n!\,T_{qp}(z)^{n-1}\,T_{kp}(z)\,T_{q\ell}(z) \quad \text{for } n \geqslant 1. \tag{21}$$

4. Large-Change Network Sensitivities

A general expression for all of the higher order derivatives of a system function with respect to a particular branch transmittance function has been derived. This expression, together with a Taylor's series expansion, can be used to derive a general relation that expresses a system function in terms of the branch transmittance function in question and 3 transfer functions in the structure. This relation constitutes both a proof and a generalization of the concepts associated with return difference in feedback control theory.[4]

To derive this relation, we begin by writing the Taylor's series expansion of the system function $T_{k\ell}(z)$ about some nominal value of the branch transmittance function $H_{pq}(z)$ denoted by a prime, $H_{pq}(z)'$.

$$\left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)' + \Delta H_{pq}(z)} = \left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)'} + \sum_{n=1}^{\infty} \frac{1}{n!} \left. \left\{ \frac{\partial^n T_{k\ell}(z)}{\partial H_{pq}(z)^n} \right\} \right|_{H_{pq}(z) = H_{pq}(z)'} [\Delta H_{pq}(z)]^n, \tag{22}$$

where $\Delta H_{pq}(z)$ represents the deviation of the branch transmittance function $H_{pq}(z)$ from its nominal value $H_{pq}(z)'$.

The general expression for the higher order derivatives (21) can now be applied to (22).

$$\left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)' + \Delta H_{pq}(z)} = \left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)'} + \sum_{n=1}^{\infty} \frac{1}{n!} \left[n!\, T_{qp}(z)^{n-1}\, T_{kp}(z)\, T_{q\ell}(z) \right] [\Delta H_{pq}(z)]^n, \tag{23}$$

where the transfer functions $T_{qp}(z)$, $T_{kp}(z)$, and $T_{q\ell}(z)$ must be evaluated for the branch transmittance function at its nominal value $H_{pq}(z)'$. By canceling the n! and making the change of variables $m = n - 1$, (23) can be written as follows.

$$\left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)' + \Delta H_{pq}(z)} = \left. T_{k\ell}(z) \right|_{H_{pq}(z) = H_{pq}(z)'} + T_{kp}(z)\, T_{q\ell}(z)\, \Delta H_{pq}(z) \sum_{m=0}^{\infty} [T_{qp}(z)\, \Delta H_{pq}(z)]^m. \tag{24}$$

If the summation in (24) is finite, then it can be written in closed form.

$$T_{k\ell}(z)\Big|_{H_{pq}(z)=H_{pq}(z)'+\Delta H_{pq}(z)} = T_{k\ell}(z)\Big|_{H_{pq}(z)=H_{pq}(z)'} + \frac{T_{kp}(z)\, T_{q\ell}(z)\, \Delta H_{pq}(z)}{1 - T_{qp}(z)\, \Delta H_{pq}(z)}. \tag{25}$$

This is the expression that is desired. The transfer functions $T_{kp}(z)$, $T_{q\ell}(z)$, and $T_{qp}(z)$ in (25) are evaluated for the branch transmittance $H_{pq}(z)$ at its nominal value $H_{pq}(z)'$. The relation is valid for both incremental changes and large changes of $\Delta H_{pq}(z)$.

For a branch between node p and node q composed of a simple coefficient, (25) can be applied directly to determine the large-change coefficient sensitivity. By letting the nominal value of the coefficient c be c' and the change in the coefficient be Δc, we obtain the expression

$$T_{k\ell}(z)\Big|_{c=c'+\Delta c} = T_{k\ell}(z)\Big|_{c=c'} + \frac{T_{kp}(z)\, T_{q\ell}(z)\, \Delta c}{1 - T_{qp}(z)\, \Delta c}. \tag{26}$$

If the branch between node p and node q is composed of a coefficient c_d and a delay, the coefficient sensitivity can be determined by treating the branch as a subnetwork of two branches in cascade. One branch is assigned to be a simple delay and the second branch is assigned to be a coefficient c_d. The system function of this subnetwork is simply the branch transmittance function $H_{pq}(z)$. The variation $\Delta H_{pq}(z)$ attributable to a variation Δc_d can then be determined by using (26).

$$\Delta H_{pq}(z) = H_{pq}(z)\Big|_{c_d=c_d'+\Delta c_d} - H_{pq}(z)\Big|_{c_d=c_d'} = z^{-1}\Delta c_d.$$

Then with the aid of (25) the coefficient sensitivity for a branch with a coefficient and delay can be expressed as

$$T_{k\ell}(z)\Big|_{c_d'+\Delta c_d} = T_{k\ell}(z)\Big|_{c_d'} + \frac{T_{kp}(z)\, T_{q\ell}(z)\, \Delta c_d}{z - T_{qp}(z)\, \Delta c_d}. \tag{27}$$

Again, $T_{kp}(z)$, $T_{q\ell}(z)$ and $T_{qp}(z)$ must be evaluated at $c_d = c_d'$.

5. Sensitivities for Nonrecursive Structures

For the special case of nonrecursive structures it has been shown that for networks with (2) in nonrecursive form the transpose of the transfer function matrix $\underline{T}(z)^t$ is lower triangular. Recall that the nonrecursive form of (2) implies that $\underline{H}(z)^t$ is lower triangular and zero on the main diagonal. From these observations it is apparent that for a nonrecursive network, if $H_{pq}(z)$ is nonzero, then $T_{qp}(z)$ must necessarily be zero. Therefore, for nonrecursive structures, (25) can be written in the form

$$T_{k\ell}(z)\Big|_{H_{pq}(z)=H_{pq}(z)'+\Delta H_{pq}(z)} = T_{k\ell}(z)\Big|_{H_{pq}(z)=H_{pq}(z)'} + \left(T_{kp}(z)\ T_{q\ell}(z)\right)\Big|_{H_{pq}(z)=H_{pq}(z)'} \Delta H_{pq}(z). \tag{28}$$

In other words, since $T_{qp}(z) = 0$, all of the higher order derivatives $(n > 1)$ must necessarily be zero. The coefficient sensitivities in a nonrecursive structure must therefore be linear functions expressible in the form of (28). Only the first-order sensitivities can be nonzero.

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References

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