PLASMA DYNAMICS

V. PLASMAS AND CONTROLLED NUCLEAR FUSION*

E. Feedback Stabilization

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1. FEEDBACK EFFECTS ON A TEARING MODE

Introduction

In a previous report,[1] we considered the effect of feedback magnetic fields on the kink modes of a current-carrying plasma column. Here we consider the effect of feedback magnetic fields on the similar tearing modes that occur when there is plasma (rather than vacuum) in the region outside the current channel. That is, we consider the following plasma configuration:

(i) hot plasma and uniform axial current density j_{zo} out to $r = r_o$.

(ii) less-hot plasma carrying no current from $r = r_o$ to $r = c$.

(iii) vacuum outside $r = c$.

(iv) feedback coils or wall at $r = d$.

We define the resonant radius r_s as the radius where $(m-nq)$ goes to zero, which is where the helical field lines coincide with the helical perturbation. Tearing modes occur when r_s lies within the boundary of the plasma, and the plasma has finite resistivity, η. For the tearing modes of this plasma, we are interested in plasma currents such that r_s falls between $r = r_o$ and $r = c$.

Resistive Plasma Profile and the Two-Region Substitute Plasma

We examine feedback effects on the tearing-mode stability of a plasma profile treated by Shafranov.[2] The profile is a uniform current channel of radius r_o surrounded by current-free plasma extending from $r = r_o$ to $r = c$. Shafranov considered that a wall was located at $r = c$; but we shall consider the free-boundary case, where there is a vacuum from $r = c$ to $r = d$ with feedback coils (or a wall) at $r = d$ $(d > c)$. Thus for the actual plasma we take

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$$j_{zo} = \begin{cases} \text{constant} & 0 < r < r_o \\ \text{zero} & r_o < r < d \end{cases}$$

$$\eta = \begin{cases} \text{finite} & 0 < r < c \\ \text{infinite} & c < r < d. \end{cases}$$

The resonant point, $r = r_s$, is where $m-nq = 0$. For this current distribution

$$q = \begin{cases} q_o & 0 < r < r_o \\ q_o \left[\frac{r}{r_o}\right]^2 & r_o < r \end{cases}$$

where $q_o = (rB_z/RB_\theta)|_{r=r_o} = r_oB_z/RB_{ro}$, $B_{ro} \equiv B_\theta|_{r=r_o}$. Consequently, $(m-nq) = m-nq_o(r/r_o)^2$ is zero when $r^2/r_o^2 = m/nq_o$. Thus $r_s = r_o(m/nq_o)^{1/2}$.

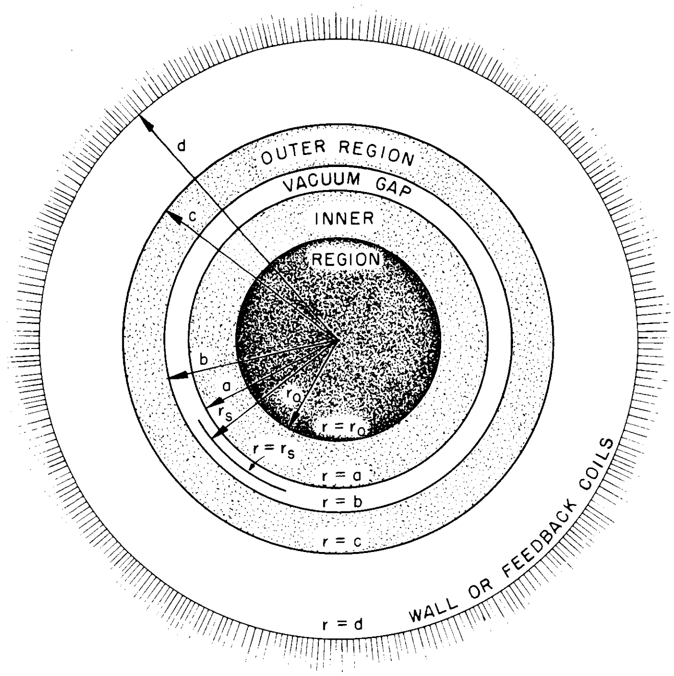

Fig. V-1. Substitute two-region plasma whose kink-mode stability bounds correspond to the tearing-mode stability bounds for the actual plasma.

To determine the stability of this plasma against tearing modes, we compute the kink-mode stability of a similar plasma with a vacuum region about $r = r_s$. We consider the two-region plasma shown in Fig. V-1, where the plasma extends from $r = 0$

to $r = a$ and from $r = b$ to $r = c$. The plasma is considered to be perfectly conducting, with kink-mode stability determined by computing the minimum value of δW, the change in potential energy arising from the presence of a kink mode. To determine tearing-mode stability, a and b are taken so that $a < r_s < b$. Then δW is found in the limit as $(b-a)$ goes to zero (still with $a < r_s < b$), with $\delta W > 0$ implying stability. For this model plasma, j_{zo} and η are

$$j_{zo} = \begin{cases} \text{constant} & 0 < r < r_o \\ \text{zero} & r_o < r < d \end{cases}$$

$$\eta = \begin{cases} \text{zero} & 0 < r < a \\ \text{infinite (vacuum)} & a < r < b \\ \text{zero} & b < r < c \\ \text{infinite (vacuum)} & c < r < d. \end{cases}$$

Value of δW for the Two-Region Plasma

We can compute δW for this two-region plasma with 3 free boundaries, where

$$\delta W = \delta W_f + \delta W_v.$$

Here δW_f is the change in potential energy within the plasma, and δW_v is the wave's potential energy within the vacuum. Now δW has four components, with $\delta W_f = \delta W_{f1} + \delta W_{f2}$ and $\delta W_v = \delta W_{v1} + \delta W_{v2}$, where

δW_{f1} is for the plasma region $r = 0$ to $r = a$

δW_{f2} is for the plasma region $r = b$ to $r = c$

δW_{v1} is for the vacuum region $r = a$ to $r = b$

δW_{v2} is for the vacuum region outside $r = c$.

Using the minimizing techniques described previously,[1] we find the minimized potential energy changes

$$\frac{\delta W_{f1}}{\pi^2 R/m\mu_o} = (a\tilde{\mathscr{B}}_a)^2 \left[\frac{-2}{m-nq_a} + \frac{a/m}{L_a}\right] \tag{1}$$

$$\frac{\delta W_{f2}}{\pi^2 R/m\mu_o} = (b\tilde{\mathscr{B}}_b)^2 \left[\frac{+2}{m-nq_b} - \frac{b/m}{L_b}\right] + (c\tilde{\mathscr{B}}_c)^2 \left[\frac{-2}{m-nq_c} + \frac{c/m}{L_c}\right] \tag{2}$$

$$\frac{\delta W_{v1}}{\pi^2 R/m\mu_o} = \left[(a\tilde{\mathscr{B}}_a)^2 + (b\tilde{\mathscr{B}}_b)^2\right]\left[\frac{1 + (a/b)^{2m}}{1 - (a/b)^{2m}}\right] + (a\tilde{\mathscr{B}}_a)(b\tilde{\mathscr{B}}_b)\left[\frac{-4(a/b)^m}{1 - (a/b)^{2m}}\right] \tag{3}$$

$$\frac{\delta W_{v2}}{\pi^2 R/m\mu_o} = (c\tilde{\mathscr{B}}_c)^2 \left[1 + 2(B_{fo}/\tilde{\mathscr{B}}_c)\right]. \tag{4}$$

Here $\tilde{\mathscr{B}}$ is $\tilde{B}_{ro}/i$ or $(B_\theta/r)(m-nq)\xi_{ro}$ and $\tilde{\mathscr{B}}_x = [\tilde{\mathscr{B}}]|_{r=x} = [(B_\theta/r)(m-nq)\xi_{ro}]_{r=x}$. For example, $\tilde{\mathscr{B}}_a = (B_a/a)(m-nq_a)\xi_{ao}$. The scale lengths L_a, L_b, and L_c are $(r\xi_r)'/(r\xi_r)$ evaluated at r = a, b, and c, respectively. B_f is the feedback field at the surface of the plasma (at r = c). (B_f is the vacuum value at r = c of the field produced by all external feedback currents.) For example, for a perfectly conducting wall at r = d, the image currents on the wall produce the feedback field $B_f = B_{fo}\exp(im\theta + ik_z z + \gamma t)$ and $B_{fo}/\tilde{\mathscr{B}}_c$ is $(c/d)^{2m}/[1-(c/d)^{2m}]$, where $\tilde{\mathscr{B}}_c = (B_c/c)(m-nq_c)\,\xi_{co}$.

a. Determination of the Scale Lengths L_a, L_b, and L_c

We now need to find the scale distances L_a, L_b, and L_c from the Euler equation for $k_z^2r^2/m^2 \ll 1$ and $B_z'r/B_z \ll 1$, where the prime denotes $\partial/\partial r$.

$$r^2\xi_r'' + \left[3 + 2\left(\frac{m}{m-nq}\right)\left(\frac{\bar{j}_z'r}{\bar{j}_z}\right)\right] r\xi_r' - (m^2-1)\xi_r = 0,$$

where $\bar{j}_z$ is the average of j_z from r = 0 out to r = r, and $\bar{j}_z = (2/\mu_o)(B_\theta/r)$.

In the region $r = r_o$ to r = a, $j_z = 0$ and $q = q_a(r^2/a^2)$. Since $j_z = \bar{j}_z + \bar{j}_z'r/2$, we have $\bar{j}_z'r/\bar{j}_z = -2$, and

$$r^2\xi_r'' + \left(1 - 2\,\frac{m + nq}{m - nq}\right) r\xi_r' - (m^2-1)\xi_r = 0$$

which, for $q = q_o\left(r^2/r_o^2\right)$, can be written

$$r^2\left[\frac{\xi_r}{r}(m-nq)\right]'' + r\left[\frac{\xi_r}{r}(m-nq)\right]' + m^2\left[\frac{\xi_r}{r}(m-nq)\right] = 0.$$

This is a recognizable equation whose solution is

$$\left[\frac{\xi_r}{r}(m-nq)\right] = Ar^{+m} + Br^{-m}.$$

In the region r = 0 to $r = r_o$, j_z is constant and $\bar{j}_z' = 0$ so that the Euler equation goes to

$$r^2[r\xi_r]'' + r[r\xi_r]' + m^2[r\xi_r] = 0,$$

whose solution is

$$[r\xi_r] = Cr^{m} + Dr^{-m}.$$

We take D = 0 in order that ξ_r be finite at r = 0 and take $\xi_r\big|_{r=r_o} = \xi_{r_o}$ to obtain ξ_r for $r < r_o$ as

$$\xi_r = \xi_{ro}\left(\frac{r}{r_o}\right)^{m-1}.$$

This solution connects with the solution in the region $r = r_o$ to $r = a$. The conditions are that ξ_r and ξ_r' be continuous at $r = r_o$, which gives

$$A = \frac{\xi_{ro}}{r_o}(m-nq_o-1)r_o^{-m}$$

$$B = (\xi_{ro}/r_o)r_o^{+m}.$$

Thus, for $r_o < r < a$, the radial dependence of ξ_r that minimizes δW_{f1} is

$$\xi_r = \xi_{ro}\left[(m-nq_o)-\left(1-(r_o/r)^{2m}\right)\right]\left(\frac{(r/r_o)^{m+1}}{m-nq}\right)$$

or, in terms of $\xi_a \equiv \xi_r\big|_{r=a}$,

$$\xi_r = \xi_a\left(\frac{r}{a}\right)^{m+1}\frac{(m-nq_a)}{(m-nq)}\left[\frac{(m-nq_o)-(1-(r_o/r)^{2m})}{(m-nq_o)-(1-(r_o/a)^{2m})}\right].$$

Thus

$$\frac{1}{L_a} \equiv \left.\frac{(r\xi_r)'}{(r\xi_r)}\right|_{r=a} = \frac{m}{a}\left[1-\frac{2(r_o/a)^{2m}}{(m-nq_o)-(1-(r_o/a)^{2m})}+\frac{2}{m-nq_a}\right].$$

Similarly, we find (for the outer plasma region) the scale distances that minimize δW_{f2},

$$\frac{1}{L_b} \equiv \left.\frac{(r\xi_r)'}{(r\xi_r)}\right|_{r=b} = \frac{m}{b}\left[\frac{2}{m-nq_b}-\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}}+\frac{(\xi_{co}/c)(m-nq_c)\,2(b/c)^{m}}{(\xi_{bo}/b)(m-nq_b)(1-(b/c)^{2m})}\right]$$

$$\frac{1}{L_c} \equiv \left.\frac{(r\xi_r)'}{(r\xi_r)}\right|_{r=c} = \frac{m}{c}\left[\frac{2}{m-nq_c}+\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}}-\frac{(\xi_{bo}/b)(m-nq_b)\,2(b/c)^{m}}{(\xi_{co}/c)(m-nq_c)(1-(b/c)^{2m})}\right].$$

We use the ratio ξ_{co}/ξ_{bo} rather than ξ_c/ξ_b, since ξ_c is seen to be either in phase or 180° out of phase with ξ_b, so as to minimize δW, where

$$\xi_c = \xi_{co} \exp(im\theta + ik_z z + \gamma t)$$

$$\xi_b = \xi_{bo} \exp(im\theta + ik_z z + \gamma t)$$

and

$$\xi_c = \xi_r \Big|_{r=c}$$

$$\xi_b = \xi_r \Big|_{r=b}.$$

We now have δW_{f1}, δW_{f2}, δW_{v1}, and δW_{v2} in terms of ξ_{ao}, ξ_{bo}, and ξ_{co} or, equivalently, in terms of $\tilde{\mathscr{B}}_a$, $\tilde{\mathscr{B}}_b$, and $\tilde{\mathscr{B}}_c$. Equations 1-4 become

$$\frac{\delta W_{f1}}{\pi^2 R/m\mu_o} = (a\tilde{\mathscr{B}}_a)^2 \left[\frac{-2}{m-nq_a} + \frac{a/m}{L_a}\right]$$

$$= (a\tilde{\mathscr{B}}_a)^2 \left[1 - \frac{2(r_o/a)^{2m}}{(m-nq_o) - (1-(r_o/a)^{2m})}\right] \tag{5}$$

$$\frac{\delta W_{f2}}{\pi^2 R/m\mu_o} = (b\tilde{\mathscr{B}}_b)^2 \left[\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{(\xi_{co}/c)(m-nq_c)\, 2(b/c)^m}{(\xi_{bo}/b)(m-nq_b)(1-(b/c)^{2m})}\right]$$

$$+ (c\tilde{\mathscr{B}}_c)^2 \left[\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{(\xi_{bo}/b)(m-nq_b)\, 2(b/c)^m}{(\xi_{co}/c)(m-nq_c)(1-(b/c)^{2m})}\right] \tag{6}$$

$$\frac{\delta W_{v1}}{\pi^2 R/m\mu_o} = \left[(a\tilde{\mathscr{B}}_a)^2 + (b\tilde{\mathscr{B}}_b)^2\right]\left[\frac{1+(a/b)^{2m}}{1-(a/b)^{2m}}\right] + (a\tilde{\mathscr{B}}_a)(b\tilde{\mathscr{B}}_b)\left[\frac{-4(a/b)^m}{1-(a/b)^{2m}}\right] \tag{7}$$

$$\frac{\delta W_{v2}}{\pi^2 R/m\mu_o} = (c\tilde{\mathscr{B}}_c)^2 \left[1+2(B_{fo}/\tilde{\mathscr{B}}_c)\right]. \tag{8}$$

Next we find the ratios of ξ_{bo} and ξ_{co} to ξ_{ao} in order to obtain δW in terms of just ξ_{ao} or $\tilde{\mathscr{B}}_a$. Equivalently, we can determine values of the ratios $\tilde{\mathscr{B}}_c/\tilde{\mathscr{B}}_b$ and $\tilde{\mathscr{B}}_b/\tilde{\mathscr{B}}_a$.

Value of δW for the Mode of Amplitude ξ_{ao}

Perturbations of the current channel provide the energy source for unstable wave growth. Thus we are interested in modes associated with the wave amplitude in the current-carrying region, and we want to hold ξ_{ao} fixed and minimize δW with respect to ξ_{bo} and ξ_{co}.

First we find the ratio of ξ_{co}/ξ_{bo} that minimizes δW by holding ξ_{bo} fixed and minimizing δW with respect to ξ_{co}. Then we hold ξ_{ao} fixed and minimize δW with respect to ξ_{bo}. (Since ξ_{xo} and $\tilde{\mathscr{B}}_x$ are related, we actually minimize with respect to the amplitude of the radial field at each plasma boundary, and obtain the ratios $\tilde{\mathscr{B}}_c/\tilde{\mathscr{B}}_b$ and $\tilde{\mathscr{B}}_b/\tilde{\mathscr{B}}_a$ that minimize δW for fixed $\tilde{\mathscr{B}}_a$.)

a. Choice of Feedback

The minimization of δW with respect to ξ_{co} depends on whether B_{fo} is proportional to ξ_{co} or to ξ_{bo}. For external sensing such as magnetic field sensing, B_{fo} is proportional (for constant $m-nq_c$) to ξ_{co}. But, in principle, internal sensing (with probes such as ion beams) could be used and B_{fo} could be made proportional to ξ_{bo}. In the former case B_{fo}/ξ_{co} is constant and

$$(c\tilde{\mathscr{B}}_c)^2\,[2B_{fo}/\tilde{\mathscr{B}}_c] = [cB_c(m-nq_c)\;2B_{fo}/\xi_{co}]\xi_{co}^2$$

which goes as ξ_{co}^2. For the latter case, the feedback term goes as $\xi_{co}\xi_{bo}$,

$$(c\tilde{\mathscr{B}}_c)^2\,[2B_{fo}/\tilde{\mathscr{B}}_c] = [cB_c(m-nq_c)\;2B_{fo}/\xi_{bo}]\xi_{co}\xi_{bo}.$$

We shall examine the former case in which external sensing is used and B_{fo}/ξ_{co} or, equivalently, $B_{fo}/\tilde{\mathscr{B}}_c$ is held constant in the partial differentiation.

b. Determination of $(c\tilde{\mathscr{B}}_c)/(b\tilde{\mathscr{B}}_b)$

Minimization with respect to ξ_{co} involves the terms δW_{f2} and δW_{v2}. Taking B_{fo}/ξ_{co} to be constant and letting $R_{cb} = (c\tilde{\mathscr{B}}_c)/(b\tilde{\mathscr{B}}_b)$, we have

$$R_{cb} \equiv \frac{c\tilde{\mathscr{B}}_c}{b\tilde{\mathscr{B}}_b} = \frac{(b/c)^m}{1 + (1-(b/c)^{2m})(B_{fo}/\tilde{\mathscr{B}}_c)} \tag{9}$$

where $c\tilde{\mathscr{B}}_c = \xi_{co}[B_\theta(m-nq)]\big|_{r=c}$ and $b\tilde{\mathscr{B}}_b = \xi_{bo}[B_\theta(m-nq)]\big|_{r=b}$.

c. Determination of $(b\tilde{\mathscr{B}}_b)/(a\tilde{\mathscr{B}}_a)$

Next we find the minimizing ξ_{bo}/ξ_{ao} ratio by holding ξ_{ao} fixed and minimizing δW with respect to ξ_{bo}. The terms that contain ξ_{bo} are δW_{f2}, δW_{v2}, and δW_{v1}.

We let $R_{ba} = (b\tilde{\mathscr{B}}_b)/(a\tilde{\mathscr{B}}_a)$, and obtain

$$R_{ba} = \frac{2(a/b)^{2m}}{1+(a/b)^{2m}} \left[\frac{1}{1 + \left[\frac{1-(a/b)^{2m}}{1+(a/b)^{2m}}\right]\left[\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{2(b/c)^m R_{cb}}{1-(b/c)^{2m}}\right]} \right]. \tag{10}$$

Clearly as $b \to a$, $R_{ba} \to 1$. Thus the $\tilde{\mathscr{B}}_b/\tilde{\mathscr{B}}_a$ ratio that minimizes the wave energy as $b \to a$ is such that $\tilde{B}_r$ remains continuous, as assumed in tearing-mode analysis.

d. Result of Minimization

We now have the minimized wave potential energy in three parts: the energy inside $r = a$ (Eqs. 1 and 5)

$$\frac{\delta W_{f1}}{\pi^2 R/m\mu_o} = (a\tilde{\mathscr{B}}_a)^2 \left[1 - \frac{2(r_o/a)^{2m}}{(m-nq_o) - (1-(r_o/a)^{2m})}\right]; \tag{11}$$

the vacuum gap energy (Eqs. 3 and 7)

$$\frac{\delta W_{v1}}{\pi^2 R/m\mu_o} = (a\tilde{\mathscr{B}}_a)^2 \left[\frac{1+(a/b)^{2m}}{1-(a/b)^{2m}}\left(1+R_{ba}^2\right) + \frac{(-4(a/b)^m R_{ba})}{1-(a/b)^{2m}}\right]; \tag{12}$$

and the energy outside $r = b$ (Eqs. 2, 4, 6, and 8)

$$\frac{\delta W_{f2} + \delta W_{v2}}{\pi^2 R/m\mu_o} = (a\tilde{\mathscr{B}}_a)^2 \left[\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{2(b/c)^m}{1-(b/c)^{2m}} R_{cb}\right] R_{ba}^2. \tag{13}$$

Note that δW_{v1} goes to zero as $b \to a$ and, correspondingly, as R_{ba} (Eq. 10) goes to one.

Thus the total energy δW for the mode of amplitude $\tilde{\mathscr{B}}_a$ (or of amplitude ξ_{ao}) is

$$\begin{aligned}\frac{\delta W}{\pi^2 R/m\mu_o} = {} & (a\tilde{\mathscr{B}}_a)^2 \left[1 - \frac{2(r_o/a)^{2m}}{(m-nq_o) - [1-(r_o/a)^{2m}]} + \frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{2(b/c)^m R_{cb}}{1-(b/c)^{2m}}\right] \\ & + (1-R_{ba}^2)\left\{(a\tilde{\mathscr{B}}_a)^2 \left[\frac{1+(b/c)^{2m}}{1-(b/c)^{2m}} - \frac{2(b/c)^m R_{cb}}{1-(b/c)^{2m}}\right]\right\} \\ & + \frac{\delta W_{v1}}{\pi^2 R/m\mu_o}. \end{aligned} \tag{14}$$

Application of δW to Tearing Modes

To determine tearing-mode stability, we want the small (b-a) limit of δW (Eq. 14), which we denote δW_T. As b goes to a, $\left(R_{ba}^2-1\right)$ goes to zero and δW_{v1} also goes to zero. Thus δW_T is just the first term in Eq. 14 (with b going to a) or

$$\frac{\delta W_T}{\pi^2 R/m\mu_o} = \frac{(a\tilde{\mathscr{B}}_a)^2\ 2}{1-(a/c)^{2m}}\left[\frac{-(r_o/a)^{2m}(1-(a/c)^{2m})}{[m-nq_o]-[1-(r_o/a)^{2m}]}+1-(a/c)^m R_{cb}\right]. \tag{15}$$

a. Fixed Boundary, $\xi_{co} = 0$

We can check part of these results by comparing them with Shafranov's result[2] for the special case of a wall at the outer plasma boundary, at r = c. For a wall at r = c, ξ_{co} is zero (the boundary condition imposed by Shafranov was $\xi_{co} = 0$) and therefore R_{cb} is zero. Taking R_{cb} (Eq. 9) to be zero, we have

$$\frac{\delta W_T}{\pi^2 R/m\mu_o} = \frac{[a\tilde{\mathscr{B}}_a]^2\ 2}{1-(a/c)^{2m}}\left[\frac{[m-nq_o]-[1-(r_o/c)^{2m}]}{[m-nq_o]-[1-(r_o/a)^{2m}]}\right]. \tag{16}$$

Thus the condition for instability (for $\delta W < 0$) is

$$1-(r_o/a)^{2m} < (m-nq_o) < 1-(r_o/c)^{2m}.$$

The radius a is related to q_o, in that our model plasma was chosen so that m-nq went to zero in the vacuum gap between r = a and r = b, at $r = r_s$. Since $q = q_o(r/r_o)^2$, the appropriate value of a is $a \approx r_s = r_o[m/nq_o]^{1/2}$. Thus r = a corresponds to a different location for each value of q_o considered. The condition for instability becomes

$$1-(nq_o/m)^m < (m-nq_o) < 1-(r_o/c)^{2m}.$$

This criterion is the same as the tearing-mode criterion obtained by Shafranov for this current and plasma distribution, and with the same $\xi_{co} = 0$ assumption.

b. Feedback Effects on This Tearing Mode

We now include the externally imposed feedback field B_{fo} so that δW_T is

$$\frac{\delta W_T}{\pi^2 R/m\mu_o} = \frac{2(a\tilde{\mathscr{B}}_a)^2}{1-(a/c)^{2m}} \left[\frac{-(r_o/a)^{2m}\left[1-(a/c)^{2m}\right]}{[m-nq_o]-\left[1-(r_o/a)^{2m}\right]} + 1-(a/c)^m\ R_{cb}\right]$$

$$= 2(a\tilde{\mathscr{B}}_a)^2 \left[\frac{-(r_o/a)^{2m}}{[m-nq_o]-\left[1-(r_o/a)^{2m}\right]} + \left(\frac{1-(a/c)^m\left[\frac{(a/c)^m}{1+(1-(a/c)^{2m})\ B_{fo}/\tilde{\mathscr{B}}_c}\right]}{1-(a/c)^{2m}}\right)\right]$$

$$= \left[\frac{2(a\tilde{\mathscr{B}}_a)^2}{1-(a/c)^{2m}\{f\}}\right] \frac{[m-nq_o]-\left[1-(r_o/c)^{2m}\{f\}\right]}{[m-nq_o]-\left[1-(r_o/a)^{2m}\right]}, \tag{17}$$

where

$$f = \left\{\frac{B_{fo}/\tilde{\mathscr{B}}_c}{1+(B_{fo}/\tilde{\mathscr{B}}_c)}\right\}.$$

Note that this expression is the same as the fixed boundary expression when $f = 1$. Note also that f is less than one for all B_{fo}, with f going to one as $(B_{fo}/\tilde{\mathscr{B}}_c) \to \infty$.

Thus as the feedback field B_{fo} is increased from zero to infinity, f goes to one and δW_T (Eq. 17) goes to the "fixed boundary" value, $\delta W_T|_{\xi_{co}=0}$ (Eq. 16).

Conclusions

We speculate that external feedback cannot fully stabilize modes that are unstable even when the outer plasma boundary is held fixed. The example above is consistent with this view. By comparing the results with feedback (Eq. 17) with the fixed boundary results (Eq. 16), we see that as $B_{fo}/\tilde{\mathscr{B}}_c$ goes to infinity, δW_T only goes to the fixed boundary value. While partial stabilization (reducing the growth rate and narrowing the range of unstable q_o values) can be achieved, $(m-nq_o)$ values from 0 to $\left[1-(r_o/c)^{2m}\right]$ remain unstable, even for very large feedback fields. The problem can best be understood by noting that regardless of what B_f is made proportional to, the externally imposed feedback contribution (Eq. 4) to δW goes as $(B_{fo}\xi_{co})$. Thus if B_{fo} is made very large, the value of ξ_{co} that minimizes δW will be very small, thereby nullifying the feedback contribution.

Consequently, we speculate that for externally imposed feedback fields, magnetic feedback stabilization may be limited (as far as doing more than reducing the growth rates) to modes (such as kink modes) that are unstable for a free boundary but are stable when ξ_r is kept equal to zero at the outer plasma boundary.

R. S. Lowder, K. I. Thomassen

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V. PLASMAS AND CONTROLLED NUCLEAR FUSION*

F. High-Temperature Plasma Physics

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1. APPARATUS FOR ION TEMPERATURE MEASUREMENT BY LYMAN ALPHA DOPPLER SHIFT

This report describes the development of an ion temperature diagnostic device for the Alcator experiment now under construction at the Francis Bitter National Magnet Laboratory. This machine is designed to produce a hydrogen plasma with a density of $\approx 10^{14}$ cm^{-3} and a particle kinetic temperature of 1-2 keV. Since the particle temperature is so high, the only probe that may be used for sensing the particle temperature is radiation. It is desirable to determine not only the ion temperature but its spatial variation.

The method under development for spatially resolving the ion temperature involves the measurement of the Doppler shift of Lyman alpha radiation emitted by neutral atoms formed by charge exchange within the plasma. The hydrogen Balmer lines, in the visible spectrum, have previously been used for this kind of measurement. The difficulty in using lines other than Lyman alpha is the very long lifetime of these other lines. A neutral formed in a level that would emit visible light can move completely across or out of the plasma before emitting a photon, thereby making spatial resolution impossible. The lifetime of the Lyman alpha transition is of the order of 10^{-8} s. A 1-keV hydrogen neutral excited to the n = 2 level will move only approximately 1 cm after a charge-exchange event before it radiates at 1215 Å. The lifetime of this level is much shorter than the characteristic time for any process that might destroy it. Neutrals excited into the n = 3 level or higher may suffer ionization or may move a great distance before radiating. The atoms produced in the n = 2 level will radiate at 1215 Å. The

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neutrals will be formed by charge exchange with a neutral hydrogen beam 1-cm wide at 10 keV. The beam will be directed along the center of the plasma column. By focusing a spectrometer on the beam at a right angle, atoms formed in the n = 2 level will be observed, while most of the atoms formed in other levels will move out of the viewing volume before radiating. By scanning the plasma radially along the beam, a spatially resolved temperature profile may be obtained.

Doppler broadening is not the only source of line broadening. Other sources are the Stark and the Zeeman effects. It has been shown that only the Zeeman effect need be considered.[1] The Zeeman effect, caused by the Alcator main toroidal field, will produce ~0.2 Å broadening.[1] This value is small and may be compensated for in the data analysis.

Spectrometer Design

The spectrometer design criteria are dispersion, detectors, physical size, available gratings, optics configurations, free spectral range, and instrumental broadening. These topics are interrelated and must be considered together. For optimum operation the spectrometer must be designed as a complete system.

For a Maxwellian plasma with Doppler broadening as the only line-broadening mechanism, the intensity distribution is

$$I(\Delta\lambda) = \frac{I_t}{\Delta\lambda_D\sqrt{\pi}} \exp\left[-(\Delta\lambda/\Delta\lambda_D)^2\right],$$

where

I_t = total line intensity

$\Delta\lambda_D = \frac{v_{th}}{c}\lambda_o$

$\Delta\lambda_s = \pm\frac{v_s}{c}\lambda_o$

v_{th} = average particle thermal velocity

v_s = component of particle velocity along the line of observation.

The full width at half maximum (FWHM) of the line is

$$\Delta\lambda_{1/2} = 2(\ln 2)^{1/2}\,\Delta\lambda_D.$$

A plot of $\Delta\lambda_{1/2}$ against T_i is shown in Fig. V-2. For a particle energy of 2 keV, the FWHM is 3 Å. In order to observe the structure of the line, a spectrometer with a minimum resolution of approximately 0.3 Å must be used. This resolution will allow reasonably accurate measurement of temperatures as low as a few hundred electron volts.

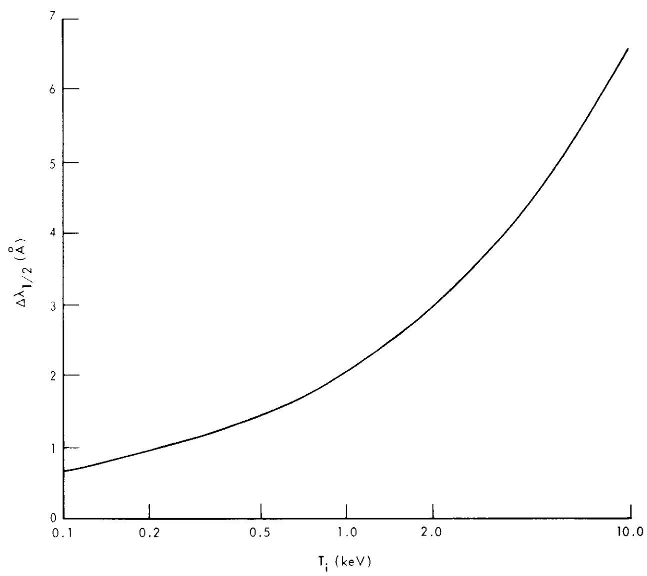

Fig. V-2. Linewidth vs ion temperature.

In order to determine the spectrometer dispersion to provide a resolution of 0.3 Å, the method of detection of light leaving the spectrometer must be examined. To obtain this resolution, the width of the detectors must be

$$L \leqslant \frac{\text{Required Resolution}}{(d\lambda/da)} \text{ mm},$$

where $d\lambda/da$ is the linear dispersion (Å/mm), or

$$L \leqslant \frac{0.3}{(d\lambda/da)} \text{ mm}.$$

For most types of photomultiplier tube detectors, with L of the order of 1 cm, a dispersion of 0.03 Å/mm would be required. This value is impractically high for a system of reasonable size and expense. Because of this consideration Channeltron electron multipliers were chosen.[2] These devices have a 1-mm aperture and a gain of $\sim 10^7$-10^8. This detector must have a dispersion of 0.3 Å/mm for the required resolution. Several channeltrons will be placed side by side so that each detector may observe a 0.3 Å segment of the line. The ratio of the signals from any two detectors will give a value of ion temperature. Values from several pairs of detectors will be compared in order to determine whether the distribution function is a Maxwellian. Because of size restrictions 1 m was set as a goal for the approximate size of the spectrometer.

The requirements of small instrument size and high dispersion led to the selection of an echelle grating as the light-dispersing element. The echelle is a coarse-ruled flat grating that is operated in a high order. Very high dispersion can be obtained with all gratings if they are operated in a high order, but with ordinary gratings very little light is dispersed into high orders. The echelle grating, however, has the property of throwing ~60% of the dispersed radiation into the particular high order for which the system is designed, with 10-20% going into the next lower order and 10-15% into the next higher order.[3] Therefore light intensity does not have to be sacrificed for high-order operation. Also, the echelle grating is blazed for all wavelengths at an angle of 63°26', which makes it very versatile.

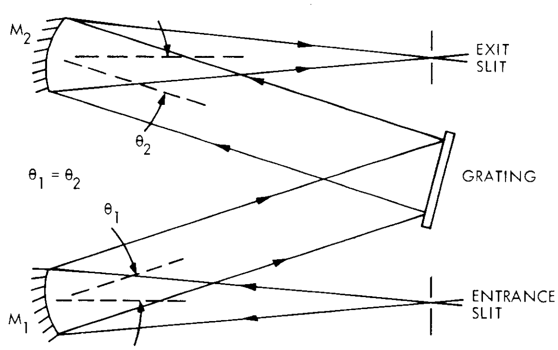

Fig. V-3. Czerney-Turner mount.

After consideration of many optical systems, the Czerney-Turner spectrometer arrangement (Fig. V-3) was chosen for this device. A Littrow mount would have been more compact but a lens would have been required instead of all reflecting optics. No lenses of high transmission efficiency are available for use in vacuum ultraviolet. By using M_gF_2 vacuum ultraviolet enhancement coatings, reflectivities of 0.8 can be obtained with mirror optics.[4] When using mirrors an off-axis reflection of a curved mirror is required somewhere in the optical train in order to get light on and off the grating.[5] The Czerney-Turner system has the property of producing zero net coma and minimizes aberration. The mirrors M_1 and M_2 were chosen to make maximum use of the grating rulings and obtain the highest possible resolving power. The diameter of each mirror is 5 cm, and the radius of curvature is 50 cm, which makes an f/5 optical system. These values were determined as optimum by mechanical considerations and by a ray-tracing computer program run at the Draper Laboratory.[6] The light-collection mirror characteristics were selected for optimum spot size at the detectors. In mounting these mirrors, the angle θ (the angle between incident light on and reflected light from the grating) was chosen to be 6°. The operating length of the spectrometer is 1 m, therefore the total distance between the grating and the detectors is 1 m; hence, the spacing between the grating and mirrors M_1 and M_2 is 75 cm. The detectors are located at the focal point of M_2, 25 cm from the mirror. Since the light is parallel between the mirrors and the grating, the distance is adjustable to suit the dispersion requirements of the system.

The echelle is operated in a reverse mode from standard gratings, using the short side as the reflecting surface (see Fig. V-4). The grating equations are

$$m\lambda = d(\sin\alpha + \sin\beta)$$

$$m\lambda = 2t - s\theta.$$

The quantities d, α, β, s, t, and θ are defined in Fig. V-4. A 316 l/mm echelle grating was chosen for use in the spectrometer. Solving the grating equations for m and $d\lambda/d\theta$,

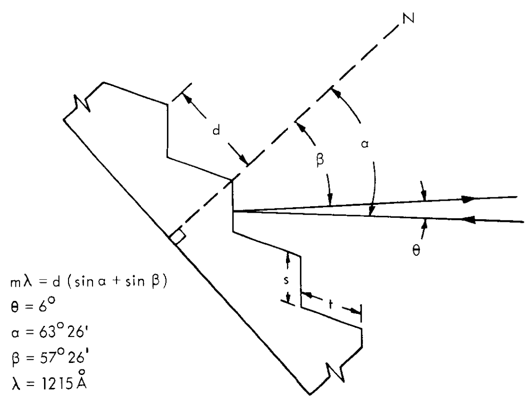

Fig. V-4. Operation of the echelle.

we obtain an order (m) of 45.2 and an angular dispersion ($d\lambda/d\theta$) of 312.9 Å/rad. The required dispersion is ~300 Å/rad. The next quantity to be considered is the free spectral range, F. This must be much wider than the linewidth under observation (~3 Å). The equation for free spectral range is $F = \lambda/m$. For λ = 1215 Å and m = 45, the free spectral range is 26.9 Å. The 316 l/mm grating meets the system requirements. The instrumental broadening determines the narrowest line that can be resolved by the spectrometer. This broadening is related to the resolving power $R_p = mN$ (where m is order and N is number of exposed lines) by $\Delta\lambda = \lambda/R_p$, with λ = 1215 Å. For the 316 l/mm grating, we chose the 45^{th} order. The number of exposed lines is determined by the ruled width in the optical system. The echalle grating will be operated at the blaze angle (63°26'). The groove length in the system is 5 cm, and the ruled width is 5 tan 63°26' or 10 cm. The number of exposed lines is therefore 3.16×10^4. The instrumental broadening, $\Delta\lambda = 0.855 \times 10^{-3}$ Å, is negligible compared with all other broadening mechanisms.

Light-Collection System

Access to the Alcator device is limited to ports 2-cm wide. The neutral beam on which the spectrometer will be focused must be observed through this slot. The 2-cm slot forms the limiting aperture of the optical system and determines the light cone emerging from the machine. The spacing between the 1-cm viewing volume and the 2-cm aperture yields an f/10 light cone. The spectrometer is designed to match an f/5 cone. If the light cone emerging from Alcator is not the same as that which the spectrometer is designed to accept, large loss of light will occur. In addition to collecting light from Alcator, the main collection mirror must change the emerging f/10 light cone to an f/5 cone. Matching these light cones allows for the most efficient operation of the optical system. The collection optics is a Cassegrainian system with a flat secondary. The entire optical system is shown in Fig. V-5. A flat secondary on the collection system is necessary, otherwise uncorrectable coma would result from off-axis reflection

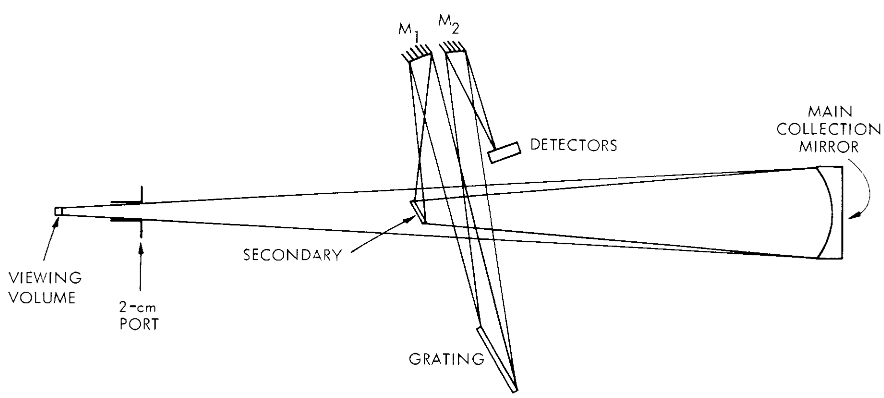

Fig. V-5.
Optical system.

from a curved mirror. After the position of the main mirror was selected according to mechanical considerations, and the diameter determined by the light cone emerging from the Alcator device, the entire optical system was analyzed by a ray-tracing computer code at the Draper Laboratory. The result of this computer run was the selection of the radius of curvature of the main mirror and the distance between the main collection mirror and the first mirror of the spectrometer for optimum spot size at the detectors. This size (~0.1 mm) is completely adequate, as it could be as large as the detector (1.0 mm) and still yield the required results. The position and size of the flat secondary is not critical. It may be placed anywhere between the main mirror and the first mirror of the spectrometer.

Vacuum System

The pressure in the spectrometer must be maintained at approximately 10^{-5} Torr. If the pressure goes below 10^{-6} Torr, oxygen will be lost by the SnO secondary emission material in the channeltron detectors. The detectors must then be brought up to atmospheric pressure for rejuvenation. If any pump oil is present and the pressure exceeds 10^{-4} Torr when the device is operating, the ultraviolet photons can cause photochemical reactions between the oil and the M_gF_2 mirror coating. This will result in a large loss in reflectivity of the mirrors. In an attempt to avoid high-pressure problems, since they are more severe than low-pressure difficulties, the pumping system will be a cryosorption pump for initial pump-down with a 100 l/s triode pump for the maintenance of operating pressure.

Neutral Beam Source

The neutral beam source is a commercial Duo-plasmatron ion source followed by a helium charge-exchange cell. The energy of the neutral beam will be 10 keV in order to maximize charge exchange in the plasma into the n = 2 level.[7] The beam source is expected to deliver a neutral equivalent current of approximately 1 mA. The beam will be chopped at a rate of 1 kHz. Chopping the beam will allow the use of tuned amplifiers following the detectors to improve the signal-to-noise ratio.

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2. CURRENT-DRIVEN MODES IN TWO-DIMENSIONAL PLASMA CONFIGURATIONS

Introduction

The presence of trapped particles in a two-dimensional configuration (such as a toroidal diffuse pinch or a multipole) has a strong effect on the characteristics of kinetic modes that are driven by an electron current along the magnetic field.[1] The reasons for this are as follows.

i. The current is carried by only a part of the electron population (the circulating part).

ii. The particles resonating with the wave have bounce frequency if they are trapped, or transit frequency if they are circulating, equal to the frequency of the wave. This process involves a different portion of velocity space than do the ordinary wave-particle resonance processes that occur in one-dimensional geometry.

iii. The periodic inhomogeneity of the magnetic field introduces a distinction between modes that are even or odd around the point where the magnetic field is minimum. The electron current, having preferential direction along the magnetic field, is shown to affect these two types of mode differently.

We notice that an assessment of the current-driven microinstabilities that can arise in toroidal high-temperature plasmas is important in order to predict their effects on the electrical resistivity and therefore on plasma heating.[2]

In the present report we confine our attention to electron distributions of the type derived by the neoclassical transport theory[3] in regimes where the electron collision frequency in smaller than the transit frequency. Consequently our conclusions are strongly limited by this assumption.

We examine the general form of toroidal modes that can be excited and give the main parameters of the particle orbits in the assumed configuration. Then we consider modes with electric potential that is odd in θ (the poloidal angle) around the point of minimum magnetic field. We decompose these modes in harmonics of the orbit periodicity and arrive at a dispersion equation. On the basis of this we conclude that, for all equilibrium distribution functions in which the term containing the electron current is odd in $v_{||}$ (the particle velocity parallel to the magnetic field), the stability of odd modes remains unaffected by this current. We also see that the odd modes are not appreciably influenced by magnetic shear.

We then consider even modes that are almost flutes in the sense that they are almost constant in the direction of the magnetic field. These modes are strongly influenced by magnetic shear and are convective in nature. That is, wave packets tend to be amplified when propagating in a radial direction that is correlated with that of the electron current. The amplification is caused by resonance (with the wave) of the circulating particles that carry the current. The range of realistic frequencies for which the relevant waves can exist, however, is such that only the electrons with energy smaller than thermal contribute appreciably to their amplification. Since there are few electrons of this kind, the amplification that is found for a measurable amount of shear is not too significant. By the same argument we also show that the ordinary current-driven drift instabilites, with wavelengths along the magnetic field that are considerably shorter than the magnetic field periodicity length, do not have an appreciable growth rate.

Finally, we consider regimes wherein the electron collisional mean-free paths are shorter than the wavelengths along the magnetic field of the even modes. In these regimes the effects of electron thermal conductivity become important and the resulting instability no longer involves wave-particle resonance processes. We then evaluate the stability criterion against convective modes in terms of the shear parameter.

Equilibrium Distribution Function

Consider an axi-symmetric toroidal configuration in which the magnetic field is represented by $B \approx B_o/[1+(r/R_o)\cos\theta]$, where r indicates the magnetic surface, and θ the poloidal angle. The regimes of interest are those in which

$$\bar{\nu}_{eT} < \bar{\omega}_{be}, \tag{1}$$

where $\bar{\nu}_{eT}$ is the average effective collision frequency, and $\bar{\omega}_{be}$ the average bounce frequency for trapped electrons. Thus $\bar{\omega}_{be} \sim v_{the}/(qR_o)(r/R_o)^{1/2}$, where the rationalized rotational transform is $1/q \approx R_oB_\theta/(rB_\zeta)$, where B_θ and B_ζ are the poloidal and toroidal magnetic fields, respectively.

We assume that the only electric field existing in equilibrium is that applied along the toroidal direction ζ and producing a current of circulating electrons. The electron distribution is taken as nearly Maxwellian, so that

$$f_e = f_{Me}(1+\hat{f}_e), \tag{2}$$

where $f_{Me} = n(r)/(2\pi T_e/m_e)^{3/2} \exp(-\epsilon/T_e)$, and $\epsilon = 1/2\, m\left(v_\perp^2+v_\parallel^2\right)$ is the particle energy. For trapped electrons $\hat{f}_e$ is the solution of Vlasov's equation so that

$$\hat{f}_{eT} = \frac{v_\parallel}{\Omega_{\theta e}}\left[\frac{1}{n}\frac{dn}{dr} - \left(\frac{3}{2} - \frac{\epsilon}{T_e}\right)\frac{1}{T_e}\frac{dT_e}{dr}\right], \tag{3}$$

where $\Omega_{\theta e} = eB_\theta/(m_ec)$. The distribution function for circulating particles is strongly dependent on the choice of the collision operator.[3,4] Here we take a solution of

Vlasov's equation in the form

$$\hat{f}_{eC} = \hat{f}_{eT} - \sigma \frac{\langle v_{\parallel} \rangle}{\nu(\epsilon)} \left(\frac{eE}{2T_e}\right) 1(\Lambda_c - \Lambda), \tag{4}$$

where ν is an effective electron-ion collision frequency such that

$$\nu(\epsilon) = \nu_o \left(\frac{T_e}{\epsilon}\right)^{3/2},$$

$1(\Lambda_c - \Lambda)$ is the step function with $\Lambda_c = 1 - r/R_o$, $\langle v_{\parallel} \rangle = (2\epsilon/m)^{1/2}$, $(\epsilon - \mu B_o)^{1/2} = (2/m)^{1/2} (1-\Lambda)^{1/2}$, $\mu = mv_{\perp}^2/(2B)$, and $\Lambda \equiv \mu B_o/\epsilon$ and $\sigma = \text{sign } v_{\parallel}$. Therefore circulating particles correspond to $0 < \Lambda < \Lambda_c$ and trapped particles to $\Lambda_c < \Lambda < \Lambda_c + 2r/R_o$. We shall make use also of the quantities defined as

$$u_o \equiv \frac{eE}{m_e \nu_o} \tag{5}$$

$$\Delta\hat{f}_e \equiv \hat{f}_{ec} - \hat{f}_{eT}. \tag{6}$$

Odd Modes

Consider the frequency range

$$\bar{\omega}_{bi} < \omega < \bar{\omega}_{be}, \tag{7}$$

where $\bar{\omega}_{bj}$ is the average bounce frequency for the species j. Since we refer to a low-β situation ($\beta \equiv 8\pi n(T_e+T_i)/B^2$), the modes of interest are electrostatic, so that $\underline{E}_1 = -\nabla\Phi_1$. In particular,

$$\Phi_1 = \tilde{\phi}_m(\theta, r) \exp(-i\omega t - im^\circ\theta + in^\circ\zeta), \tag{8}$$

and we consider modes localized around a rational surface such that $q(r_o) = m^\circ/n^\circ$, where $q(r_o) = 2\pi/\iota$, with ι the rotational transform. An important physical and topological distinction[5] among various possible modes is given by the parity of $\tilde{\phi}_m(\theta)$ around $\theta = 0$. Therefore we look first for odd modes; in particular, we consider the limit

$$\frac{\partial\tilde{\phi}_m}{\partial\theta} \gg \left[n^\circ \frac{dq}{dr_o} (r-r_o)\right]\tilde{\phi}_m \tag{9}$$

so that the effects of magnetic shear, corresponding to $dq/dr_o \neq 0$, can be neglected.

Here $r - r_o$ is of the order of the width of localization of the modes in question about r_o. In this limit the perturbed ion density can be written

$$\frac{\tilde{n}_i}{n} = \frac{m^\circ}{r} \frac{c}{B} \frac{1}{\omega} \frac{dn}{dr} \frac{1}{n} \tilde{\phi}_m \tag{10}$$

if we neglect the contributions of finite Larmor radius and magnetic curvature effects.

In order to find the electron distribution we decompose $\tilde{\phi}_m$ in harmonics of the orbit periodicity.[5] That is,

$$\tilde{\phi}_m(\theta) = \sum_p \tilde{\Phi}_m^{(p)}(\Lambda)\, e^{ip\omega_b \hat{t}}. \tag{11}$$

Here ω_b indicates the bounce frequency for trapped particles and the transit frequency for circulating particles. That is, $\omega_b = 2\pi/\tau_b$ where

$$\tau_b = \frac{1}{2} R_o q_o \oint \frac{d\theta}{|v_{\parallel}|} \tag{12}$$

for trapped particles, and $\omega_b = 2\pi/\tau_t$, where

$$\tau_t = R_o q_o \int_0^{2\pi} \frac{d\theta}{|v_{\parallel}|} \tag{13}$$

for circulating particles. In addition,

$$\hat{t} = R_o q_o \int^{\theta} \frac{d\theta'}{|v_{\parallel}|}.$$

We take the guiding-center approximation, integrate the linearized perturbed Vlasov equation along unperturbed orbits, and obtain

$$\tilde{f}_{em} = -e\left\{\tilde{\phi}_m \frac{\partial f_e}{\partial \epsilon} + i\left(\omega \frac{\partial f_e}{\partial \epsilon} - \frac{n^\circ}{R_o} \frac{c}{eB_p} \frac{\partial f_e}{\partial r}\right) \int_{-\infty}^{t} dt'\, \tilde{\phi}_m[\theta(t')] \exp[-i\omega(t'-t)]\right\}. \tag{14}$$

We have not included terms in (14) corresponding to the magnetic curvature drift. Such terms, as can be verified a posteriori, are unimportant for the modes of interest. Then we derive the perturbed electron density in the form

$$\tilde{n}_e = \frac{en}{T_e}\left\{\tilde{\phi}_m - \frac{1}{n}\frac{\pi}{2}\sum_\sigma \left(\frac{2}{m_e}\right)^2 \iint d\epsilon d\mu \, \frac{B}{|v_{\|}|} f_{Me} \times \left[\omega - \omega_{*e} + \omega_{T_e}\left(\frac{3}{2} - \frac{\epsilon}{T_e}\right) - \sigma\omega \frac{\partial|\Delta\hat{f}_e|}{\partial(\epsilon/T_e)}\right] \sum_{p\neq 0} \frac{\tilde{\Phi}_m^{(p)}(\Lambda)}{\omega - p\omega_b} e^{i\omega_b \hat{t}}\right\}, \tag{15}$$

where

$$\omega_{*e} = \frac{m^{\circ}}{r_o}\frac{cT_e}{eB}\frac{1}{n}\frac{dn}{dr}, \quad \omega_{T_e} = \omega_{*e}\frac{d\ln T_e}{dr}\bigg/\left(\frac{d\ln n}{dr}\right) \tag{16}$$

and $\sigma = \pm 1$.

We write the quasi-neutrality condition $\tilde{n}_i = \tilde{n}_e$ and construct the quadratic form

$$\frac{R_oq_o}{B_o}\int_0^{2\pi} d\theta \, \tilde{\phi}_m^*(\theta)(\tilde{n}_i - \tilde{n}_e) = 0. \tag{17}$$

Then we obtain

$$\left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{R_oq}{B_o}\oint d\theta \, |\tilde{\phi}_m|^2 - \frac{\pi}{2n}\iint d\epsilon d\mu \, |\tau| \, f_{M_e}\left(\frac{2}{m_e}\right)^2 \times \sum_\sigma \left[\omega - \omega_{*e} + \omega_{T_e}\left(\frac{3}{2} - \frac{\epsilon}{T_e}\right) - \sigma\omega\frac{\partial|\Delta\hat{f}_e|}{\partial(\epsilon/T_e)}\right] \sum_{p\neq 0}\frac{|\tilde{\Phi}_m^{(p)}|^2}{\omega - p\omega_b} = 0. \tag{18}$$

The significant contribution of the last term comes from the resonance $\omega = p\omega_b$, where for trapped particles

$$\omega_b = \left(\frac{\epsilon}{m}\frac{r_o}{R_o}\right)^{1/2}\bigg/\left[R_oq_o\mathscr{L}(\chi^2)\right], \qquad 2\chi^2 \equiv 1 + (1-\Lambda)\, R_o/r,$$

so that $0 \leq \chi \leq 1$, and $\mathscr{L}(\chi^2) = \frac{1}{4\pi}\oint d\theta \, [1+(r/R_o)\cos\theta]^{1/2}/(2\chi^2 - 1 + \cos\theta)^{1/2}$ is a weak function of χ. Therefore nearly all resonating particles have energy $\epsilon < T_e$, and we have retained the lowest order contribution of $\Delta\hat{f}_e$ through the term $\partial\Delta\hat{f}_e/\partial(\epsilon/T_e)$. Since $|\tilde{\Phi}_m^{(p)}|^2 = |\tilde{\Phi}_m^{(-p)}|^2$,

$$\sum_{p\neq 0}\frac{|\tilde{\Phi}_m^{(p)}|^2}{\omega - p\omega_b} = \sum_{p>0}|\Phi_m^{(p)}|^2 \frac{2\omega}{\omega^2 - p^2\omega_b^2}.$$

The summation over σ of this term multiplied by $\sigma\partial|\Delta\hat{f}_e|/\partial(\epsilon/T_e)$ gives zero contribution.

Thus we reach the important conclusion that the stability of <u>odd</u> modes is not affected by the presence of a relatively small electron current such that the corresponding electron distribution is represented by an odd term in $v_{\|}$, as exemplified by the expression (4). The evident reason for this is that standing modes can be regarded as resulting from the interaction of waves traveling in opposite directions along the magnetic field and having equal amplitudes. While one type of wave would tend to be damped by the current flow in a given direction, the other would tend to grow and the result of their interaction is a marginally stable mode (with respect to the effects of current).

We can rewrite Eq. 18, considering that $\omega = \omega_{*e} + \delta\omega$, as

$$\frac{\delta\omega}{\omega_{*e}}\left(\int_0^{2\pi} d\theta\, |\tilde{\Phi}_m|^2\right) + i\,\frac{3}{2}\,\omega_{T_e}\,\frac{\omega_{*e}^2}{\bar{\omega}_{te}^3}\left(\frac{R}{r}\right)\frac{\sqrt{\pi}}{4\pi^3}$$

$$\left[\int_0^1 d\chi^2\, \mathscr{L}^4(\chi^2) \sum_{p>0} \frac{|\tilde{\Phi}_m^{(p)}(\chi^2)|^2}{p^3}\right] = 0.$$

Therefore the significant instability that can be found in this case corresponds to an <u>inverted</u> gradient of the temperature profile for the trapped electron population. In fact, this can be invoked as a factor contributing to the destruction of a skin layer of electron temperature under conditions $\bar{\nu}_{eT} < \bar{\omega}_{be}$, which is relevant to the present treatment. Here we have defined $\bar{\omega}_{te} = v_{the}/(qR)$.

<u>Even Modes (in θ)</u>

Consider the opposite limit of that treated so far; that is, <u>flutelike</u> modes such that $\tilde{\phi}_m(\theta, r)$ is nearly independent of θ, in the sense that

$$\frac{\partial\tilde{\phi}_m}{\partial\theta} \ll n^\circ \frac{dq}{dr}(r-r_o) = \frac{m^\circ}{q}\frac{dq}{dr}(r-r_o) \equiv S. \tag{19}$$

Now we have

$$\Phi_1 = \tilde{\phi}_m(r)\exp(i\omega t + iS\theta) \tag{20}$$

and in the case of trapped particles the quantity $\exp[iS\theta(t')]$ entering the integration of Vlasov's equation along particle orbits can be decomposed in harmonics of the orbit periodicity so that

$$e^{iS\theta(\hat{t})} = \sum_p D_T^{(p)}(\Lambda, S)\, e^{ip\omega_b\hat{t}}. \tag{21}$$

For circulating particles we can write, for $\omega_t = 2\pi/\tau_t$,

$$e^{iS\theta(t)} = e^{iS\omega_t \hat{t}} \sum_p D_C^{(p)}(\Lambda, S)\, e^{ip\omega_t \hat{t}}. \tag{22}$$

If we assume that

$$S \frac{v_{thi}}{qR} < \omega, \tag{23}$$

the perturbed ion density can be obtained from moment equations,[6] so that, for $\rho_i^2 = T_i/(m_i\Omega_i^2)$ and $\Omega_i = eB/(m_ic)$,

$$\frac{\tilde{n}_i}{n} = -\frac{e}{T_i}\left\{\frac{\omega_{*i}}{\omega} - \frac{1}{2}\rho_i^2\left(1 - \frac{\omega_{*i}}{\omega} - \frac{\omega_{T_i}}{\omega}\right)\left(\frac{\partial^2}{\partial r^2} - \frac{m^{\circ 2}}{r_o^2}\right)\right.$$
$$\left. - S^2 \frac{T_i}{m_iR_o^2q_o^2\omega^2}\left(1 - \frac{\omega_{*i}}{\omega} - \frac{\omega_{T_i}}{\omega}\right)\right\}\tilde{\phi}_m(r). \tag{24}$$

Notice that the quantity $S^2/\left(R_o^2q_o^2\right)$ can be written $\left(m^\circ/r_o\right)^2\ (r-r_o)^2/L_s^2$, since we define

$$\frac{1}{L_s} = \frac{B_\theta}{B_\zeta}\frac{1}{q}\frac{dq}{dr} \tag{25}$$

and $B_\theta/B_\zeta \approx r_o/(R_oq_o)$. Also, $\omega_{*i} = -\omega_{*e}\ T_i/T_e$ and $\omega_{T_i} = \omega_{*i}(d\ln T_i/dr)/(d\ln n/dr)$. We ignore the contributions of the terms $D_T^{(p)}$ and $D_C^{(p)}$ for $p \neq 0$, and write the perturbed electron density as

$$\frac{\tilde{n}_e}{n} = \frac{e}{T_e}\tilde{\phi}_m(r)\left\{1 - \left(1 - \frac{\omega_{*e}}{\omega}\right)\frac{n_T}{n} - \frac{1}{2n}\int_C d^3v\, f_{M_e}\sum_\sigma\right.$$
$$\left.\times \frac{\omega - \omega_{*e} + \omega_{T_e}\left(\frac{3}{2} - \frac{\epsilon}{T_e}\right) + \sigma\left\{\omega\left[1 + \frac{\partial}{\partial(\epsilon/T_e)}\right]\right\}\left(\frac{\langle v_\parallel\rangle}{\nu(\epsilon/T_e)}\frac{eE}{2T_e}\right)}{\omega - S|\omega_t|\sigma}\right\}. \tag{26}$$

If we consider the limit

$$\omega < S\frac{v_{the}}{qR} < \overline{\omega_{te}}, \overline{\omega_{be}}, \tag{27}$$

we can retain only the resonant contribution of the last term. We recall that $\omega_t \propto (\epsilon)^{1/2} F(\Lambda)$, where $F(\Lambda)$ is a weak function of Λ, and we appropriately ignore the contribution of the barely circulating particles. Therefore the resonant electrons will correspond to

$$\frac{\epsilon}{T_e} \sim \left(\frac{\omega}{\bar{\omega}_{te} S}\right)^2.$$

Thus we can rewrite the relevant integral after summing over σ, as

$$\int_C \frac{\left[\omega - \omega_{*e} + \left(\frac{3}{2} - \frac{\epsilon}{T_e}\right)\omega_{T_e}\right] \omega - \omega S|\omega_t| \frac{2u_o}{v_{the}} \frac{\epsilon}{T_e} (1-\Lambda)^{1/2} 1(\Lambda_c - \Lambda)}{\omega^2 - S^2\omega_t^2} f_{Me} \, d^3\vec{v} \tag{28}$$

Now the term in square brackets is of order $\omega_{Te}\omega_{*e}$, while the term containing the current is of order $(u_o/v_{the}) \, \omega_{*e}^4/(S\bar{\omega}_{te})^2$. Therefore, instability could occur only if

$$\frac{u_o}{v_{the}} > \frac{S^2\bar{\omega}_{te}^2 \omega_{T_e}}{\omega_{*e}^3} \tag{29}$$

which is impossible with $\omega_{T_e} \sim \omega_{*e}$.

A significant growth rate can be obtained for current-driven instabilities if the current-carrying distribution can be represented by the model

$$\hat{f}_{eC} = \hat{f}_{eT} - \sigma\langle v_{\|}\rangle \frac{u_o}{v_{the}^2} 1(\Lambda_c - \Lambda) \tag{30}$$

and the electron temperature gradient can be ignored so that $\omega_{*e} \gg \omega_{T_e}$. These are rather unrealistic assumptions. So, as we shall see, current-driven modes of the type under consideration acquire practical significance only in the limit where the effects of electron collisions become important and replace those of wave-particle resonances. Then, for the sake of completeness, we point out that the equation for $\tilde{\phi}_m(r)$ corresponding to the expression (30) for f_{eC} is

$$-\frac{\delta\omega}{\omega_*}\left(1-\frac{n_T}{n}\right)\tilde{\phi}_m(r-r_o)+\left(1+\frac{T_i}{T_e}\right)\left[\frac{T_eS^2}{m_i\omega^2R^2q^2}-\frac{T_e}{m_i\Omega_i^2}\left(\frac{\partial^2}{\partial r^2}-\frac{m^{\circ 2}}{r_o^2}\right)\right]$$

$$\tilde{\phi}_m(r-r_o)+i\frac{u_o}{v_{the}}(\text{sign } S)\frac{\omega^2}{S^2\bar{\omega}_{te}^2}\tilde{\phi}_m(r-r_o)=0. \tag{31}$$

It is possible to see that the relevant modes are of the convective type, with respect to the $r-r_o$ variable. We follow a procedure described elsewhere[7] and consider a wave packet propagating in the direction where the sign of S is such as to give instability. Thus assuming a WKB solution of the form

$$\tilde{\phi}_m(r-r_o)\approx\tilde{\tilde{\phi}}_m\, e^{ik_r(r-r_o)}, \tag{32}$$

where k_r is complex and ω is real, we evaluate the amplification

$$e^{i\int \text{Im}\, k_r(r-r_o)\, d(r-r_o)}. \tag{33}$$

Unlike the type of convective modes treated by Coppi et al.[7] current-driven modes tend to be amplified when propagating in one direction along r and to be damped when propagating in the opposite direction. If we consider the limit $k \gg m^\circ/r_o$, we obtain

$$k_I=\frac{1}{\rho_i^{(e)}}\left\{\frac{1}{2}\left[\left(\frac{(r-r_o)^2L_s^2}{\rho_i^{(e)2}r_n^2}-\delta\hat{\omega}\right)^2+\left(\frac{\bar{\bar{u}}_o}{v_s}\right)^2\left(\frac{m_e}{m_i}\right)^3\left(\frac{(r-r_o)^2L_s^2}{\rho_i^{(e)2}r_n^2}\right)^2\right]^{1/2}-\frac{1}{2}\left(\frac{(r-r_o)^2L_s^2}{\rho_i^{(e)2}r_n^2}-\delta\hat{\omega}\right)\right\}, \tag{34}$$

where $r_n\equiv -n/(dn/dr)$, $\rho_i^{(e)}\equiv v_s/\Omega_i$, $v_s=(T_e/m_i)^{1/2}$, $\delta\hat{\omega}\equiv(\delta\omega/\omega_{*e})(1-n_T/n)\,T_e/(T_e+T_i)$, $\bar{u}_o=u_e[T_e/(T_e+T_i)]$, $\bar{\omega}_{te}=v_{the}/(R_oq)$, and L_s is defined by Eq. 25. If we choose for simplicity $\delta\hat{\omega}<\left[(r-r_o)L_s/\left(\rho_i^{(e)}r_n\right)\right]^2$, we obtain the rough criterion

$$\frac{r_n}{L_s}>\left(\frac{\bar{\bar{u}}_o}{v_s}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^{3/4} \tag{35}$$

by demanding that the rate of energy extraction from resonating electrons be smaller than that of convection.

Finally, we consider the case of nonconvective drift modes, with relatively short wavelengths along the magnetic field, which are not affected by magnetic shear.[8,2] Since the frequency of these modes for realistic parameters of toroidal plasma experiments is less than $\bar{\omega}_{be}$, the relevant frequency range is

$$\frac{v_{thi}}{qR_o} < k_\zeta v_{thi} < \omega < \frac{v_{the}}{qR_o} < k_\zeta v_{the},$$

where, in particular, $k_\zeta qR > R/r$. Therefore, the electrons resonating with these waves have velocity $v_{\|}' = \omega/k_\zeta \ll v_{the}$. In particular the total energy of these electrons is less than the thermal energy and they do not carry a significant fraction of the current. Therefore these electrons do not contribute a significant growth rate to the instability.

Collisional Instabilities Arising from Finite Thermal Conductivity

The even modes that we have considered have wavelengths considerably longer than the magnetic field periodicity length. It is therefore realistic to consider regimes in which the mean-free path is shorter than the mode longitudinal wavelengths. That is, if $k_{\|} = \left(m^\circ/r_o\right)(r-r_o)/L_s$, we treat the limit

$$\frac{\omega}{k_{\|} v_{the}} < k_{\|}\lambda_e < 1, \tag{36}$$

where λ_e is the electron-electron collision mean-free path.

In this limit, a current-driven instability associated with finite electron thermal conductivity has been found theoretically.[2] This has been confirmed experimentally by Ellis and Motley.[9] In the simplified treatment here we omit the damping effects of ion viscosity and thermal conductivity[2] because the instability criterion that we derive from the influence of magnetic shear is sufficiently stringent to make the former effects unimportant. The momentum-conservation equation for circulating electrons is

$$-ik_{\|}(\tilde{n}_C T_e + n_C \tilde{T}_{eC}) + ik_{\|} e n_C \tilde{\phi} \tag{37}$$

and the corresponding energy-conservation equation is

$$\frac{3}{2} n_C \frac{dT_{eC}}{dt} + ik_{\|} T_e \tilde{u}_{e\|} = -k_{\|}^2 \chi_e \frac{T_e}{m\nu_e} n_C \tilde{T}_{eC}, \tag{38}$$

where in the limit of (36) the first term on the left-hand side is negligible, ν_e is the average electron-electron collision frequency, and χ_e is a numerical coefficient specified, for instance, by Braginskii.[10] The mass-conservation equation is

$$-i\omega\tilde{n}_{eC} - i\,\frac{m^\circ}{r}\,\frac{c\tilde{\phi}}{B}\,\frac{dn}{dr} + ik_{\|}n_C\tilde{u}_{e\|} = 0. \tag{39}$$

Then we find the perturbed density of circulating electrons as

$$\tilde{n}_{eC} = \frac{e\tilde{\phi}}{T_e}\,\tilde{n}_C\left\{1 - i\,\frac{\nu_e(\omega-\omega_*-k_{\|}u_e)}{\chi_e k_{\|}^2 T_e/m_e}\right\}. \tag{40}$$

For simplicity, we have omitted in (40) the term corresponding to a transverse gradient of the electron equilibrium temperature. We see that if $\omega \approx \omega_{*e}$, the sign of the imaginary term changes with the sign of $(r-r_o)/r_o$. Therefore amplification of wave packets occurs only when they propagate in the $k_{\|}u_e$ positive direction. The perturbed density for trapped electrons, with the effects of collisions ignored, is simply given (recall Eq. 26) by the mass conservation equation in the form[5]

$$-i\omega\tilde{n}_{eT} + i\,\frac{m^\circ}{r}\,\frac{\tilde{\phi}c}{B}\,\frac{dn_T}{dr} = 0. \tag{41}$$

The perturbed ion density, under the assumptions previously specified, is given by Eq. 24. For simplicity, we neglect the terms in ω_{Ti} containing the ion temperature gradient.

Imposing the quasi-neutrality condition, we finally arrive at the dispersion equation

$$\left(1 - \frac{n_T}{n}\right)\left(1 - \frac{\omega_{*e}}{\omega}\right)\tilde{\phi} - \left(1 + \frac{\omega_{*e}}{\omega}\,\frac{T_i}{T_e}\right)\frac{T_e}{m_i}\left[\frac{1}{\Omega_i^2}\left(\frac{\partial^2}{\partial r^2} - \frac{m_\circ^2}{r_\circ^2}\right) + \frac{1}{\omega^2}\,k_{\|}^2\right]\tilde{\phi} - i$$

$$\times\,\frac{\nu_e(\omega-\omega_{*e}-k_{\|}u_e)}{\chi_e k_{\|}^2 T_e/m_e}\,\tilde{\phi} = 0. \tag{42}$$

Then considering the limit $\partial/\partial r > m^\circ/r_o$, for $\omega = \omega_{*e} + \delta\omega$ we have

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{(r-r_o)^2}{\delta^4} - \left[\frac{\delta\omega}{\omega_*} + \frac{i}{\chi_e}\,\frac{r_o L_s \nu_e u_{e\|}}{m^\circ(r-r_o)T_e/m_e}\right]\frac{m_i\Omega_i^2}{(T_e+T_i)(1-n_T/n)}\right\}\tilde{\phi} = 0, \tag{43}$$

where $\delta^2 = \left[L_s T_e/\left(r_n m_i \Omega_i^2\right)\right]$. Now we can argue that stability will occur when the rate at which energy is extracted from the electrons is equal to the rate of convection,[7,11] which (refer to Eq. 43) corresponds to

$$\frac{1}{\delta^2} \gtrsim \frac{1}{\chi_e} \frac{r_o L_s \nu_e u_{e\parallel}}{m^\circ (T_e/m_e)\,\delta} \left(\frac{m_i \Omega_i^2}{(T_e+T_i)(1-n_T/n)}\right), \tag{44}$$

and leads to the condition

$$\frac{r_n}{L_s} > \left(\frac{u_{e\parallel}}{v_s} \frac{m_e}{m_i} \frac{\nu_e}{\omega_{*e}}\right)^{2/3} \left[\frac{T_e}{\chi_e (T_e+T_i)(1-n_T/n)}\right]^{2/3} \tag{45}$$

which is considerably more severe than that obtainable for the collisionless regime. A more precise criterion can be derived by considering and evaluating the full amplification integral $\int k_I(r)\,dr$. From Eq. 42 we have

$$k_I = \left\{\frac{1}{2}\left[\left(\frac{(r-r_o)^2}{\delta^4} - \frac{\delta\omega}{\omega_{*e}\hat{\rho}_i^2}\right)^2 + \left(\frac{r_o L_s \nu_e u_e}{\chi_e \hat{\rho}_i^2 m^\circ (r-r_o)\, T_e/m_e}\right)^2\right]^{1/2} - \frac{1}{2}\left(\frac{(r-r_o)^2}{\delta^4} - \frac{\delta\omega}{\omega_{*e}\hat{\rho}_i^2}\right)^2\right\}, \tag{46}$$

where $\hat{\rho}_i \equiv (T_e+T_i)(1-n_T/n)/\left(m_i \Omega_i^2\right)$, and the value of $\delta\omega/\omega_{*e}$ is so chosen that the wave-packet amplification is maximized.[7]

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3. ELECTRON NEGATIVE-ENERGY MODES IN TWO-DIMENSIONAL PLASMAS

To understand the macroscopic transport properties of two-dimensional confined plasmas, we need knowledge of the modes that can be excited in them.[1] In particular, an important question is whether the orbit of deeply trapped electrons in a toroidal confinement configuration can be significantly altered by the collective modes[2] to which the plasma is subject. Analysis of such modes leads to identification of the following requirements: (i) they exist at frequencies $\omega \sim \hat{\omega}_{be}$, where $\hat{\omega}_{be}$ is the average bounce frequency of trapped electrons; (ii) the profile of the resulting electric-field fluctuations is correlated with the periodic variation of the magnetic field and is nonzero and even around the point of minimum magnetic field; (iii) they cannot be damped by the process of resonant interaction with trapped electrons. This last requirement can be met, for instance, if the relevant modes are of negative energy,[3] in the sense that they tend to grow when positive energy is transferred to the resonant particles from the wave.

In a symmetric torus in which the magnetic field can be represented as $B \approx B_o/[1+(r/R_o)\cos\phi]$, with r and R_o the minor and major radii of a magnetic surface, $\hat{\omega}_{be}$ is of order $(r/R_o)^{1/2}\, v_{the}/(2qR_o)$, where v_{the} is the electron thermal velocity, and $q \approx rB_T(R_oB_p)$, B_T and B_p being the toroidal and poloidal magnetic field components, respectively. An analysis of the possible modes that can be excited with frequency $\omega \sim \hat{\omega}_{be}$ on the basis of the microinstabilities known to occur in one-dimensional equilibrium configurations leads to unrealistic results. For instance, if we consider drift modes with electric potential $\tilde{\Phi} = \tilde{\phi}(x)\exp\left(-i\omega t - im^o\theta + ik_\zeta\zeta\right)$, where ζ is the direction of the magnetic field, we obtain

$$\omega \approx \omega_{*e}\, \frac{F(b_i)}{1+(T_e/T_i)[1+F(b_i)]} \lesssim \frac{1}{2\sqrt{\pi}}\left(\frac{T_e}{T_e+T_i}\right)\frac{v_{thi}}{r_n}. \tag{1}$$

Here $\omega_{*e} = (m^o/r)[cT_e/(eBn)]\, dn/dr = m^o(\rho_e/r)\, v_{the}/(2r_n)$ with ρ_e the electron Larmor radius, r_n indicates the density gradient scale distance, $b_i = \frac{1}{2}\left(m^o\rho_i/r\right)^2$ with ρ_i the ion Larmor radius, v_{thi} is the ion thermal velocity, $F(b_i) \equiv I_o(b_i)\exp(-b_i)$ with I_o the known modified Bessel function. Therefore the condition $\omega \gtrsim \omega_{be}$ would require

$$\frac{r_n}{R_o q}\left(\frac{r}{R}\right)^{1/2} < \left(\frac{m_e}{m_i}\frac{T_i}{\pi T_e}\right)^{1/2} \frac{T_e}{T_e + T_i},$$

which is not satisfied in realistic diffuse-pinch configurations.

We could also consider ion-sound waves, but these are not likely to be excited in experiments in which T_e is not much larger than T_i and the electron drift velocity is much less than v_{the}, as is usually the case.

The modes that we shall find here satisfy all three requirements that we have indicated and, unlike the known drift modes, they can be made unstable by a gradient of the electron temperature in the direction of the density gradient. For a two-dimensional toroidal configuration that is inhomogeneous and periodic in θ the appropriate normal modes are of the form $\tilde{\Phi} = \tilde{\phi}_{m^o,n^o}(\theta, r)\exp(-i\omega t - im^o\theta + in^o\zeta)$. We consider in particular those modes that are radially localized around a rational surface $r = r_o$ such that $q(r_o) = m^o/n^o$. The longitudinal electric field, which is important for the resonant mode-particle interaction, is $\tilde{E}_{\parallel} = \vec{E}_1 \cdot \vec{B}/B = -(1/r_o)(\partial/\partial\theta)\tilde{\phi}_m(\theta)(B_P/B_T)$, where $\vec{E}_1 = -\nabla\tilde{\Phi}$ and $\tilde{\phi}_m(\theta) \equiv \tilde{\phi}_{m^o,n^o}(r_o, \theta)$. Since we are interested in the interactions with deeply trapped electrons, we shall consider modes with $\tilde{\phi}_m(\theta)$ odd (in θ) around $\theta = 0$, so that $\tilde{E}_{\parallel}$ is even and nonzero around the same point.

We consider the frequency range $\hat{\omega}_{bi} < \hat{\omega} \lesssim \hat{\omega}_{be}$ and short transverse wavelengths so that $\omega_{*e} \gtrsim \hat{\omega}_{be}$. This implies $b_i \gg 1$ and the perturbed ion distribution function is then[1]

$$\tilde{n}_i = -\frac{en_i}{T_i}\tilde{\phi}_m. \tag{2}$$

To determine the perturbed electron density $\tilde{n}_e$, we derive the perturbed electron distribution, in the guiding-center approximation, by integrating the linearized Vlasov equation along particle orbits.[1] In order to perform this integration, we decompose $\tilde{\phi}_m(\theta)$ in harmonics of the orbit periodicity so that $\tilde{\phi}_m(\theta) = \sum_p \tilde{\Phi}_m^{(p)}(\Lambda)\exp(ip\omega_b\hat{t})$, where $\Lambda = \mu B_o/\epsilon$ with $\mu = mv_\perp^2/(2B)$ the particle magnetic moment and $\epsilon = m\left(v_\parallel^2 + v_\perp^2\right)/2$ the energy, $\omega_b\hat{t} \equiv 4\pi\left(\int^\theta d\theta'/v_\parallel\right)/\oint(d\theta'/v_\parallel)$ with $v_\parallel = (2\epsilon/m)^{1/2}[1 - \Lambda B(\theta)/B_o]^{1/2}$, so that $\omega_b\hat{t}$ is a function of θ and Λ only.

The unperturbed electron distribution is assumed to be of the form $f_e = f_{M_e}(1+\hat{f}_e)$, where f_{M_e} is the Maxwellian with temperature T_e, and $\hat{f}_e = (v_\parallel/|\Omega_{pe}|)[(dn_e/dr)/n_e - (dT_e/dr)(3/2 - \epsilon/T_e)/T_e]$. This form of $\hat{f}_e$ is appropriate for trapped electrons in regimes in which their average collision frequency $\langle \nu_e \rangle$ is smaller than $\hat{\omega}_{be}$, and we do not consider a different form for the circulating electron,[4] since this will not influence the result, in the limit $h_o/r_o \gg 1$. Here $|\Omega_{pe}| = eB_p/(m_e c)$. Then we are led[1,2] to

$$\tilde{n}_e = (en/T_e)\left\{\tilde{\phi}_m - n^{-1}\int d^3\vec{v}\, f_{M_e}\left[\omega - \omega_{*e} + \omega_{T_e}(3/2 - \epsilon/T_i)\right]\right.$$

$$\left.\cdot \sum_p \left[\tilde{\Phi}_m^{(p)}(\Lambda)\exp(ip\omega_b\hat{t})\right]/(\omega - p\omega_b)\right\}, \tag{3}$$

where $\omega_{Te} = \omega_{*e}(d\ln T_e/dr)/(d\ln n/dr)$.

We consider Poisson's equation $\left(m^o/r_o\right)^2\tilde{\Phi} = 4\pi e(\tilde{n}_i - \tilde{n}_e)$ and take the quadratic form

$$\left(m^o/r_o\right)^2 \oint d\ell\ |\tilde{\phi}_m|^2/B - 4\pi e\oint d\ell\ \tilde{\phi}_m^*(\tilde{n}_i - \tilde{n}_e)/B = 0, \tag{4}$$

where $d\ell = R_o q_o d\theta$. Then, for $\left(m^o\lambda_D/r_o\right)^2 < 1$, we obtain[1]

$$(1+T_e/T_i)\, n\oint d\ell\ |\tilde{\phi}_m|^2/B - (\pi/2)(2/m_e)^2\iint d\epsilon d\mu\ f_{M_e}(\epsilon)|\tau|$$

$$\left[\omega - \omega_{*e} + \omega_{T_e}(3/2 - \epsilon/T_e)\right]\sum_{p\neq 0}\left[|\tilde{\Phi}_m^{(p)}(\Lambda)|^2/(\omega - p\omega_b)\right] = 0, \tag{5}$$

where λ_D is the Debye length, $\tau = \frac{1}{2} R_o q_o \oint d\theta/|v_{\parallel}|$ for trapped particles, $\tau = \frac{1}{2} R_o q_o \int_0^{2\pi} d\theta/v_{\parallel}$ for circulating particles, and $\omega_b = 2\pi/\tau$. We have also expressed $\int d^3\vec{v}$ as $(\pi/2)\int d\mu d\epsilon\ B/|v_{\parallel}|$ with the convention that contributions from positive and negative values of $v_{\parallel}$ are to be added, and have made use of the fact that $\int d\ell/B \iint d\mu d\epsilon\, B/|v_{\parallel}| = \iint d\mu d\epsilon\ |\tau|$.

For simplicity, we consider the limits $\omega/\hat{\omega}_{be} < 1$ and $R_o/r_o > 1$ (large aspect ratio). We expand Eq. 5 in these two parameters and carry out the integration over ϵ to obtain

$$\left(1 + \frac{T_e}{T_i}\right)\int_0^{2\pi} d\theta\ |\tilde{\phi}_m|^2 - \left(\frac{R_o}{r_o}\right)^{1/2}\frac{\left(\omega_{*e} - \omega_{T_e}\right)\omega}{2\pi^2\hat{\omega}_{ce}^2}\int_0^1 d\chi^2\mathscr{L}^3(\chi^2)\sum_{p>0}|\tilde{\Phi}_m^{(p)}(\chi^2)|^2/p^2$$

$$+\ i\sqrt{\pi}\left(\frac{R_o}{r_o}\right)\frac{\left(\frac{3}{2}\omega_{T_e} - \omega_{*e}\right)\omega^2}{4\pi^3\hat{\omega}_{ce}^3}\int_0^1 d\chi^2\mathscr{L}^4(\chi^2)\sum_{p>0}|\tilde{\Phi}^{(p)}(\chi^2)|^2/p^3 = 0. \tag{6}$$

In order to derive Eq. 6 from Eq. 5 we have observed that $\sum_p |\tilde{\Phi}_m^{(p)}|^2/(\omega-p\omega_b) = \sum_{p>0} |\tilde{\Phi}_m^{(p)}|^2\ 2\omega/\left(\omega^2-p^2\omega_b^2\right)$, $(\omega-p\omega_b)^{-1} = P(\omega-p\omega_b)^{-1} + i\pi\delta(\omega-p\omega_b)$, where $\delta(\omega-p\omega_b) = (\epsilon/|\omega|)\ \delta(\epsilon-p\epsilon(\chi))$, $\epsilon(\chi) = (\omega R_o q_o)^2 \mathscr{L}^2(\chi)(R_o r_o)$, and $P\left[2\omega/\left(\omega^2-p^2\omega_b^2\right)\right] \approx -2\omega/(p\omega_b)^2$ in the considered limit $\omega/\hat{\omega}_{be} < 1$. We also have defined $\hat{\omega}_{ce} = v_{the}/(R_o q_o)$ as the average transit frequency, chosen m^o so that $\omega > 0$, and taken $\chi^2 \equiv \frac{1}{2}\left[1+(1-\Lambda)R_o/r_o\right]$ so that $0 \leqslant \chi \leqslant 1$ is equal to half of the excursion amplitude in θ of deeply trapped particles. Also, $\mathscr{L}(\chi^2) \approx \frac{1}{2} \oint d\theta/(2\chi^2-1+\cos\theta)^{1/2}$.

The last term in Eq. 6 results from resonances of the considered wave with trapped particles that have bounce frequency $\omega_b = \omega/p$. The second term, which is larger by a factor of order $\hat{\omega}_{be}/\omega$, has no correspondence in the stability theory of one-dimensional plasmas involving resonances of the form $\omega = k_\zeta v_{\parallel}$, so that, in the limit $\omega < k_\zeta v_{th}$, the resonant contribution to the dispersion relation is of order $\omega/k_\zeta v_{th}$ instead of $\omega^3/\hat{\omega}_{be}^3$, as it is in the present case.

If we neglect the last (resonant) term in Eq. 6, we can use the remaining quadratic form as a variational form in order to evaluate ω. Thus the imaginary part of ω is obtained as a perturbation, and in this sense we can estimate the stability of the modes under consideration by

$$\left(1+\frac{T_e}{T_i}\right) - \left(\frac{R_o}{r_o}\right)^{1/2} \frac{\left(\omega_{*e}-\omega_{T_e}\right)\omega}{\hat{\omega}_{ce}^2} \mathscr{T}_1 - i\left(\frac{R_o}{r}\right) \frac{\left(\frac{3}{2}\ \omega_{T_e}-\omega_{*e}\right)\omega^2}{\hat{\omega}_{ce}^3} \mathscr{T}_2 = 0. \qquad (7)$$

Here $\mathscr{T}_1$ and $\mathscr{T}_2$ are positive numbers resulting from the evaluation of the integrals in Eq. 6 when $\tilde{\phi}$ is replaced by a trial function $\tilde{\tilde{\phi}}$ which is found by applying the variational principle.

Now we see that instability is found for $2/3\ (d\ln n/dr) < (d\ln T_e/dr) < (d\ln n/dr)$, and this is compatible with the assumption that $\omega < \hat{\omega}_{be}$. The general quadratic form of Eq. 5, however, furnishes no evidence that these modes disappear for $\omega \sim \hat{\omega}_{be}$. When $d\ln T_e/dr \to d\ln n/dr$, $\omega \to \hat{\omega}_{be}$ and a simple analytical treatment is no longer possible. We also recall that the known electron drift modes[1] are damped by the contribution of ω_{T_e} when $(d\ln T_e/dr)/(d\ln n/dr) > 0$ and are such that $\tilde{n}$ and $\tilde{\phi}$ are out of phase. Instead, for the modes present here, $\tilde{n}$ and $\tilde{\phi}$ are in phase, as indicated by Eq. 2, and we expect, on the basis of quasi-linear theory, that they lead to electron thermal-energy transport across the magnetic field without a corresponding particle transport.

We refer to the quadratic form (4) and define an effective dielectric constant ϵ in terms of the integrals $\mathscr{T}_1$ and $\mathscr{T}_2$ in Eq. 7. Then we may argue that the wave energy is proportional to $\omega\partial\epsilon/\partial\omega = -\left[r_o/\left(m^o\lambda_D\right)\right]^2 (1+T_e/T_i)$ which is evidently negative. We can also see that most of the electrons (linearly) resonating with this wave are barely trapped.

The influence of well-developed modes of the type considered here on the orbit of deeply trapped ions[5] has been studied analytically and numerically.[6] It has been found[6] that, depending on the value of the parameter $\alpha^{1/3}$, where $\alpha = e\tilde{\phi}_c R_o/(\mu B_o r) \sim (R_o/r)(e\tilde{\phi}_c/T)$ and $\tilde{\phi}_c$ is the characteristic mode amplitude, the trapped particle excursions can be amplified up to $\sim 2(16\alpha)^{1/3}$ but still remain trapped if α is sufficiently small. If α is larger than a reasonable value, [such as $\alpha \approx 0.15$ which has been obtained[6] with an appropriate choice of $\tilde{\phi}_m(\theta)$] the resonating particles can be untrapped. Correspondingly, the radial particle excursions can be considerably larger than the known banana width of trapped particles, and the average magnetic curvature drift be more favorable to the stability of interchange[7] modes than in the case with fluctuations absent.

Therefore, in the presence of modes that are odd in $\tilde{\phi}_m$ and have frequency close to the average bounce frequency of the ions and the electrons, we can have "quasi-banana" orbits, with relatively large amplitudes for the particles that remain trapped, or circulating orbits for a considerably larger fraction of the particle population than has been estimated from theories ignoring the effects of fluctuations. We can infer that the stability of lower frequency modes[7,8] such as the trapped particle interchange modes, and the particle and energy transport[9] across the magnetic field will have to be reevaluated by taking these effects into proper account.

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