V. QUANTUM ELECTRONICS

A. Laser Applications

Academic Research Staff

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1. LONG-TERM LASER FREQUENCY STABILIZATION USING A MOLECULAR BEAM REFERENCE

U. S. Air Force Office of Scientific Research (Contract F44620-71-C-0051)

Shaoul Ezekiel, Lloyd A. Hackel, Douglas G. Youmans

We have recently completed the construction of two I_2 molecular beam stabilized argon lasers. The beat data confirmed the excellent long-term stability of a few parts in 10^{14}, which we have measured earlier by using only one laser and observing the laser frequency drift in an independent I_2 molecular beam. Laser intensity shift was shown to be less than five parts in 10^{14} for a 10% change in intensity. The reproducibility of the laser frequency at present is around one part in 10^{11}.

With the two independently stabilized argon lasers, we made a direct measurement of the frequency spacings between the I_2^{127} hyperfine-structure transitions associated with the P(13) and R(15) (0-43) lines. The data have been fitted to obtain improved values for the nuclear quadrupole coupling constants and the spin-rotation interaction strengths. The standard deviation of the best fit was 300 kHz, which is large compared with the uncertainty in our measurement of the line spacings. The effect of magnetic octopole interactions is now being considered.

Using one stabilized laser and an external acousto-optic frequency shifter, we are making precision measurements of the natural line shape of I_2 hyperfine structure in a molecular beam. At present, the instrument-limited linewidth of our technique is approximately one part in 10^{10} and further refinements are needed to make a 10% measurement of the natural line shape, which is expected to be 100 kHz wide.

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2. SINGLE-FREQUENCY CONTINUOUS-WAVE DYE LASER

Joint Services Electronics Program (Contract DAAB07-71-C-0300)

Shaoul Ezekiel, Robert E. Grove, Frederick Y-F. Wu

In the dye laser program we have reduced the linewidth in a jet stream cw dye laser to 100 kHz rms by locking the laser to an external Fabry-Perot cavity. With this laser we have demonstrated ultrahigh resolution spectra of hyperfine structure in a molecular beam of I_2^{127}. Linewidths of 350 kHz (HWHM) have been observed which are within a factor of 2 of the natural width as determined by lifetime measurements.

Furthermore, the dye laser has been stabilized to an I_2 transition in a molecular beam and a long-term stability of 6 parts in 10^{13} has been achieved.

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V. QUANTUM ELECTRONICS

B. Gaseous Lasers

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1. HIGH-PRESSURE CO_2 LASERS

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Joint Services Electronics Program (Contract DAAB07-71-C-0300)

John L. Miller, Arthur H. M. Ross, E. Victor George

A transverse discharge, spark-preionized carbon dioxide laser has been operated at total pressures above 12 atm. Pressure broadening of the 10.4 μm vibration-rotation transitions blends the Lorentzian line shapes into one another, thereby giving a small-signal gain characteristic with only small fluctuations from one line to the next. Net gain exists over the P and R branches of the 10.4 μm band, which makes broadband tuning and multiple-line operation of oscillators possible.

The discharge chamber was constructed of acrylic plastic with polished aluminum electrodes and KCl Brewster-angle windows. The active discharge volume between the modified Rogowski profile electrodes was estimated to be approximately 0.7 cm (electrode separation) $\times$ 1.0 $\times$ 23 cm. Electrical excitation was provided by either a discrete capacitive discharge circuit or a distributed Blumlein circuit, depending on the pressure. Preionization was provided by a single row of arcs of 2 mm length and 10 mm separation situated approximately 2 cm from the optical axis. Operation in a stable optical cavity, with output coupling from 10% to 70%, produced pulses of order 100 mJ. Pulse duration decreased with pressure from 150 ns at 2 atm to 70 ns at 8 atm. The output pulses were observed to have significant (5%) spectral content on as many as four P branch lines; the strongest line was P(20).

Comparison of the experimental results with a simple theory based on a four-temperature model of the molecular kinetics gives good agreement with respect to pulse shape, energy, and timing. These properties of the pulses can be explained entirely by the buildup of energy in the cavity modes by amplification of spontaneous emission. The spectral distribution of the output, however, is not in accord with the simple theory. JS

JS In particular, the lack of multiple-line oscillation at lower pressures is not adequately explained. Also, much more intense R lines would be expected than are observed. Since the relative amplitudes of the various rotational lines depend exponentially on the relative small-signal gain, small differences in gain can have appreciable effects on the spectral content, although the total intensity is insensitive to it.

In the simple theory, the gain spectrum arises from the assumption of a Boltzmann distribution over rotational states, and of a Lorentzian line shape for each transition. Two causes of deviation from this model are thought to be significant. First, all bands of the form $n_1 n_2^{\ell} n_3 \rightleftharpoons n_1+1,\ n_2^{\ell},\ n_3-1$ (of which $01^11 \rightleftharpoons 11^10$ is the principal contributor) fall at approximately the same wavelength, and therefore overlap some of the 001-100 lines. Second, at high densities the free rotation of the CO_2 molecules is hindered by their environment; hence, the vibration-rotation band, which arises (classically) from modulation of the molecular vibration by rotation, is narrowed.

Nevertheless, these effects, which contribute no more than a few percent of the overall gain, are sufficient to alter the spectral distribution significantly from that given by naive models.

2. ULTRAVIOLET LASERS

Joint Services Electronics Program (DAAB07-71-C-0300)
University of California, Livermore (Subcontract No. 7877409)

Charles W. Werner, E. Victor George, Paul W. Hoff,
Charles K. Rhodes

We have been studying the emission characteristics of high-pressure rare-gas systems. A model has been developed to explain the salient features of bound-free lasers with particular emphasis on high-pressure rare-gas excimer systems excited by an electron beam. Included are important kinetic processes, laser field evolution, and interaction of the field with the excimer population. The kinetic model encompasses such reactions as three-body formation of both a $^1\Sigma$ and $^3\Sigma$ molecular state, collapse from $^1\Sigma$ to $^3\Sigma$ via atomic or electron spin-exchange collisions, Penning ionization, dissociative recombination from the molecular ion, and feeding of the ^3P atomic manifold from higher lying atomic states. Energy balance equations were written for appropriate particle species and the resulting temperatures were incorporated into the kinetic reactions, as well as the emission and absorption processes.

The energy lost by the primary electron beam was partitioned in a manner similar to that described by Bass and Green.[1] In order to simplify the calculation of the various excitation rates and energy balance, the secondary electron distribution was assumed to be Maxwellian. The predicted temporal evolution of the spontaneous ultraviolet emission in the absence of laser action is seen to be in agreement with experiment.

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A random-phase approximation was used to obtain the temporal and spectral profiles of the laser pulse. Rate equations were written for the energy of a single mode in terms of gain, loss, and noise at the mode frequency. By considering the total spectral energy in a narrow band $d\nu$, we arrive at JS

$$\frac{dF(\nu)}{dt} = cG(\nu) - L(\nu)\,F(\nu) + S(\nu).$$

Here, $F(\nu)\,d\nu$ is the total field energy in the interval ν, $\nu+d\nu$ and is equal to the mode density $2L/c$ multiplied by the individual mode energy at ν, $G(\nu)$ is the net gain, $S(\nu)$ is the noise, and $L(\nu)$ is a loss that includes photoionization, and mirror and scattering losses. Similiarly, the induced depopulation rate may be obtained by considering the contribution from a single mode, summing over modes, and converting the sum to an integral in terms of $F(\nu)$. Thus we obtain

$$\left.\frac{dN}{dt}\right|_{\text{stimulated}} = -c\int_0^\infty \frac{G(\nu)}{h\nu}\,F(\nu)\,d\nu.$$

The gain and noise expressions in our model contained an energy-dependent dipole moment. Overlap integrals were constructed by using Morse potential functions to determine the variation of the dipole transition moment with ground-state energy. These calculations are used to compute the ground-state absorption, and comparisons with recent experiments are in good qualitative agreement.

The total dynamic model is used to determine both the temporal and spectral properties of the laser radiation. The predicted line narrowing and short temporal pulses agree well with experimental results.

A paper, entitled "Dynamic Model of High-Pressure Ultraviolet Lasers," by C. W. Werner, E. V. George, P. W. Hoff and C. K. Rhodes, was presented at the IEEE Quantum Electronics Conference, San Francisco, California, June 10-13, 1974, and has been accepted for publication in Applied Physics Letters.

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V. QUANTUM ELECTRONICS

C. Nonlinear Phenomena

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1. SHORT LASER PULSES: SLOW SATURABLE ABSORBER MODE-LOCKING SOLUTION

Joint Services Electronics Program (Contract DAAB07-71-C-0300)

U. S. Army Research Office – Durham (Contract DAHC04-72-C-0044)

Hermann A. Haus, Christopher P. Ausschnitt, Peter L. Hagelstein

[Peter L. Hagelstein is an undergraduate student in the Department of Electrical Engineering.]

We have reported previously a closed-form theory for mode locking a homogeneously broadened laser by a "fast" saturable absorber.[1] The mode-locked pulse was found to be a hyperbolic secant in time. In this report we investigate mode locking by a "slow" saturable absorber; that is, one in which the response time τ_a of the absorber is comparable to or slower than the rate of change of intensity in the laser cavity. In the limit of a "very slow" absorber we find a closed-form solution for the mode-locked pulse which is also a hyperbolic secant in time. The power of the pulse produced by the very slow absorber decreases with τ_a, whereas initially the pulse width approaches a constant independent of τ_a and much shorter than τ_a. As $\tau_a \to \infty$ the decrease in pulse power leads to quenching of the mode locking when the negative resistance of the laser medium, which is required to be below threshold for successful mode locking,[1] reaches threshold.

We can treat slow-absorber mode locking by a modification of the differential equation developed for the equivalent cavity current I(t) in the case of fast-absorber mode locking,[1] which is rewritten

$$\frac{Q}{Q_a^o}\left|\frac{I(t)}{I_a}\right|^2 I(t) = \left[1 - r\left(1 + \frac{1}{\omega_M^2}\frac{d^2}{dt^2}\right) + \frac{\delta}{\omega_m}\frac{d}{dt}\right] I. \tag{1}$$

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We define

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Q = cavity quality factor

Q_a^o = absorber quality factor in the absence of power

I_a^2 = saturation power of the absorber

ω_M = laser medium bandwidth

r = laser medium saturated negative resistance

δ = detuning parameter

ω_m = frequency spacing of mode-locked cavity modes.

The left-hand side of (1) is the injection voltage generated by the flow of current through the nonlinear current-dependent impedance of the absorber. The right-hand side represents the voltage across the cavity and laser medium impedance, where the laser line has been expanded to second-order about line center. The detuning parameter δ allows for a difference between cavity-mode and laser-medium reactances. Specifically,

$$\delta = \frac{\omega_m - \omega_{mo}}{\Delta\omega_c}, \tag{2}$$

where $\Delta\omega_c$ is the cavity mode bandwidth and ω_{mo} is the "tuned" mode separation frequency defined as the empty-cavity mode spacing $\Delta\omega$ modified by the laser medium dispersion

$$\omega_{mo} = \frac{\Delta\omega}{1 + \frac{r}{\omega_M}\frac{\omega_o}{2Q}}. \tag{3}$$

One of the conditions of the fast-absorber solution is that $\omega_m = \omega_{mo}$. In other words, the reactive components of the cavity and laser medium cancel in fast-absorber mode locking.[1] As we shall see, this is not the case for the slow absorber.

Equation 1 can be adapted to the case of the slow absorber by making the substitution

$$\left|\frac{I(t)}{I_a}\right|^2 \Longrightarrow e^{-t/\tau_a} \int_{-\infty}^{t} e^{t/\tau_a} \left|\frac{I(t)}{I_a}\right|^2 \frac{dt}{\tau_a} \tag{4}$$

for the response of the absorber impedance to the current, where τ_a is the relaxation time of the absorber. The right-hand side of (4) follows from our rate equation model of the absorber as a slow two-level system in which the fractional change in the lower level population is small. In the limit of large τ_a we can approximate the right-hand side of (4) by

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$$\int_{-\infty}^{t} \left|\frac{I(t)}{I_a}\right|^2 \frac{dt}{\tau_a} \tag{5}$$

so that the mode-locking equation (1) becomes

$$\frac{Q}{Q_a^o} I(t) \int_{-\infty}^{t} \left|\frac{I(t)}{I_a}\right|^2 \frac{dt}{\tau_a} = \left[(1-r)I + \frac{\delta}{\omega_m}\frac{d}{dt} + \frac{r}{\omega_M^2}\frac{d^2}{dt^2}\right]I. \tag{6}$$

If we now assume a pulse width τ_p such that $\tau_a \gg \tau_p \gg 1/\omega_M$, then the pulse spectrum is sufficiently narrow that the laser line can be assumed flat. Thus the last term in (6), which originates from the parabolic frequency dependence of the laser medium negative resistance near line center, can be neglected.

Deletion of the second-derivative term from (6) permits a solution of the form

$$I(t) = \frac{A}{\cosh\left(\frac{t}{\tau_p}\right)}. \tag{7}$$

If we assume a repetition rate T of the pulses, A^2 is related to the power by

$$P = \frac{1}{T}\int_{-\infty}^{\infty} \frac{A^2}{\cosh^2\left(\frac{t}{\tau_p}\right)}\, dt = \frac{2\tau_p}{T} A^2. \tag{8}$$

In order to trace the evolution of I with increasing τ_a we normalize (7) such that

$$\bar{I}(t) = \frac{1}{N} \frac{1}{\cosh\left(\frac{t}{\tau_p}\right)}, \tag{9}$$

where N is specified by

$$N^2 \frac{Q}{Q_a^o} \frac{A^2}{I_a^2} \frac{1}{(1-r)} = 1. \tag{10}$$

Introducing (9) in (6) and balancing the coefficients of the hyperbolic secant and its first derivative gives the relations

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$$\frac{1}{N^2} = \frac{\tau_a}{\tau_p} \tag{11}$$

$$\frac{\delta}{\omega_m(1-r)} = -\tau_p \tag{12}$$

which are supplemented by the negative resistance power dependence

$$r = \frac{r_o}{1+\frac{P}{P_s}}. \tag{13}$$

Equation 11 shows that an increase in τ_a must be accompanied by either an increase in τ_p or a decrease in N, or both. The decrease in N is equivalent to a decrease in pulse amplitude, since N determines the amplitude scale. If we assume for the sake of argument that τ_p remains constant, then the power P must vary inversely with τ_a From Eq. 12 we note that the detuning parameter δ will always be negative (recall from the fast-absorber analysis that the laser medium must be below threshold, hence 1−r is positive if mode locking is successful[1]). A negative δ implies $\omega_m < \omega_{mo}$, so that the round-trip transit time of the pulse in the laser cavity mode locked by the slow absorber is longer than in the case of the fast absorber. Furthermore, Eq. 12 tells us that as the laser medium approaches threshold ($r \to 1$) the pulse width τ_p increases rapidly. The mode-locking solution will be quenched when threshold is reached and the laser will revert to free-running oscillation.

Equation 6 can be viewed differently if we assume that as τ_a increases $I_a^2\tau_a$ remains constant; i.e., the slower the absorber the more easily it saturates. The mode-locking strength of the absorber becomes independent of τ_a for large τ_a. A τ_a invariant solution to (6) is obtained for each value of the parameter $I_a^2\tau_a$. Because the mode-locking strength approaches a constant, rather than zero, as $\tau_a \to \infty$ the mode-locking solution is never quenched.

The computer solution of the complete slow-absorber mode-locking equation obtained by substituting (4) in (1) substantiates the features contained in the closed-form expressions (11-13). Figure V-1 is a plot of the computed pulse amplitude, pulse width, and the parameter

$$a \equiv -\frac{1}{\tau_{po}}\frac{\delta}{\omega_m}\frac{1}{(1-r)} \tag{14}$$

against the absorber relaxation time τ_a. Pulse amplitude and width have been normalized to the peak current A_o and width τ_{po} of the fast absorber pulse. In the region of small τ_a

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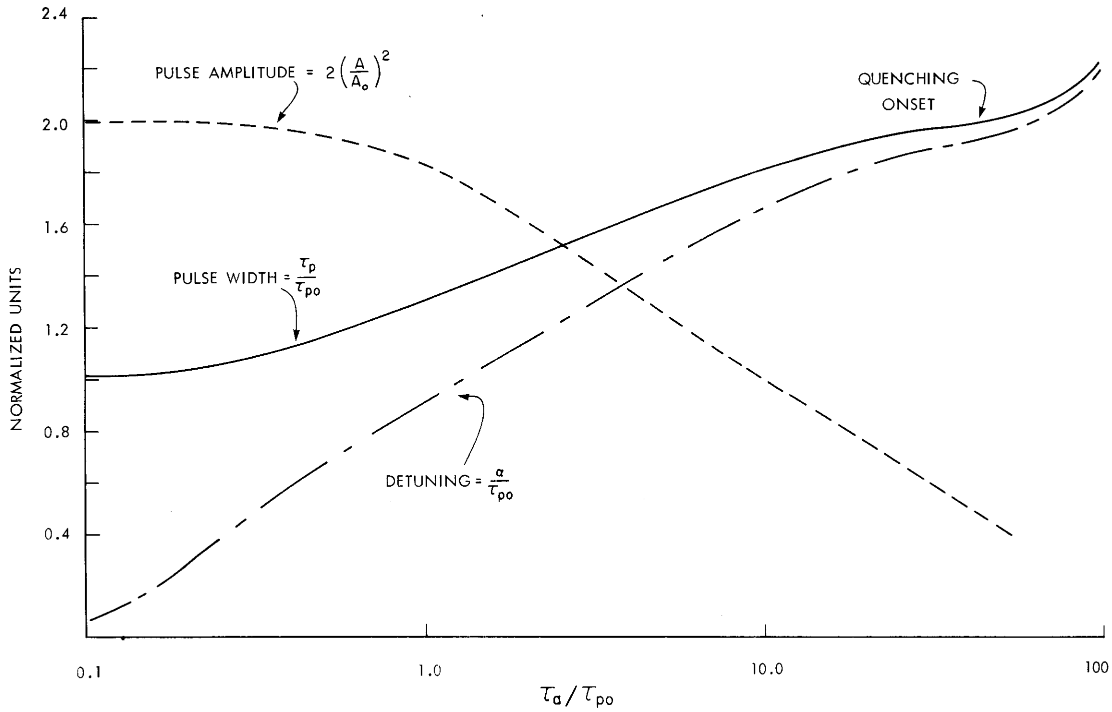

Fig. V-1. Mode-locked pulse amplitude, width, and detuning as a function of saturable absorber relaxation time.

the magnitude of the detuning parameter $|\delta|$ increases more rapidly with τ_a than the pulse width τ_p, while the pulse amplitude decreases slightly. As τ_a increases, α and τ_p approach each other asymptotically as predicted by (12). In keeping with Eq. 11 the slowing of the increase in τ_p is accompanied by a sharper decrease in A^2. In the region defined by $10 < \tau_a/\tau_{po} < 50$, Eqs. 11 and 12 are further verified because α and τ_p approach a constant, while A^2 is roughly inversely proportional to τ_a. The onset of quenching predicted by (11-13) is apparent when $\tau_a/\tau_{po} = 100$.

The computed pulse shapes show only a slight asymmetry for large τ_a (a lengthening of the front of the pulse relative to the back). In fact, for $1 < \tau_a/\tau_{po} < 50$ the pulse shape is virtually invariant. These results are in agreement with the symmetric closed-form solution (7). The slight asymmetry is a consequence of the second-derivative term in (6) which was neglected in deriving (7).

The results of the slow-absorber mode-locking analysis have a simple physical origin. The slow response of the absorber retards the propagation of the mode-locked pulse, hence "pulling" it off the round-trip transit time imposed by the laser cavity. This results in a negative detuning of the cavity modes. The effects produced by the slow absorber, therefore, are analogous to the effects of detuning the intracavity modulator in forced mode locking.[2] Forcing the modes off resonance introduces additional loss

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into the cavity. Consequently, the pulse power decreases, the pulse spectrum narrows, and the pulse width increases. The pulse width, however, is not directly related to the absorber relaxation time. The pulse "terminates" not because the absorption has recovered, but because the modes covering a finite mode-locked spectral width begin to interfere destructively. JS

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2. SHORT LASER PULSES: FLUCTUATION OF MODE-LOCKED PULSES

Joint Services Electronics Program (Contract DAAB07-71-C-0300)
U. S. Army Research Office – Durham (Contract DAHC04-72-C-0044)

Christopher P. Ausschnitt, Hermann A. Haus

Introduction

We have developed a theory of forced mode locking in the frequency domain.[1] In this report we apply our formalism to evaluate amplitude, phase, frequency, and timing jitter of a train of forced mode-locked pulses. AM and FM mode locking are treated simultaneously. The analysis is then specialized to consider the response of an AM mode-locked laser to spontaneous emission noise.

Steady-State Mode-Locking Equation

We consider a cavity with a set of axial modes, evenly spaced by $\Delta\omega$ in frequency, of normalized impedance $1 + jx_c$. A homogeneously broadened laser medium of normalized impedance

$$-r\left(1 + j\,\frac{\omega}{\omega_M} + \frac{\omega^2}{\omega_M^2}\right) \tag{1}$$

fills the cavity, where ω is measured from the line center frequency ω_o of the medium, ω_M is a measure of the medium linewidth, and the Lorentzian denominator has been expanded to second order. The saturated negative resistance r has the power dependence of the homogeneous line, JS

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$$r = \frac{r_o}{1 + \frac{P}{P_s}} \tag{2}$$

To achieve mode locking, we introduce a normalized modulated impedance into the cavity

$$Z_m(t) = (1 - \cos \omega_m t) \frac{1}{R_c} (R_m + jX_m). \tag{3}$$

The modulated impedance operates on the current of the oscillating modes to generate injection voltages in the set of modes at frequencies spaced $k\omega_m$ from the mode at line center, where k ($\lesseqgtr 0$) is an integer that counts the modes from the mode nearest line center denoted by $k = 0$. In the steady state the equivalent voltage of each cavity mode is balanced by the injection locking voltage produced by the interaction of the equivalent cavity current with the modulator. Because of the assumed sinusoidal form of the modulation, the injection voltage in any given axial mode is caused by the currents in the adjacent modes. A difference equation in k results for the current in the axial modes that oscillate at frequencies $\omega_o + k\omega_m$ if the mode locking is successful.

To simplify the solution, the difference equation is approximated by a differential equation; that is, we approximate k and hence the cavity mode spectrum by a continuum

$$M \frac{d^2 I}{dk^2} = \left\{ 1 + jx_c(k) - r\left[1 - \left(\frac{\omega_m}{\omega_M}\right)^2 k^2\right] + jr \frac{\omega_m}{\omega_M} k \right\}, \tag{4}$$

where $I(k)$ is the distribution of current over the cavity modes. The left-hand side is the set of injection voltages produced by the current I flowing through $Z_m(t)$, where we have defined

$$M = |M|\, e^{j\phi} \equiv \frac{R_m + jX_m}{2R_c}. \tag{5}$$

The cavity reactance jx_c in (4) is a function of k because modes at a different "distance" from line center, in general, will oscillate at different detunings from cavity resonance. In the free-running laser the axial modes prefer to oscillate where the net reactance of the cavity and the medium is minimum:

$$x_c(k) + r \frac{\omega_m}{\omega_M} k = \frac{2Q}{\omega_o} \delta\omega_o, \tag{6}$$

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where Q is the cavity quality factor, and the constant term on the right arises because the mode at line center may oscillate at a frequency displaced from line center by $\delta\omega_o$. Equation 6 determines a "tuned" modulation frequency ω_{mo} equal to the cavity mode spacing $\Delta\omega$ as modified by the dielectric constant of the laser medium. For the present, we assume that the applied modulation frequency ω_m is equal to ω_{mo}. Later we shall show that detuning ω_m from ω_{mo} has no influence on the noise response of the mode-locked laser. JS

With the use of (6), Eq. 4 can be recast in the form of the harmonic oscillator equation of quantum mechanics. Thus the eigenfunctions of (4) are the well-known Hermite-Gaussian functions:

$$u_n(k) = \frac{1}{\sqrt[4]{\pi}} \left(\frac{\gamma}{n!\,2^n}\right)^{1/2} H_n(\gamma k) \exp\left[-\frac{1}{2}(\gamma k)^2\right], \tag{7}$$

where we have defined

$$\gamma \equiv \frac{\omega_m}{\omega_p} \exp -j\frac{\phi}{4} \tag{8}$$

in terms of a measure of the bandwidth of the mode-locked spectrum

$$\omega_p = \sqrt[4]{\frac{|M|}{r}}\ \sqrt{\omega_m\omega_M}. \tag{9}$$

Because they describe the collective oscillation of many cavity modes, the eigenfunctions (7) are called "supermodes" of the cavity. As defined in (7) the supermodes are orthonormal, that is,

$$\int_{-\infty}^{\infty} u_n(k)\, u_{n'}^*(k)\, dk = \delta_{nn'}. \tag{10}$$

The eigenvalues of (4) are given by

$$E_n = \left(r-1-j\frac{2Q}{\omega_o}\delta\omega_o\right)n = 2\frac{\gamma^2 r}{\omega_M^2}\left(n+\frac{1}{2}\right), \tag{11}$$

where the real part of E_n determines the excess gain of the mode-locked laser and the imaginary part determines the spectrum shift off line center $\delta\omega_o$.

Haus[1] has shown that only the lowest order n = 0 supermode is stable. Thus, in the steady state, the current distribution over the cavity modes is given by JS

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$$I(k) = \sqrt{P}\ u_o(k), \tag{12}$$

where we have used a normalization such that

$$\int_{-\infty}^{\infty} |I|^2\, dk = P. \tag{13}$$

Here P is the total power in the spectrum.

Perturbations of the Steady State

The steady-state supermode is a discrete set of equally spaced spectral lines under a Gaussian envelope given by (7). The perturbed supermode becomes

$$I(k) + \delta I(k) = \sqrt{P + \delta P}\ u_o\left(k + \frac{\delta\omega}{\omega_m}\right). \tag{14}$$

The perturbation current can be expressed as a superposition of the envelope and individual mode perturbations

$$\delta I(k) = \frac{1}{2}\sqrt{P}\ \frac{\delta P}{P}\ u_o(k) + \sqrt{P}\ \frac{\delta\omega}{\omega_m}\ \frac{du_o}{dk}. \tag{15}$$

To account for both amplitude and phase fluctuations, the perturbations must be taken to be complex. The physical significance of the perturbations in the frequency domain can be described as follows:

(a) $\dfrac{\delta P_r}{P}$ = Fractional fluctuation of total power in the spectrum.

(b) $\delta\theta \equiv \dfrac{\delta P_i}{P}$ = Phase fluctuation of the total spectrum.

(c) $\delta\omega_r$ = Uniform fluctuation of the frequency of the individual modes.

(d) $\dfrac{\omega_m}{\omega_p^2}\,\delta\omega_i$ = Uniform fluctuation of the relative phase of the discrete modes.

Here the subscripts r, i denote the real and imaginary parts of the perturbations. Frequency-domain fluctuations (a), (b), and (c) correspond to fluctuations in the time domain of pulse power, carrier phase, and carrier frequency, respectively. From the Fourier transform of (14) we find that (c) and (d) cause a pulse timing fluctuation, which when normalized to a measure of the pulse width $\tau_p = 1/\omega_p$, is given by

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$$\frac{\delta t}{\tau_p} = \frac{\delta\omega_i}{\omega_p}\cos\frac{\phi}{2} - \frac{\delta\omega_r}{\omega_p}\sin\frac{\phi}{2}. \tag{16}$$

It is convenient to recast Eq. 14 in the form of an expansion in terms of the complete set of cavity supermodes,

$$\delta I = \sum_n A_n u_n(k). \tag{17}$$

We make use of the fact that the derivative of the zero-order supermode is proportional to the first-order supermode to match the coefficients of (14) with those of (17). The zero- and first-order supermodes of (17) contain the information on δP and $\delta\omega$:

$$\frac{\delta P_r}{P} = \frac{2 \text{ Re } A_o}{\sqrt{P}} \tag{18a}$$

$$\delta\theta = \frac{2 \text{ Im } A_o}{\sqrt{P}} \tag{18b}$$

$$\frac{\delta\omega_r}{\omega_p} = -\sqrt{\frac{2}{P}}\left[\text{Re } A_1 \cos\frac{\phi}{4} - \text{Im } A_1 \sin\frac{\phi}{4}\right] \tag{18c}$$

$$\frac{\delta\omega_i}{\omega_p} = -\sqrt{\frac{2}{P}}\left[\text{Re } A_1 \sin\frac{\phi}{4} + \text{Im } A_1 \cos\frac{\phi}{4}\right]. \tag{18d}$$

The coefficients of the higher order ($n > 1$) supermodes of (17) describe higher order effects such as pulse distortion and the fine structure of the phase fluctuations. We shall concentrate, therefore, on the response of A_o and A_1 to a noise source.

The modification of the mode-locking equation (4) to include a noise source proceeds as follows:

(a) A noise source voltage v, which we shall describe in detail, is introduced on the left-hand side with the mode-locking injection signal term.

(b) The steady-state Gaussian supermode is replaced by the perturbed supermode $I(k) + \delta I(k)$, where $\delta I(k)$ is described by Eq. 15.

(c) The saturated gain (negative resistance) r is replaced by $r + \delta r$, where δr is the change in gain caused by the power fluctuations δP_r. Using (15), (10), and (18a), we obtain

$$\delta r = \frac{-2r}{1 + \frac{P}{P_s}} \frac{P}{P_s} \text{ Re } A_o. \tag{19}$$

(d) The cavity mode reactance seen by the perturbation current δI is included by

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expanding to first order about the steady-state reactance:

$$x_c(k,\Omega) = x_c(k) + \Omega \frac{\partial x_c}{\partial \Omega} = x_c(k) + \Omega x_c'$$

$$= x_c(k) + \Omega\left(\frac{2}{\Delta\omega_c}\right), \tag{20}$$

where $\Delta\omega_c$ is the cavity mode bandwidth. We have introduced the frequency Ω to denote the deviation of the k^{th} mode from its steady-state oscillating frequency $k\omega_m$. The variation of all other parameters of (4) with Ω is neglected, an approximation that disregards the energy storage associated with the medium in comparison with the energy storage of the cavity modes.

Thus, to first order in the perturbation, the equation governing the response of the steady-state supermode perturbed by a noise source becomes

$$M \frac{d^2}{dk^2} \delta I + v = \left[-E_o + r\left(\frac{\omega_m}{\omega_M}\right)^2 k^2\right] \delta I - \delta r \left[1 - \left(\frac{\omega_m}{\omega_M}\right)^2 k^2 - j \frac{\omega_m}{\omega_M} k\right] I + j\Omega x_c' \delta I. \tag{21}$$

This is the fundamental equation which we shall now analyze. First, we characterize the noise source v.

Noise Source

We restrict our attention to noise within the fractionally narrow bandwidth of the mode-locked spectrum. Furthermore, the narrow linewidth of each cavity mode $\Delta\omega_c$ relative to the mode spacing $\Delta\omega$ enables us to treat the total noise source as a superposition of independent sources in each of the cavity modes. Thus the noise source is described by a set of fluctuating voltages $v(k,\Omega)$, where k specifies the axial mode, and Ω is the frequency deviation of the noise source in the k^{th} mode from the steady-state oscillating frequency $k\omega_m$ of the mode. The narrow linewidth of the cavity modes also tells us that both the amplitude and phase of the noise source in the k^{th} mode fluctuate slowly compared with ω_m. In other words, fluctuations occur on a time scale which is long compared with the pulse separation $T_R = 2\pi/\omega_m$. Each pulse in the mode-locked train has a spectrum given by the superposition of the steady-state spectrum and the total noise spectrum which does not vary during the pulse.

Since we are interested in the fluctuations of the steady-state supermode oscillation, we expand the noise source in the supermodes of the cavity.

$$v(k,\Omega) = \sum_n v_n(\Omega)\, u_n(k). \tag{22}$$

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The voltages $v_n(\Omega)$ now represent a slow modulation of the n^{th}-order supermode. In general, they are complex. The real components represent in-phase fluctuations with respect to the steady-state supermode and the imaginary components represent quadrature-phase fluctuations. JS

Supermode Fluctuations Caused by Noise

We are now equipped to analyze the response of the mode-locked spectrum to noise perturbations. We express both the noise source and the perturbation current as expansions in supermodes of the cavity and, by making use of (19), Eq. 21 becomes

$$M \sum_n A_n \frac{d^2}{dk^2} u_n + \sum_n v_n u_n = \left[-E_o + r\left(\frac{\omega_m}{\omega_M}\right)^2 k^2\right] \sum_n A_n u_n$$

$$+ \frac{2r}{1+\frac{P}{P_s}} \frac{P}{P_s} \left[1 - \left(\frac{\omega_m}{\omega_M}\right)^2 k^2 - j\frac{\omega_m}{\omega_M} k\right] \sqrt{P}\, \mathrm{Re}\, A_o u_o$$

$$+ j\Omega x_c' \sum_n A_n u_n. \tag{23}$$

To obtain the response function of $A_o(\Omega)$, we multiply (23) by $u_o(k)$ and integrate over k. We make use of the orthogonality condition (10) and the fact that $u_n(k)$ obeys the eigenvalue equation (4) to obtain

$$j\Omega x_c' A_o(\Omega) + \frac{2r}{1+\frac{P}{P_s}} \frac{P}{P_s} \left[1 - \frac{1}{2}\left(\frac{\omega_p}{\omega_M}\right)^2 \exp j\frac{\phi}{2}\right] \mathrm{Re}\, A_o = v_o(\Omega). \tag{24}$$

Likewise, we can obtain the equation governing the response of A_1 through multiplication of (23) by $u_1(k)$ and integration over k:

$$j\Omega x_c' A_1(\Omega) + 2r\left(\frac{\omega_p}{\omega_M}\right)^2 \exp j\frac{\phi}{2} A_1 - j\frac{2r}{1+\frac{P}{P_s}} \frac{P}{P_s} \frac{1}{\sqrt{2}} \frac{\omega_p}{\omega_M} \exp j\frac{\phi}{2} \mathrm{Re}\, A_o = v_1(\Omega), \tag{25}$$

where we have used (11). As we have noted, Eqs. 24 and 25 specify the response to noise of the first-order perturbations of the steady-state supermode. In order to JS

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transform the coefficients A_o and A_1 back to observable effects via (18), we must separate (24) and (25) into real and imaginary parts. Thus far, we have carried out the analysis for combined AM and FM modulation within the laser cavity. For the sake of brevity, we limit further analysis to the case of the AM mode-locked laser where the equations are simplified because $\phi = 0$. The extension to FM or combined AM and FM mode locking is obvious.

At this juncture we also note that we need not alter our analysis to consider detuning. A consequence of detuning[1] is to transform the index k to:

$$k' = k + j \frac{d\omega_M^2}{r\omega_m^2}, \tag{26}$$

where we define the detuning parameter as

$$d \equiv \frac{\omega_m - \omega_{mo}}{\Delta\omega_c}. \tag{27}$$

The detuned supermodes generated by (26) are still orthonormal. Thus the derivations of (24) and (25) are not affected by detuning. The change in the steady-state saturated gain r caused by detuning,[1] however, will affect the noise response.

<u>Noise Response of the AM Mode-Locked Laser</u>

For the AM mode-locked laser M is pure real, $\phi = 0$, and the separation of (24) and (25) into real and imaginary parts yields

$$\text{Re } A_o = \frac{v_o^r(\Omega)}{x_c'} \frac{1}{j\Omega + \frac{1}{\tau_o}} \tag{28}$$

$$\text{Im } A_o = \frac{v_o^i(\Omega)}{x_c'} \frac{1}{j\Omega} \tag{29}$$

$$\text{Re } A_1 = \frac{v_1^r(\Omega)}{x_c'} \frac{1}{j\Omega + \frac{1}{\tau_1}} \tag{30}$$

$$\text{Im } A_1 = \left[\frac{v_1^i(\Omega)}{x_c'} + \frac{1}{\tau_2} \text{ Re } A_o\right] \frac{1}{j\Omega + \frac{1}{\tau_1}}, \tag{31}$$

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where we have defined

$$\frac{1}{\tau_o} \equiv \frac{1}{x'_c} \frac{2r}{1+\frac{1}{P_s}} \frac{P}{P_s} \left[1 - \frac{1}{2}\left(\frac{\omega_p}{\omega_M}\right)^2 \right] \tag{32}$$

$$\frac{1}{\tau_1} \equiv \frac{1}{x'_c} 2r\left(\frac{\omega_p}{\omega_M}\right)^2 \tag{33}$$

$$\frac{1}{\tau_2} \equiv \frac{1}{x'_c} \frac{2r}{1+\frac{P}{P_s}} \frac{P}{P_s} \frac{1}{\sqrt{2}} \frac{\omega_p}{\omega_M}. \tag{34}$$

The interpretation of these equations is straightforward. Equation 28, which governs the response of the power fluctuations of the mode-locked pulse to noise, is a simple relaxation response to an applied source. The "restoring force" is provided by the saturation of the laser medium negative resistance. Equation 29, which governs the carrier phase fluctuations, experiences no such restoring force; that is, it has an infinite relaxation time. Thus we find that the phase fluctuations of the AM mode-locked laser behave similarly to those of a conventional van der Pol oscillator, which obey an equation similar to (29).

Equations 30 and 31 are both in the form of a relaxation response to an applied source. In both cases the restoring force is provided by the fact that the zero-order supermode is stable with respect to the first-order supermode perturbation.[1] The stability is dictated by the requirement of the first-order supermode that the excess gain $r-1$ of the laser be higher than that of the zero-order supermode by an amount $E_1 - E_o$. The quadrature-phase noise source in (31) that is responsible for the timing jitter is augmented by a term dependent on the power fluctuation of the pulse. This is a consequence of the fact that fluctuations of the pulse power modulate the dielectric susceptibility of the laser medium, and hence the cavity mode spacing.

The power spectral densities of the pulse energy, carrier phase, carrier frequency, and pulse timing fluctuations can be obtained by inspection from (18) and (28-31):

$$\frac{\overline{|\delta P|^2}}{P^2} = \frac{4}{Px'^2_c} \frac{\overline{|v_o^r(\Omega)|^2}}{\Omega^2 + \frac{1}{\tau_o^2}} \tag{35}$$

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$$\overline{|\delta\theta|^2} = \frac{4}{Px_c'^2} \frac{\overline{|v_o^i(\Omega)|^2}}{\Omega^2} \tag{36}$$

$$\frac{\overline{|\delta\omega|^2}}{\omega_p^2} = \frac{2}{Px_c'^2} \frac{\overline{|v_1^r(\Omega)|^2}}{\Omega^2 + \frac{1}{\tau_1^2}} \tag{37}$$

$$\frac{\overline{|\delta t|^2}}{\tau_p^2} = \frac{\frac{2}{Px_c'^2} \overline{|v_1^i(\Omega)|^2} + \frac{1}{2\tau_2^2} \frac{\overline{|\delta P|^2}}{P^2}}{\Omega^2 + \frac{1}{\tau_1^2}} \tag{38}$$

The mean-square fluctuations may be obtained by integrating (35), (37), and (38). But (36) is not integrable. The spectrum of $\overline{|\delta\theta|^2}$ suggests that $\delta\theta$ experiences a spread that is like the spread in distance covered by a one-dimensional random walk.[2]

Fluctuation Caused by Spontaneous Emission Noise

In order to obtain specific results, we shall now concentrate on the spectrum of the noise source caused by spontaneous emission noise. The voltage source $v(k, \Omega)$ obeys the Nyquist formula generalized to the quantum case

$$\overline{|v(k, \Omega)|^2} = \overline{|v(\Omega)|^2} = 2ar\hbar\omega_o \frac{\Delta\Omega}{2\pi}, \tag{39}$$

where $a = N_2 - \frac{g_2}{g_1} N_1$. We have neglected the k dependence of the laser medium negative resistance because the mode-locked spectrum occupies only a small portion of the overall laser line within which the k dependence of the line is negligible. This assumption implies that the noise spectrum is "white"; that is, each mode is subject to the same mean-square noise. The spectrum of $v_n(\Omega)$ is

$$\overline{v_n v_{n'}^*} = \delta_{nn'} 2ar\hbar\omega_o \frac{\Delta\Omega}{2\pi}. \tag{40}$$

Furthermore, stationarity dictates that the noise power be divided equally between the in-phase and quadrature-phase components:

$$\begin{aligned} &\overline{v_n^r v_n^i} = 0 \\ &\overline{|v_n^r|^2} = \overline{|v_n^i|^2} = \frac{1}{2}\overline{|v_n|^2}. \end{aligned} \tag{41}$$

Using (40) and (41) and integrating (35) over the spectrum, we obtain the mean-square power fluctuations JS

$$\frac{\overline{|\delta P|^2}}{P^2} = \frac{2}{Px_c'^2} \tau_o a r \hbar \omega_o. \tag{42}$$

Recognizing that the energy in the mode-locked pulse is given by

$$W = PT_R = 2\pi \frac{P}{\omega_m}, \tag{43}$$

we can rewrite (42) as

$$\frac{\overline{|\delta P|^2}}{P^2} = \pi \left(\frac{\Delta\omega_c}{\omega_m}\right)^2 \omega_m \tau_o a r \frac{\hbar\omega_o}{W}. \tag{44}$$

The last factor is the inverse number of photons in the cavity. The factor $\Delta\omega_c/\omega_m$ is generally much less than unity, whereas $\omega_m \tau_o$ measures the relaxation time in terms of the pulse repetition rate and is generally much greater than unity. On the whole the two factors tend to compensate. Thus (44) shows that the mean-square power fluctuations arising from spontaneous emission noise will be extremely small.

In a similar manner we can obtain expressions for the mean-square carrier frequency and pulse timing fluctuations,

$$\frac{\overline{|\delta\omega|^2}}{\omega_p^2} = \frac{\overline{|\delta t|^2}}{\tau_p^2} = \frac{\pi}{2} \frac{\Delta\omega_c}{\omega_m} \left(\frac{\omega_M}{\omega_p}\right)^2 a \frac{\hbar\omega_o}{W}, \tag{45}$$

where we have neglected the influence of the power fluctuations on the pulse timing. As in (44), the last factor $\hbar\omega_o/W$ dictates that the fluctuations will be small. The carrier phase experiences a random walk, which obeys the "spreading" formula[2]

$$|\delta\theta|^2 = \frac{1}{2} a r \hbar \omega_o \tau, \tag{46}$$

where τ is the time between two phase measurements.

Conclusion

The effects of spontaneous emission on the response of a mode-locked laser are small. But the response of the mode-locked laser to perturbation sources which can be much larger than spontaneous emission noise, such as cavity length or cavity JS

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Q fluctuations, can be treated with the present analysis. Moreover, the noise analysis brings into focus the similarities between the mode-locked laser and the van der Pol oscillator. In the absence of noise the mode-locked laser oscillates on a set of discrete modes – of fixed relative amplitude, frequency, and phase – distributed about a carrier frequency determined by the laser medium line center. The introduction of noise causes a power fluctuation restrained by the saturation characteristic of the medium and an unrestrained "random walk" phase fluctuation that is analogous to the noise response of the van der Pol oscillator. The fact that the mode-locked spectrum contains many spectral lines allows for fluctuations of pulse timing and carrier frequency that are not encountered in the van der Pol oscillator. These fluctuations are restrained by the steady-state injection signals generated by the modulated impedance $Z_m(t)$ whose frequency ω_m is assumed fixed.

References

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