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A. AN OPTIMUM RECEIVER FOR THE BINARY COHERENT STATE QUANTUM CHANNEL

National Aeronautics and Space Administration (Grant NGL 22-009-013)

S. J. Dolinar, Jr.

Recently an (ideally) implementable receiver for the detection of binary coherent-state signals has been discovered[1] whose performance is exponentially optimum, differing from that of the optimum quantum receiver by at most a factor of two in the error probability. By suitably generalizing the structure of that receiver, we have been able to obtain one that achieves quantum optimal performance precisely for this particular detection problem.

Our problem is to decide, with minimum probability of error, between two possible messages m_0 and m_1, with a priori probabilities π_0 and π_1, respectively, when the received field, conditioned on m_j being sent, is a linearly polarized narrow-band plane wave corresponding to the quantum coherent state $|s_j\rangle$, $j = 0, 1$. The minimum error probability for this problem is well known,[2]

$$P(\epsilon) = \frac{1}{2}\left[1 - \sqrt{1 - 4\pi_0\pi_1 |\langle s_0|s_1\rangle|^2}\right], \tag{1}$$

where $\langle s_0|s_1\rangle$ is the inner product of $|s_0\rangle$ with $|s_1\rangle$. For coherent states, the magnitude squared of this inner product may be simply expressed in terms of the complex envelopes $S_0(t)$, $S_1(t)$ of the classical received fields corresponding to the states $|s_0\rangle$, $|s_1\rangle$.

$$|\langle s_0|s_1\rangle|^2 = \exp\left(-\int_0^T \lambda(t)\, dt\right), \tag{2}$$

where we have defined $\lambda(t)$ to be the photon-counting rate for direct detection of a plane wave of complex envelope $S_o(t) - S_1(t)$, and $[0, T]$ is the signaling interval.

$$\lambda(t) = C|S_o(t) - S_1(t)|^2, \tag{3}$$

where C is a constant related to the aperture area, the impedance of space, and the energy of photons at the carrier frequency.

The near-optimum receiver processes the received field by adding to it a fixed linearly polarized local reference plane wave $\ell(t)$, equal to the negative of one of the signals $S_o(t)$ or $S_1(t)$, and direct-detecting the result with a photon counter. We generalize this concept by allowing the choice of the local reference field added at the receiver to depend causally on the actual output of the photon counter, via a feedback arrangement. This generalization immediately introduces complexities in the analysis of the photon counter output, since it is no longer a Poisson process conditioned on the message sent. It remains at least a regular point process, and so it is characterized by the set of times $t_1 < t_2 < \dots$ at which counts occur. Thus, the class of receivers that we are considering is identifiable with the set of all possible feedback functions, $\{\ell(t:\underline{t}),\ \underline{t} = (t_1, \dots, t_n)\colon 0 < t_1 < \dots < t_n < t < T,\ n = 0, 1, 2, \dots\}$. Here, $\ell(t:\underline{t})$ represents the feedback field at time t when the vector of counts observed prior to t is $\underline{t}$. The structure of such a receiver is illustrated in Fig. VII-1.

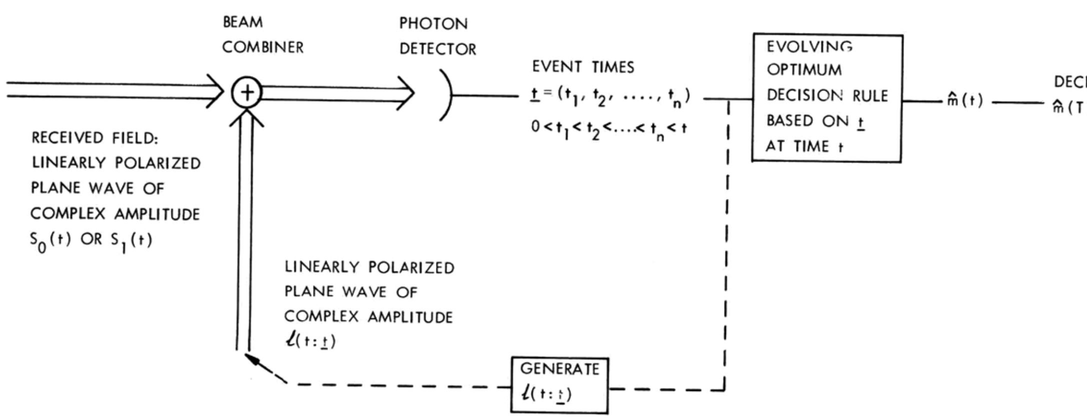

Fig. VII-1. Receiver structure considered here.

When a particular feedback function $\ell(\cdot)$ is specified, the statistics of the photon counter output may be determined from the following observation. Conditioned on the message m_j being sent and also on the entire history of event times $\underline{t}$ of the photon count process up to time t, the probability that a single count will occur in the small interval $(t, t+\Delta)$ is $\lambda_j(t:\underline{t})\Delta$, within $o(\Delta)$, and the probability of multiple counts in this interval is $o(\Delta)$, where

$$\lambda_j(t:\underline{t}) = C\left|S_j(t)+\ell(t:\underline{t})\right|^2. \tag{4}$$

This incremental analysis may be easily extended to yield the probability density $p_j(\underline{t}:T)$ for the set of event times $\underline{t}$ within the entire observation interval $[0, T]$, conditioned on m_j.

$$p_j(\underline{t}:T) = \left[\prod_{i=1}^{n} \lambda_j(t_i:\underline{t}_{i-1})\right] \prod_{i=1}^{n+1} \exp\left[-\int_{t_{i-1}}^{t_i} \lambda_j(\sigma:\underline{t}_{i-1})\, d\sigma\right], \tag{5}$$

where

$$\underline{t} = (t_1, \ldots, t_n) \qquad t_o \equiv 0$$

$$\underline{t}_{i-1} = (t_1, \ldots, t_{i-1}) \qquad t_{n+1} \equiv T.$$

The optimum decision rule is obvious from (5), and in principle the probability of error may be obtained by integrating the expressions in (5) over the appropriate decision regions in $\underline{t}$, for any feedback function $\ell(\cdot)$. As expected, explicit evaluation of performance is possible only for certain special cases, one of which is the optimum feedback function $\hat{\ell}(\cdot)$.

In this report, we shall describe, rather than derive, the optimum feedback $\hat{\ell}(\cdot)$, and then calculate performance and verify that it is quantum-optimal. We first define a deterministic weighting function $f_\gamma(t)$, which depends on $S_o(\cdot)$, $S_1(\cdot)$, and the ratio $\gamma \equiv \pi_1/\pi_o$. Without loss of generality, we assume $\gamma \geq 1$.

$$f_\gamma(t) = \frac{(1+\gamma)^2}{2\gamma} e^{m(t)} - 1 + \frac{1+\gamma}{2\gamma}\sqrt{(1+\gamma)^2 e^{2m(t)} - 4\gamma e^{m(t)}}, \tag{6}$$

where $m(t) = \int_0^t \lambda(\sigma)\, d\sigma = C \int_0^t \left|S_o(\sigma)-S_1(\sigma)\right|^2 d\sigma$. Next, we define two weighted combinations of the signals $S_o(t)$, $S_1(t)$.

$$\ell_1(t) = -\frac{S_1(t)\, f_\gamma(t) - S_o(t)}{f_\gamma(t) - 1} \tag{7a}$$

$$\ell_o(t) = -\frac{S_o(t)\, f_\gamma(t) - S_1(t)}{f_\gamma(t) - 1}. \tag{7b}$$

The optimum feedback function $\hat{\ell}(\cdot)$ depends very trivially on the observations $\underline{t}$:

$$\hat{\ell}(t:\underline{t}) = \begin{cases} \ell_1(t), \underline{t} = (t_1, \ldots, t_n), & n \text{ even} \\ \ell_o(t), \underline{t} = (t_1, \ldots, t_n), & n \text{ odd} \end{cases} \tag{8}$$

If we use this feedback function in (4) and (5) and make use of the fact that $f_\gamma(\cdot)$ satisfies the integral equation

$$\ln \left[\frac{f_\gamma(t)}{f_\gamma(t_o)}\right] = \int_{t_o}^{t} \lambda(\sigma) \left[\frac{f_\gamma(\sigma) + 1}{f_\gamma(\sigma) - 1}\right] d\sigma, \qquad 0 < t_o < t < T, \tag{9}$$

it is straightforward, albeit tedious, to obtain an expression for the likelihood ratio $\Lambda(\underline{t}: T) = p_1(\underline{t}: T)/p_o(\underline{t}: T)$.

$$\gamma\Lambda(\underline{t}: T) = [f_\gamma(T)]^{(-1)^n}, \qquad \underline{t} = (t, \ldots, t_n). \tag{10}$$

Since $f_\gamma(T) \geq 1$, the decision regions for minimum probability of error are very simply defined:

$$D_1 = \{\underline{t} = (t_1, \ldots, t_n) : \gamma\Lambda(\underline{t}: T) \geq 1\} = \{\underline{t} = (t_1, \ldots, t_n) : n \text{ even}\} \tag{11a}$$

$$D_o = \{\underline{t} = (t_1, \ldots, t_n) : \gamma\Lambda(\underline{t}: T) < 1\} = \{\underline{t} = (t_1, \ldots, t_n) : n \text{ odd}\}. \tag{11b}$$

The probability of error is given by

$$\begin{aligned} P(\epsilon) &= \int_{D_o} \pi_1 p_1(\underline{t}: T)\, d\underline{t} + \int_{D_1} \pi_o p_o(\underline{t}: T)\, d\underline{t} \\ &= \int_{D_o} \gamma\Lambda(\underline{t}: T)\, \pi_o p_o(\underline{t}: T)\, d\underline{t} + \int_{D_1} \frac{1}{\gamma\Lambda(\underline{t}: T)}\, \pi_1 p_1(\underline{t}: T)\, d\underline{t} \\ &= \int_{D_o} \frac{1}{f_\gamma(T)}\, \pi_o p_o(\underline{t}: T)\, d\underline{t} + \int_{D_1} \frac{1}{f_\gamma(T)}\, \pi_1 p_1(\underline{t}: T)\, d\underline{t} \\ &= \frac{1}{f_\gamma(T)} \left[\int_{D_o} \pi_o p_o(\underline{t}: T)\, d\underline{t} + \int_{D_1} \pi_1 p_1(\underline{t}: T)\, d\underline{t}\right]. \end{aligned} \tag{12}$$

The quantity in brackets is the probability of correct detection, or $1 - P(\epsilon)$. Solving for $P(\epsilon)$, we obtain

$$P(\epsilon) = \frac{1}{1+f_\gamma(T)}$$

$$= \frac{1}{2}\left[1 - \sqrt{1 - \frac{4\gamma}{(1+\gamma)^2}\, e^{-m(T)}}\right]$$

$$= \frac{1}{2}\left[1 - \sqrt{1 - 4\pi_o\pi_1\, e^{-m(T)}}\right]. \tag{13}$$

Recalling the definition of m(T), we see that this is identical to the quantum-optimal results in (1) and (2).

Let us briefly discuss some aspects of the behavior of our optimum receiver (Fig. VII-2). The feedback field alternates between $\ell_1(t)$ and $\ell_o(t)$ with each count. At the same time, $\ln \gamma\Lambda(\underline{t}:t)$ changes sign at each count, but its magnitude is a deterministically increasing function of time, $\ln f_\gamma(t)$. Thus the occurrence of each successive count constitutes a more and more conclusive negative test of the hypothesis last considered more probable. As time goes on, the optimum feedback

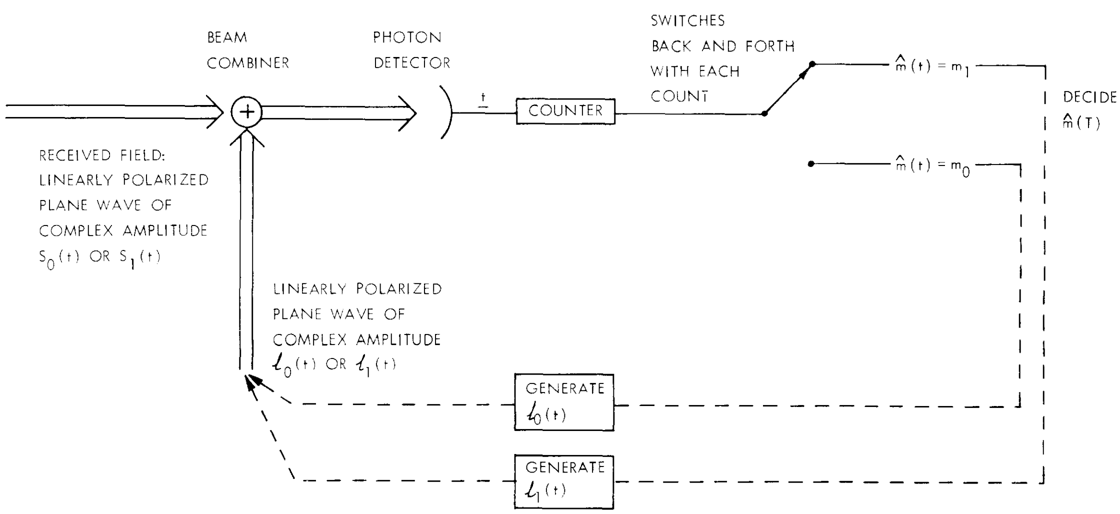

Fig. VII-2. Optimum receiver for known $S_o(t)$, $S_1(t)$.

alternates less frequently between $\ell_1(t)$ and $\ell_o(t)$ and with increasing probability tends toward nulling whichever signal was actually sent. In fact, if $\lim_{t\to\infty} m(t) = \infty$, there is, with probability one, a final count even in the case of an infinite signaling interval $(T=\infty)$.

We remark that in the case of equal a priori probabilities $(\gamma = 1)$, Eqs. 4, 7, and 8 provide only a formal description of the optimum processing, since the required feedback field, and hence the photon counting rate, are infinite at $t = 0$. It is possible to define rigorously the random process which represents the photodetector output for this optimum processor, but it is not a regular point process on $[0, T]$ and so the optimum receiver technically does not belong to the class we are considering. We point out, however, that we may approach optimum performance as closely as desired with bounded feedback fields by appropriately redefining $\hat{\ell}(t:\underline{t})$ for t in some small interval $[0, \epsilon)$, $0 < \epsilon < T$, and letting $\epsilon \downarrow 0$. One redefinition that has this convergence property is

$$\hat{\ell}_\epsilon(t:\underline{t}) = \begin{cases} -S_1(t), & t \in [0, \epsilon) \\ \hat{\ell}(t:\underline{t}), & t \in [\epsilon, T] \end{cases} \tag{14}$$

From a practical point of view, the foregoing result might seem insignificant, since an exponentially optimum receiver for this particular detection problem is already available. The attainment of precisely optimum performance, however, warrants an investigation into the possible connection between the type of feedback receiver considered here and quantum-optimal measurements for more general problems for which exponentially optimum receivers have not yet been found. Further study is being conducted on this question.

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