PLASMA OSCILLATIONS IN A UNIFORM MAGNETIC FIELD

by

JOHN P. HOLDREN
S.B., Massachusetts Institute of Technology
(1965)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1966

Signature of Author ______________________
Department of Aeronautics and
Astronautics, June 1966

Certified by ______________________
Thesis Supervisor

Accepted by ______________________
Chairman, Departmental
Graduate Committee

ABSTRACT

Longitudinal electrostatic plasma oscillations in a uniform external magnetic field are analyzed. Collisions and ion motion are ignored, and the zeroth order electron distribution function is assumed Maxwellian, although the treatment is not dependent on the last assumption. The dispersion relation as given by Harris for arbitrary direction of propagation with respect to the external field is transformed to permit a calculation analogous to Landau's. A sixth-order polynomial in the square of the real part of the frequency is obtained, and the imaginary part, or Landau damping, is calculated in terms of the roots. Numerical values are given for representative cases. The results are valid for arbitrary direction of propagation and arbitrary magnetic field strength, with the restrictions that the product of the perpendicular component of the wave number times the electron gyro radius, and the ratio of the parallel component of the wave number times the thermal speed of the electrons to the real part of the frequency must be small. The analysis also fails if the real part of the frequency is very near the first or second multiple of the cyclotron frequency. When the restrictions are not violated, the real part of the frequency is in good agreement with the previously known results for limiting cases, and the imaginary part has exactly the expected behavior in the perpendicular and parallel propagation limits.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to the following persons: Professor James E. McCune for his patient guidance and many insights, Professor Sheila Widnall for her advice on the use of the computer, and Mrs. Barbara Marks who typed the manuscript.

This work was done while the author was supported under a National Science Foundation Cooperative Graduate Fellowship. The computer work was done at the M.I.T. Computation Center.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF SYMBOLS

Symbol	Definition	Page Introduced
a	dummy variable	22
$\vec{B}$,B	magnetic field	4
e	electron charge	4
$\vec{E}$,E	electric field	4
f	electron distribution function	6
i	$(-1)^{1/2}$	5
$I_n(x)$	modified Bessel function	22
$I_1 \ldots I_3$	integrals	26
$J_n(x)$	Bessel function	17
J_1, J_2	integrals	43
$\vec{k}$,k	wave vector, wave number	5
K	Boltzmann's constant	10
L_D	Debye length	36
m	electron mass	4
n	electron number density	4
n	summation index	17
p	electron pressure	5
s	perpendicular component of wave no. times electron gyro radius	24
t	time	4
T	temperature	10
$\vec{u}$	electron bulk velocity	4
u	velocity component along B	8
u	transformation variable	20
$\vec{v}$	electron particle velocity	6
$\vec{x}$	position vector	4

LIST OF SYMBOLS (Continued)

Symbol	Definition	Page Introduced
x	dummy variable	
x,y,z	Cartesian coordinate system with z along $\vec{B}$	13
X,Y	numerator, denominator of ω_I expression	40
α	most probable electron thermal speed	10
γ	ratio of specific heats for electrons	5
∇	gradient in position space	4
∇_v	gradient in velocity space	6
ϵ_0	permittivity of free space	5
ρ	dummy variable	22
θ	propagation angle with respect to $\vec{B}$	13
ω	characteristic frequency of oscillation	5
ω_R	real part of characteristic frequency	10
ω_I	imaginary part of characteristic frequency	10
ω_p	plasma frequency	6
ω_c	electron cyclotron frequency	12
$\omega_{R1\ldots5}$	computer solutions for frequency	48
$(\ )_0$	equilibrium quantity	5
$(\ )^{(1)}$	perturbation quantity	5
$(\ )_z$	component parallel to $\vec{B}$	17
$(\ )_\perp$	component perpendicular to $\vec{B}$	17

CHAPTER 1

INTRODUCTION

The basic features of plasma oscillations are easily visualized by considering a simple electrically neutral plasma under the influence of no external fields, and composed of equal numbers of ions and electrons. Any disturbance in the plasma involving a local separation of charge creates powerful electrostatic restoring forces, whose end result is a high frequency oscillation of the electrons against the background of relatively immobile ions. If the effects of finite electron temperature are considered, these oscillations in the electron density and the attendant local electric field cease being a mere standing wave and propagate through the plasma; in fact, the plasma becomes a dispersive medium. Since the oscillating electric field is aligned parallel to the direction of propagation and the induced magnetic field is usually negligible, the phenomenon is described as longitudinal, electrostatic plasma oscillations. If the characteristic frequency of these oscillations greatly exceeds the collision frequency in the plasma, collisions can be ignored. This assumption proves valid in thermonuclear and many astrophysical applications.

The elementary behavior of plasma oscillations is most readily analyzed by what will be referred to in this thesis as the macroscopic approach; specifically, use of the hydrodynamic continuity and momentum equations and Maxwell's equations. A more sophisticated, microscopic approach, based on kinetic theory and introduced by Landau[1] in 1946, yields similar results but includes the possibility of damping of the oscillations even in the absence of collisions. This phenomenon, generally called Landau damping, can, under certain circumstances, represent a growth in the oscillation amplitude instead of a decay. Although agreement on the physical mechanism underlying this growth or damping process is not yet unanimous,[2,3,4] the existence of Landau damping has been experimentally verified,[5] and its importance in predicting the behavior of plasmas in which longitudinal oscillations can occur is apparent.

Landau's original calculation was done for a plasma under the influence of no external fields. In most cases of practical interest, however, some sort of imposed magnetic field is present. The physical and mathematical complications arising from this addition are considerable, even when the magnetic field is uniform and there is no imposed electric field. Previous work on this case has yielded characteristic frequencies and expressions for the damping in certain limiting cases, such as propagation of the disturbance perpendicular and parallel to the magnetic field, and for arbitrary direction of propagation if the magnetic field is very weak or the electron temperature very low.[6] A more general understanding of the behavior of plasma oscillations in a constant magnetic field is clearly desirable. In this thesis, expressions for the characteristic frequencies and Landau damping for arbitrary field strength and arbitrary direction of propagation are obtained, interpreted, and compared with the previously known limiting cases.

The work is developed as follows: Chapter 2 introduces the plasma dispersion relation in the absence of external fields, using the macroscopic approach. A description of the Landau calculation is given, since the thesis treatment of oscillations in a magnetic field relies heavily on Landau's approach. Chapter 3 extends the macroscopic treatment to include an imposed magnetic field and examines previous work on the microscopic solution for this case. Chapters 4 and 5 use the Harris dispersion relation[7] for a plasma in a uniform magnetic field and a transformation introduced by McCune[8] to derive general expressions for the characteristic frequencies and damping. Chapter 6 evaluates the results for cases of interest.

CHAPTER 2

PLASMA OSCILLATIONS IN THE ABSENCE OF EXTERNAL FIELDS

2.1 The Macroscopic Approach

In considering the high frequency behavior of a fully ionized plasma it is reasonable to assume that only the electrons move, while the much more massive ions simply form a neutralizing background of positive charge. The classical hydrodynamic continuity and momentum equations for the electron fluid are then:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \tag{2.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \frac{e}{m}(\vec{E} + \vec{u} \times \vec{B}) - \frac{1}{mn}\nabla p \tag{2.2}$$

where $n(\vec{x},t)$ is the local electron number density, $\vec{u}(\vec{x},t)$ is the electron bulk velocity, and p is the pressure, assumed scalar. Assuming that the equilibrium state is the electron fluid at rest with no fields present, these equations can be linearized to give the perturbations on that equilibrium resulting from some initial disturbance:

$$\frac{\partial n^{(1)}}{\partial t} + n_0 \nabla \cdot \vec{u}^{(1)} = 0 \tag{2.3}$$

$$\frac{\partial \vec{u}^{(1)}}{\partial t} - \frac{e}{m}\vec{E}^{(1)} + \frac{1}{mn_o}\left(\frac{\partial p}{\partial n}\right)_o \nabla n^{(1)} = 0 \tag{2.4}$$

where $(\ )^{(1)}$ denotes a perturbation quantity and $(\ )_o$ an equilibrium quantity. If the perturbation magnetic field is assumed negligible, then the perturbation electric field is essentially curl free and is determined by

$$\nabla \times \vec{E}^{(1)} = 0 \qquad\qquad \nabla \cdot \vec{E}^{(1)} = \frac{e}{\epsilon_o}(n^{(1)} - n_o) \tag{2.5}$$

The last assumption is called the electrostatic approximation.

With the further assumption that the perturbation quantities vary as $\exp[i(\vec{k}\cdot\vec{x} - \omega t)]$, equations (2.3) through (2.5) become

$$-\omega n^{(1)} + n_o \vec{k}\cdot\vec{u}^{(1)} = 0 \tag{2.6}$$

$$-imn_o\omega\vec{u}^{(1)} + \frac{ip_o n^{(1)}\gamma\vec{k}}{n_o} + n_o e\vec{E}^{(1)} = 0 \tag{2.7}$$

$$\vec{E}^{(1)} = \frac{ien^{(1)}\vec{k}}{\epsilon_o k^2} \tag{2.8}$$

where the adiabatic relation $p/p_o = (n/n_o)^\gamma$ has been used in obtaining (2.7). Substituting (2.8) into (2.7) yields a pair of homogeneous algebraic equations in the quantities n and u. The condition for a nontrivial solution is the vanishing of the determinant of the coefficients, which gives the dispersion relation:

$$\omega^2 = \frac{n_0 e^2}{\epsilon_0 m} + \frac{p_0 \gamma k^2}{n_0 m} \qquad (2.9)$$

If the last term vanishes, either because $p_0 = 0$ (no electron thermal motion) or $k = 0$ (disturbance of infinite wavelength), equation (2.9) describes free oscillations of the electron number density, bulk velocity, and electric field at the plasma frequency

$$\omega_p^2 \equiv \frac{n_0 e^2}{\epsilon_0 m} \qquad (2.10)$$

If the last term in (2.9) is finite, a nonzero group velocity, $d\omega/dk$, is defined for each k; the waves propagate and undergo dispersion. It should be noted that the linearization procedure which led to equation (2.9) restricts the results to small amplitude motions.

2.2 The Landau Calculation

The behavior of an unbounded plasma can be completely determined if the velocity distribution functions, $f(\vec{x},\vec{v},t)$, for its components are known. For the present case, in which the ions are assumed to comprise a positive immobile background, only the electron distribution function is of interest, and the plasma is described by the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f - \vec{F}\cdot\nabla_v f = \left(\frac{\delta f}{\delta t}\right)_{\text{collisions}} \qquad (2.11)$$

where $\vec{F}$ describes the force field acting on the electrons and

∇_v is the gradient in velocity space. Since the phenomenon of interest here proves to have characteristic frequencies which exceed the collision frequency by 10^6 or more, the collision term on the right hand side of (2.11) can be ignored.

Landau solves the resulting collisionless Boltzmann, or Vlasov, equation as an initial value problem. That is, given a small initial perturbation, $f^{(1)}(\vec{x},\vec{v},t=0)$, on a uniform equilibrium specified by some distribution, $f_o(\vec{v})$ (satisfying the Vlasov equation), he assumes:

$$f(\vec{x},\vec{v},t) = f_o(\vec{v}) + f^{(1)}(\vec{x},\vec{v},t) \tag{2.12}$$

Using this expression in (2.11), subtracting out the equilibrium solution, and keeping only first-order terms in the small perturbation, $f^{(1)}$, gives a linearized equation. With external fields assumed absent, it is

$$\frac{\partial f^{(1)}}{\partial t} + \vec{v}\cdot\nabla f^{(1)} - \frac{e}{m}\vec{E}^{(1)}\cdot\nabla_v f_o = 0 \tag{2.13}$$

where

$$\vec{E}^{(1)} = -e\iint f^{(1)}d\vec{v}\,d\vec{x} \tag{2.14}$$

Equations (2.13) and (2.14) are now Fourier analyzed in space and Laplace analyzed in time, and the components of velocity transverse to $\vec{k}$ are integrated out for simplicity. (For details of these and following procedures in the calculation, the reader is referred to Landau[1] or Montgomery and Tidman[9].) The resulting equations are readily solved for the Fourier transforms of the perturbation electric field and the distribution function along the direction of propagation, $\vec{k}$. The electric field is of more immediate physical interest; its transform is:

$$E^{(1)}(k,\omega) = \frac{n_0 e}{kD(k,\omega)} \int_{-\infty}^{\infty} \frac{F^{(1)}(k,u,t=0)}{ku-\omega} du \tag{2.15}$$

where u is the component of $\vec{v}$ parallel to $\vec{k}$, $F^{(1)}(k,u,t=0)$ is the spatial transform of the initial perturbation distribution function along $\vec{k}$, and $D(k,\omega)$ is given by

$$D(k,\omega) = 1 - \frac{n_0 e^2}{\epsilon_0 mk} \int_{-\infty}^{\infty} \frac{\frac{\partial F_0(u)}{\partial u}}{ku-\omega} du \tag{2.16}$$

The quantity ω is in general complex, and by the nature of the transform used in obtaining (2.15) and (2.16), these expressions are valid only where $\mathrm{Im}(\omega) \geq 0$.

The time behavior of the kth spatial component of the electric field follows from the Laplace inversion formula:

$$E^{(1)}(k,t) = \frac{1}{2\pi} \int_{i\sigma-\infty}^{i\sigma+\infty} \exp(-i\omega t)\, E^{(1)}(k,\omega)\, d\omega \tag{2.17}$$

where the contour of integration in the complex ω plane is a line parallel to the $\mathrm{Re}(\omega)$ axis and above all the singularities of the integrand. Although equation (2.17) is the formal solution to the original initial value problem, it cannot be evaluated for distribution functions of practical interest due to the intractability of the integrals involved.

Landau therefore resorts to examining the behavior of the electric field in the $t \to \infty$ limit. By deforming the contour of integration downward into the $\mathrm{Im}(\omega) < 0$ half plane, (2.17) can be rewritten:

$$E^{(1)}(k,t) = \sum_i R_i e^{-i\omega_i t} + \frac{1}{2\pi} \int_{-i\alpha-\infty}^{-i\alpha+\infty} e^{-i\omega t} E^{(1)}(k,\omega)\, d\omega \qquad (2.18)$$

where the R_i are the residues of the singularities of the integrand located at the points ω_i and encircled by the deformed contour (see Figure 1), and the function $E^{(1)}(k,\omega)$ must now be analytically continued into the lower half plane. The last procedure requires that $F^{(1)}(k,u,t=0)$ and $F_o(u)$ be analytic functions of u everywhere.

Since it can be shown that the only singularities which exist for $|\omega| < \infty$ are poles where the denominator, $D(k,\omega)$, vanishes, and since the integral term in (2.18) will always be strongly damped compared to the uppermost pole terms in the sum, it is clear that the long-time behavior of the field will be dominated by these poles. Thus in the $t \to \infty$ limit, the equation $D(k,\omega) = 0$ becomes a dispersion relation:

$$1 - \frac{\omega_p^2}{k} \int_{-\infty}^{\infty} \frac{\partial F_0(u)/\partial u}{ku - \omega}\, du - \frac{\omega_p^2}{k^2} 2\pi i F_0'\left(\frac{\omega}{k}\right) = 0 \qquad (2.19)$$

where $D(k,\omega)$ has been analytically continued so that (2.19) is valid for $\mathrm{Im}(\omega) < 0$. The real parts of the roots of the dispersion relation give the characteristic frequencies of oscillation of the field's spatial components, and the imaginary parts give their exponential growth or damping rates. It can be shown, however, that if the initial electron velocity distribution is Maxwellian, then (2.19) has no roots in the $\mathrm{Im}(\omega) > 0$ half plane, so that all roots are damped, or stable.

Remembering that this entire treatment has been for the $t \to \infty$ limit, it is apparent that the most interesting pole is the one nearest the real ω axis, since its contribution is least strongly damped. Landau's approach is to expand the denominator in the integral of (2.19) in a power series, assuming k small, and obtain $\omega(k)$ by successive approximations. The successive approximation technique is also used to obtain the correction due to the analytic continuation term. This correction proves to be the small imaginary part of the frequency, or Landau damping decrement, which vanishes when k goes to zero. Landau's result, to second order in k, is

$$\omega_R^2 = \omega_p^2 + \frac{3}{2}k^2\alpha^2 \tag{2.20}$$

$$\omega_I = -\pi^{1/2}\omega_p\left(\frac{\omega_p}{k\alpha}\right)^3 \exp\left(-\omega_p^2/k^2\alpha^2\right) \tag{2.21}$$

where ω_R is the real part of the frequency, ω_I is the imaginary part, $\alpha \equiv (2KT/m)^{1/2}$ is the most probable thermal speed for the electrons, and ω_p is the plasma frequency.

Jackson[16] has noted that a more accurate result for the imaginary part has ω_R^2 in place of ω_p^2 in the exponential. In the same paper, Jackson analyzes the problem numerically.

He finds that the expression (2.21) for the damping decrement, with the correction just noted, is in essentially perfect agreement with numerical results for $k^2\alpha^2/\omega_p^2 \lesssim .125$ and is within about 10 per cent up until $k^2\alpha^2/\omega_p^2 \approx .25$. Above this value, the small k approximation upon which (2.21) depends is apparently no longer satisfied. This result will be of some use in estimating the range of validity of the thesis result for the damping in a magnetic field.

Two further points should be noted before leaving the Landau calculation. First, if the state equation, $p_0 = n_0 KT$ and the value of γ appropriate to one translational degree of freedom are substituted into the macroscopic result for the frequency, (2.9), the microscopic result for the real part of the frequency, (2.21), is recovered. Thus the two approaches agree to order k^2, except, of course, that the damping decrement is inaccessible to the macroscopic approach. This result suggests that it will be worthwhile to treat the problem in a magnetic field macroscopically, too, as a means of checking the new microscopic results to first order.

Finally, it should be remembered that collisions were neglected entirely in the derivation of (2.20) and (2.21). If an evaluation of (2.21) for a given component of the field were to indicate that the damping time is of the order of or longer than the mean collision time (which, for very small k, is not unlikely), then it should be apparent on physical grounds that this component would be collisionally damped before Landau damping became important at all.

CHAPTER 3

ADDITION OF AN EXTERNAL MAGNETIC FIELD

3.1 Some Physical Effects

The physical complications arising from the addition of a uniform external magnetic field are readily imagined by considering the motion of a single electron. The interaction of its thermal velocity with the magnetic field yields a helical motion; the motion along the field direction is unchanged while in the plane perpendicular to the field the electron travels a circular path with frequency $\omega_c = eB/m$. The oscillating electric field of the plasma oscillations complicates this situation as follows: the electron sees the component of oscillating electric field parallel to the magnetic field just as if the magnetic field were not present at all, and responds accordingly. But the interaction of the magnetic field with the perpendicular component of the electric field produces two additional electron motions. The first is a drift in the direction perpendicular to both the electric and magnetic fields, due to the instantaneous value of $\vec{E}$. The second is a drift along the direction of $\vec{E}$, due to its time rate of change, $\partial\vec{E}/\partial t$. As will become apparent, the complexity of the mathematics describing plasma oscillations increases appropriately with these physical complications.

3.2 The Macroscopic Calculation with a Magnetic Field

The linearized hydrodynamic equations of motion describing the exponentially varying electron number density, bulk velocity and electric field in a plasma with a uniform magnetic field imposed differ from (2.6) and (2.7) only by a $\vec{u}^{(1)} \times \vec{B}_0$ force term in the momentum equation. As in Chapter 2, the oscillating perturbation magnetic field is neglected, giving the electrostatic approximation. The equations are:

$$-\omega n^{(1)} + n_0 \vec{k} \cdot \vec{u}^{(1)} = 0 \tag{3.1}$$

$$-imn_0\omega\vec{u}^{(1)} + i\frac{n^{(1)}}{n_0} p_0 \gamma \vec{k} + n_0 e(\vec{E}^{(1)} + \vec{u}^{(1)} \times \vec{B}_0) = 0 \tag{3.2}$$

with

$$\vec{E}^{(1)} = \frac{ien^{(1)}}{\epsilon_0 k^2} \vec{k} \tag{3.3}$$

Combining (3.3) and (3.2) gives one scalar and one vector, or equivalently, four scalar homogeneous algebraic equations in $n^{(1)}$ and the three components of $u^{(1)}$.

For their evaluation, a Cartesian coordinate system is chosen in which $\vec{B}_0$ lies along the z axis, $\vec{k}$ lies in the x-z plane, and θ is the polar angle between $\vec{B}_0$ and $\vec{k}$ (see Figure 2). In this coordinate frame, the determinant of the coefficients of $n^{(1)}$ and the components of $u^{(1)}$ is:

$$\begin{vmatrix} -\omega & n_0 k\sin\theta & 0 & n_0 k\cos\theta \\ ik\sin\theta\left(\frac{p_0\gamma}{n_0}+\frac{n_0e^2}{\epsilon_0k^2}\right) & -imn_0\omega & n_0eB_0 & 0 \\ 0 & -n_0eB_0 & -imn_0\omega & 0 \\ ik\cos\theta\left(\frac{p_0\gamma}{n_0}+\frac{n_0e^2}{\epsilon_0k^2}\right) & 0 & 0 & mn_0\omega \end{vmatrix}$$

The dispersion relation resulting from setting the determinant equal to zero as the condition for a nontrivial solution reduces to:

$$m\omega^2\left[(m\omega)^2-(eB_0)^2\right] + \left[(m\omega)^2-(eB_0\cos\theta)^2\right]\left(\frac{p_0\gamma k^2}{n_0}+\frac{n_0e^2}{\epsilon_0}\right)=0 \qquad (3.4)$$

Using the state equation and the definitions of ω_p and α as in Chapter 2, and noting that eB_0/m is the cyclotron frequency, ω_c, equation (3.4) becomes:

$$\omega^2(\omega^2-\omega_c^2) + (\omega^2-\omega_c^2\cos^2\theta)(\omega_p^2+\tfrac{3}{2}k^2\alpha^2)=0 \qquad (3.5)$$

The dispersion relation (3.5) is quadratic in ω^2 and is easily solved by the formula:

$$2\omega^2 = \omega_p^2+\omega_c^2+\tfrac{3}{2}k^2\alpha^2 \pm\left[(\omega_p^2+\omega_c^2+\tfrac{3}{2}k^2\alpha^2)-4\omega_c^2\cos^2\theta\left(\tfrac{3}{2}k^2\alpha^2+\omega_p^2\right)\right]^{1/2} \qquad (3.6)$$

Several limiting cases are readily recovered from this relation.

For B_o and hence ω_c equal to zero, the familiar result of the no-field analysis in Chapter 2 is recovered (together with a trivial root at $\omega^2 = 0$):

$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2\alpha^2 \tag{3.7}$$

For propagation parallel to a finite magnetic field ($\vec{k} \parallel \vec{B}_o$, $\cos\theta = 1$) the root (3.7) is again obtained. This result is not surprising since velocity components along the magnetic field are unaffected by it. The second root obtained in this case is $\omega^2 = \omega_c{}^2$, accounting for the effect of the magnetic field on velocity components perpendicular to it (when a magnetic field is present, the bulk velocity vector $\vec{u}^{(1)}$ is no longer aligned, in general, with the propagation vector $\vec{k}$). A closer look at the equations of motion, (3.1) through (3.3), shows that the cyclotron motion in the plane perpendicular to $\vec{B}_o$, described by the $\omega^2 = \omega_c{}^2$ root, is not coupled with the oscillating electric field at all for the case of purely parallel propagation. That is, in this case the component equations relating $E^{(1)}$, $n^{(1)}$ and u_z can be completely uncoupled from those relating u_x, u_y, and B_o, so that the bulk electron motion in the x-y plane can neither excite nor be excited by an electric field. As soon as the propagation vector $\vec{k}$ is even slightly out of line with $\vec{B}_o$, however, the cyclotron motion and the electric field are coupled.

Finally, for propagation perpendicular to the magnetic field, a hybrid root is obtained (along with the trivial $\omega^2 = 0$ solution):

$$\omega^2 = \omega_p^2 + \omega_c^2 + \frac{3}{2}k^2\alpha^2 \tag{3.8}$$

For arbitrary direction of propagation, a clearer picture of the roots than is provided by (3.6) as it stands can be obtained by expanding the square root term in that expression for $\omega_p^2 \gg \omega_c^2$ or vice versa. For $\omega_p^2 \gg \omega_c^2$, expanding the square root in powers of ω_c^2/ω_p^2 and keeping terms up to ω_c^4/ω_p^4 and $k^4\alpha^4/\omega_p^4$ in the expansion yields for the positive root:

$$\begin{aligned}\omega^2 = \omega_p^2 + \omega_c^2\sin^2\theta + \frac{3}{2}k^2\alpha^2 \\ + \omega_c^2(\omega_c^2/\omega_p^2)\sin^2\theta\cos^2\theta\end{aligned} \tag{3.9}$$

and for the negative root:

$$\omega^2 = \omega_c^2\cos^2\theta - \omega_c^2(\omega_c^2/\omega_p^2)\sin^2\theta\cos^2\theta \tag{3.10}$$

For $\omega_c^2 \gg \omega_p^2$ the equivalent expansion gives for the positive root:

$$\begin{aligned}\omega^2 = \omega_c^2 + \omega_p^2\sin^2\theta + \frac{3}{2}k^2\alpha^2\sin^2\theta \\ + \omega_p^2(\omega_p^2/\omega_c^2)\sin^2\theta\cos^2\theta \\ + (3\omega_p^2 + \frac{9}{4}k^2\alpha^2)(k^2\alpha^2/\omega_c^2)\sin^2\theta\cos^2\theta\end{aligned} \tag{3.11}$$

and for the negative root:

$$\omega^2 = \omega_p^2 \cos^2\theta + \frac{3}{2} k^2 a^2 \cos^2\theta - \omega_p^2 (\omega_p^2/\omega_c^2) \sin^2\theta \cos^2\theta - (3\omega_p^2 + \frac{9}{4} k^2 a^2)(k^2 a^2/\omega_c^2) \sin^2\theta \cos^2\theta \quad (3.12)$$

Note that these results have the previously discussed values in the parallel ($\sin\theta = 0$) and perpendicular ($\cos\theta = 0$) propagation limits.

3.3 Previous Work on the Microscopic Approach with a Magnetic Field

Although the mathematics is more difficult, the general approach to the solution of the linearized Vlasov equation as an initial value problem is of course still appropriate when a uniform magnetic field is present. A dispersion relation analogous to (2.19) can be obtained, whose roots are the characteristic frequencies of oscillation of the plasma in the $t \to \infty$ limit. This dispersion relation has been published in several equivalent forms; for representative derivations the reader may consult Montgomery and Tidman[9] or Bernstein[6].

The version treated here is that published by Harris[7] for the electrostatic limit. With ion motion assumed negligible, it is

$$1 = \frac{1}{k^2} \int_{-\infty}^{\infty} dv_z \int_0^{\infty} 2\pi v_\perp dv_\perp \sum_{n=-\infty}^{\infty} \omega_p^2 J_n^2(k_\perp v_\perp/\omega_c) \frac{\times\left[\frac{k_z}{k}\left(\frac{\partial f_0}{\partial v_z}\right) + \frac{n\omega_c}{k v_\perp}\left(\frac{\partial f_0}{\partial v_\perp}\right)\right]}{\frac{k_z v_z}{k} + \frac{n\omega_c}{k} - \frac{\omega}{k}} \qquad \mathrm{Im}(\omega) > 0 \quad (3.13)$$

In this equation $(\ )_\perp$ and $(\ )_z$ denote quantities perpendicular and parallel to the magnetic field, respectively, f_0 is the equilibrium electron distribution function, and $J_n(x)$ is a Bessel function of the first kind with real argument. It is not difficult to show that equation (3.13) is equivalent to the dispersion relation of Bernstein in the electrostatic limit. It has been solved in certain limiting cases by Bernstein and others; thus these special results provide a useful comparison for the solutions presented here.

It is well known that in the limiting case of propagation parallel to the magnetic field the familiar Landau result is recovered, both for the real part of the frequency and for the damping decrement. It is also known that for propagation exactly perpendicular to the magnetic field the damping must vanish completely.[6]

In the case of perpendicular propagation, Bernstein's dispersion relation can be simplified in the limit $k^2\alpha^2/\omega_c^2 \ll 1$ to yield (except for a presumed misprint):

$$2\omega^2 = 5\omega_c^2 + \omega_p^2 - \left[(3\omega_c^2 - \omega_p^2)^2 + 6k^2\alpha^2\omega_p^2 \right]^{1/2} \tag{3.14}$$

which for $\omega_p^2 \gg \omega_c^2$ is approximately:

$$\omega^2 = 4\omega_c^2 - \frac{3}{2}k^2\alpha^2 \tag{3.15}$$

and for $\omega_c^2 \gg \omega_p^2$:

$$\omega^2 = \omega_c^2 + \omega_p^2 - \frac{1}{2}k^2\alpha^2(\omega_p^2/\omega_c^2) \tag{3.16}$$

Bernstein has also solved his dispersion relation for arbitrary direction of propagation in the special cases of weak magnetic field and low temperature ($\omega^2 \gg \omega_c^2, k^2\alpha^2$) and strong magnetic field and low temperature ($\omega_c^2 \gg \omega_p^2, k^2\alpha^2$). In the weak field case he obtains for the real part of the frequency:

$$\omega_R^2 = \omega_p^2 + \omega_c^2 \sin^2\theta + \frac{3}{2} k^2\alpha^2 \tag{3.17}$$

and for the imaginary part, or damping:

$$\omega_I = -\pi^{1/2}\omega_R(\omega_R/k\alpha)^3\left(1 + \frac{\sin^2\theta}{3}\frac{\omega_R^4\omega_c^2}{k^6\alpha^6}\right)\exp\left(-\frac{\omega_R^2}{k^2\alpha^2}\right) \tag{3.18}$$

Bernstein points out that the expression (3.18) reduces to Landau's result in the limit $\omega_c = 0$. He fails to mention that (3.18) does not vanish for perpendicular propagation, as would seem to be required if $\omega_c \neq 0$. For the strong field case Bernstein gets for the real part of the frequency:

$$\omega_R^2 = \omega_p^2\cos^2\theta + \frac{3}{2}k^2\alpha^2\cos^2\theta - \frac{1}{2}\frac{\omega_p^2}{\omega_c^2}(\cos^2\theta)\omega_p^2 \tag{3.19}$$

and for the imaginary part:

$$\omega_I = -\pi^{1/2}\omega_R(\omega_R/k\alpha)^3\cos\theta\,\exp\left(-\frac{\omega_R^2}{k^2\alpha^2}\right) \tag{3.20}$$

The damping decrement (3.20) has the required behavior in both the parallel and perpendicular propagation limits.

CHAPTER 4

A NEW APPROACH TO THE MICROSCOPIC DISPERSION RELATION

4.1 The McCune Transformation

From the rather formidable appearance of equation (3.13), it is not difficult to imagine why this dispersion relation and its relatives have resisted more general treatment. In the form in which (3.13) is written, the infinite sum of Bessel functions and the attendant infinite sequence of singular denominations make Landau's denominator expansion and successive approximation technique appear hopeless, to say nothing of the question of analytic continuation into the $-\omega_I$ half-plane.

The key to reducing these difficulties to the point of tractability is a transformation introduced by McCune[8], in the context of establishing a general criterion for electrostatic plasma instabilities in a magnetic field. For each n he lets:

$$\frac{k_z v_z}{k} + \frac{n\omega_c}{k} = u \tag{4.1}$$

and

$$f_0(v_z, v_\perp) = f_0\left[\frac{k}{k_z}\left(u - \frac{n\omega_c}{k}\right), v_\perp\right] \tag{4.2}$$

so that the dispersion relation (3.13) can be rewritten in the form

$$1 = \frac{1}{k^2}\int_{-\infty}^{\infty} \frac{P(u)\,du}{u-\omega/k} \qquad \mathrm{Im}(\omega) > 0 \tag{4.3}$$

where

$$P(u,k_z,k_\perp) = \int_0^\infty 2\pi v_\perp dv_\perp \sum_{n=-\infty}^{\infty} \omega_p^2 J_n^2(k_\perp v_\perp/\omega_c)$$
$$\times\left\{ \frac{\partial f_0(v_z,v_\perp)}{\partial v_z}\bigg|_{v_z = \frac{k}{k_z}\left(u-\frac{n\omega_c}{k}\right)} + \frac{n\omega_c}{k_z v_\perp}\frac{\partial f_0}{\partial v_\perp}\left[\frac{k}{k_z}\left(u-\frac{n\omega_c}{k}\right), v_\perp\right]\right\} \mathrm{sgn}(k_z) \tag{4.4}$$

The transformation is not without physical content: ku is a Doppler-shifted cyclotron frequency or one of its harmonics, and P(u) is a sort of generalized projection of the derivatives of f_0 . McCune notes that when $k_z = 0$ or $k_\perp = 0$ (propagation perpendicular or parallel to the magnetic field), (4.3) and (4.4) reduce to the usual forms for those limiting cases. The particular relevance of this transformation to the present problem, of course, is that the dispersion relation (4.3) has exactly the familiar, one-dimensional form treated by Landau.

To determine how much real advantage has been gained by these manipulations, it is necessary to look closely at the properties of the function P(u). For the calculations in this thesis, the equilibrium electron distribution function is chosen to be Maxwellian; there is no reason why the work cannot be carried out for other reasonably well behaved distributions. With

$$f_0(v_z, v_\perp) = \left(\frac{1}{\pi\alpha^2}\right)^{3/2} \exp\left[-(v_z^2 + v_\perp^2)/\alpha^2\right] \tag{4.5}$$

where $\alpha = (2KT/m)^{1/2}$, equation (4.4) for P(u) becomes:

$$P(u, k_\perp, k_z) = -\frac{4\pi u}{\alpha^2}\left(\frac{1}{\pi\alpha^2}\right)^{3/2}\left|\frac{k}{k_z}\right|\omega_p^2 \sum_{n=-\infty}^{\infty} \exp\left[-\left(\frac{ku - n\omega_c}{k_z\alpha}\right)^2\right]$$

$$\times \int_0^\infty v_\perp J_n^2(k_\perp v_\perp/\omega_c) \exp(-v_\perp^2/\alpha^2)\, dv_\perp \tag{4.6}$$

This expression simplifies with the use of a well known formula relating J_n and the modified Bessel function of the first kind, I_n [10]:

$$\int_0^\infty \rho J_n^2(a\rho) e^{-\rho^2} d\rho = \frac{1}{2} e^{-a^2/2} I_n(a^2/2) \tag{4.7}$$

Thus with $\rho = v_\perp/\alpha$ and $a = k_\perp\alpha/\omega_c$, there results:

$$\int_0^\infty v_\perp J_n^2(k_\perp v_\perp/\omega_c) \exp(-v_\perp^2/\alpha^2)\, dv_\perp$$
$$= \frac{\alpha^2}{2} \exp(-k_\perp^2\alpha^2/2\omega_c^2)\, I_n(k_\perp^2\alpha^2/2\omega_c^2) \tag{4.8}$$

and noting that $k_\perp = k\sin\theta$, $k_z = k\cos\theta$, P(u) becomes:

$$P(u,k,\theta) = -\frac{2u\omega_p^2}{\pi^{1/2}\alpha^3|\cos\theta|}\exp\left(-\frac{k^2\alpha^2\sin^2\theta}{2\omega_c^2}\right)$$

$$\times\sum_{n=-\infty}^{\infty} I_n\left(\frac{k^2\alpha^2\sin^2\theta}{2\omega_c^2}\right)\exp\left[-\left(\frac{ku-n\omega_c}{k\alpha\cos\theta}\right)^2\right]$$

(4.9)

This expression for P(u) is not difficult to interpret: for fixed k and θ the terms of the sum are exponential pulses, each centered at u = $n\omega_c/k$, having equal characteristic spread $\alpha\cos\theta$, and weighted by the modified Bessel functions, I_n . Since the Bessel functions are symmetric in n $[I_{-n}(x) = I_n(x)]$, so is the weighting. It is apparent from the series definition of the modified Bessel functions,

$$I_n(x) = \sum_{p=0}^{\infty} \left(\frac{x}{2}\right)^{2p+n} \Big/ p!\,(p+n)! \qquad (4.10)$$

that for small argument, x, the Bessel functions of higher order, n, rapidly approach zero. Thus if the argument $k^2\alpha^2\sin^2\theta/2\omega_c^2$ in the expression (4.9) is small, only the several terms in the sum near n = 0 will be important in determining P(u). The only variable outside the sum in (4.8) is -u, which dominates the behavior of P(u) near u = 0, but is dominated by the vanishing of the weighted exponentials for large u (see Figure 3). Thus the behavior of P(u), at least for $k^2\alpha^2\sin^2\theta/2\omega_c^2 < 1$, is quite like that of the derivative of a distribution function, in particular approaching zero from below at $u = +\infty$ and from above at $u = -\infty$. This property of P(u) can be proven from expression (4.4) in general; i.e., without choosing any special form for the initial distribution, f_o[8].

4.2 Simplification of the Transformed Dispersion Relation

It is notationally convenient to define the relation

$$F'(u) = P(u)/\omega_p^2 \tag{4.11}$$

so that the dispersion relation (4.3) becomes:

$$1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - \omega/k} \qquad \mathrm{Im}(\omega) > 0 \tag{4.12}$$

with

$$F'(u) = -\frac{2ue^{-s^2}}{\pi^{1/2}\alpha^3|\cos\theta|} \sum_{n=-\infty}^{\infty} I_n(s^2)\exp\left[-\left(\frac{ku - n\omega_c}{k\alpha\cos\theta}\right)^2\right] \tag{4.13}$$

$$s^2 = k^2\alpha^2\sin^2\theta / 2\omega_c^2 \tag{4.14}$$

so that the parameter s^2 is proportional to the square of the product of $k_\perp$ with the electron gyro radius. If the real part of the integral in (4.12) is interpreted in the Cauchy principle value sense in the limit $\omega_I \to 0$ where $\omega = \omega_R + i\omega_I$, i.e.,

$$\lim_{\omega \to \omega_R + i0} \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - \omega/k}\right) = P\!\!\int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - \omega_R/k}$$

$$= \lim_{\delta \to 0}\left[\left(\int_{-\infty}^{\omega_R/k - \delta} + \int_{\omega_R/k + \delta}^{\infty}\right) \frac{F'(u)\,du}{u - \omega_R/k}\right] \tag{4.15}$$

then the roots are the real parts of the desired characteristic frequencies of oscillation. In the Landau calculation, the real part, ω_R , was determined in this way from the expression analogous to (4.12), and the additional term resulting from the analytic continuation of that expression into the $\omega_I < 0$ half-plane was then used to obtain the small imaginary part, ω_I , by successive approximations. Although a somewhat less cumbersome technique will be used to obtain ω_I here, the requirement that ω_R be obtained first is the same. This section is devoted to solving equation (4.12) for ω_R , assuming k and ω_I small.

To reduce F'(u) to a more manageable form, all functions of s^2 are expanded in power series:

$$e^{-s^2} = 1 - s^2 + s^4/2 + O(s^6) \tag{4.16}$$

and from (4.10):

$$I_0(s^2) = 1 + s^4/4 + O(s^8) \tag{4.17}$$

$$I_1(s^2) = I_{-1}(s^2) = s^2/2 + O(s^6) \tag{4.18}$$

$$I_2(s^2) = I_{-2}(s^2) = s^4/8 + O(s^8) \tag{4.19}$$

$$I_3(s^2) = I_{-3}(s^2) = O(s^6) \tag{4.20}$$

Because of the k^2 term in the denominator of equation (4.12), it is necessary initially to keep terms to order s^4 in the expansions in order to solve (4.12) to order k^2. Five terms in the infinite sum of Bessel functions must therefore be retained, namely $n = -2, -1, 0, 1, 2$. Thus with expressions (4.16) through (4.19) the dispersion relation becomes, to order s^4:

$$1 = -\frac{2(1-s^2+s^4/2)}{\pi^{1/2}\alpha^3|\cos\theta|}\Bigg(\left\{1+\frac{s^4}{4}\right\}\frac{\omega_p^2}{k^2}\,P\!\!\int_{-\infty}^{\infty}\frac{u}{u-\omega_R/k}\exp\left\{-\left[\frac{u}{\alpha\cos\theta}\right]^2\right\}du$$

$$+\frac{s^2\omega_p^2}{2k^2}\,P\!\!\int_{-\infty}^{\infty}\frac{u}{u-\omega_R/k}\left\{\exp\left[-\left(\frac{ku-\omega_c}{k\alpha\cos\theta}\right)^2\right]+\exp\left[-\left(\frac{ku+\omega_c}{k\alpha\cos\theta}\right)^2\right]\right\}du$$

$$+\frac{s^4\omega_p^2}{8k^2}\,P\!\!\int_{-\infty}^{\infty}\frac{u}{u-\omega_R/k}\left\{\exp\left[-\left(\frac{ku-2\omega_c}{k\alpha\cos\theta}\right)^2\right]+\exp\left[-\left(\frac{ku+2\omega_c}{k\alpha\cos\theta}\right)^2\right]\right\}du\Bigg)$$

(4.21)

Let the three integrals in equation (4.21) be I_1, I_2, and I_3. The first is treated by expanding the denominator in powers of ku/ω_R, as in Landau's treatment. Thus

$$I_1 = \frac{\omega_p^2}{k^2}\,P\!\!\int_{-\infty}^{\infty}\frac{u\exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right]}{u-\omega_R/k}du$$

$$\approx -\frac{\omega_p^2}{\omega k}\int_{-\infty}^{\infty}u\exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right]\left[1+\frac{ku}{\omega_R}-\frac{k^2u^2}{\omega_R^2}+\frac{k^3u^3}{\omega_R^3}-O(k^4)\right]du$$

(4.22)

It should be recognized that the number of terms which must be kept to make the expansion in (4.22) a reasonable approximation increases with u, and that for u large enough, the expansion will not converge at all. This difficulty is not serious if the exponential vanishes rapidly enough with increasing u to insure that the dominant contribution to the integral occurs where u is small. Such an assumption is equivalent to requiring that, near the denominator's singularity at $u = \omega_R/k$, the term $\exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right]$ is sufficiently constant and near zero that the contribution of the singularity in the Cauchy principle value integral is negligible. Putting $u = \omega_R/k$ in the exponential makes this condition approximately

$$\omega_R^2/k^2\alpha^2\cos^2\theta > 1 \tag{4.23}$$

The required size of this parameter can be determined more exactly by considering the results of Jackson's numerical treatment of the Landau calculation without magnetic field.[4] In the Landau treatment, a condition exactly equivalent to (4.23) arises, namely:

$$\omega_R^2/k^2\alpha^2 > 1 \tag{4.24}$$

Jackson's results show that the approximate solution for the real part of the frequency is in good agreement with the numerical results if this parameter is greater than 2 (the corresponding number for the validity of the imaginary part is 4). The condition from the evaluation of the integral I_1 in the present analysis, for the real part of the frequency in a magnetic field, is therefore

$$\omega_R^2/k^2\alpha^2\cos^2\theta \geq 2 \tag{4.25}$$

Now returning to the evaluation of I_1 and noting that

$u^n \exp\left[-(u/\alpha\cos\theta)^2\right]$ is an odd function of u whenever n is odd, it is clear that the integrals over even powers of k in equation (4.22) all vanish. Thus of the first four terms in the expansion, only ku/ω_R and k^3u^3/ω_R^3 contribute, and there results:

$$I_1 \approx -\frac{\omega_p^2}{\omega_R^2}\int_{-\infty}^{\infty} u^2 \exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right] du - \frac{\omega_p^2 k^2}{\omega_R^4}\int_{-\infty}^{\infty} u^4 \exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right] du \tag{4.26}$$

Using the well known values of these definite integrals[11] gives:

$$I_1 \approx -\frac{\omega_p^2}{\omega_R^2}\frac{\sqrt{\pi}\,\alpha^3|\cos\theta|}{2}\left(\cos^2\theta + \frac{3}{2}\frac{k^2\alpha^2\cos^4\theta}{\omega_R^2}\right) \tag{4.27}$$

The treatment of the integral I_2 is somewhat more involved. It is apparent that the integrand of

$$I_2 = \frac{s^2\omega_p^2}{2k^2} P\!\!\int_{-\infty}^{\infty} \frac{u}{u-\omega_R/k}\left\{\exp\left[-\left(\frac{u-\omega_c/k}{\alpha\cos\theta}\right)^2\right] + \exp\left[-\left(\frac{u+\omega_c/k}{\alpha\cos\theta}\right)^2\right]\right\} du \tag{4.28}$$

consists, for small k, of two rather widely separated exponential pulses, centered at $u = \pm\omega_c/k$, and weighted by the term $u/(u-\omega_R/k)$. These components of the integrand are plotted schematically as Figure 4. The pulses differ substantially from zero only over a distance of the order of $\alpha\cos\theta$ from their centers, so that the major contributions to the integral occur in the vicinities where the term $u/(u-\omega_R/k)$ is reasonably approximated by Taylor series expansions about $u = \pm\omega_c/k$. If ω_R is much different from ω_c , the denominator's singularity at $u = \omega_R/k$ occurs where the numerator has gone exponentially to zero; since the integral is taken in the Cauchy principle value sense, the effect of the singularity is negligible in this case. The restriction on the size of $|\omega_R - \omega_c|$ appears more explicitly when the actual expansions are written down. Near $u = +\omega_c/k$, the Taylor series is

$$\frac{u}{u-\omega_R/k} = -\frac{\omega_c}{\omega_R-\omega_c} - \frac{(ku-\omega_c)}{(\omega_R-\omega_c)^2}\omega_R - \frac{(ku-\omega_c)^2}{(\omega_R-\omega_c)^3}\omega_R - \cdots \tag{4.29}$$

As ω_R approaches ω_c, the number of terms in the series which must be retained to represent $u/(u-\omega_R/k)$ accurately in the region of interest increases, correctly reflecting the fact that this factor is varying increasingly rapidly with u there. For ω_R very close to ω_c, (4.29) becomes a series representation of infinity—the singularity has moved into the region described by the expansion. For u near $-\omega_c/k$, the Taylor expansion is

$$\frac{u}{u-\omega_R/k} = \frac{\omega_c}{\omega_R+\omega_c} - \frac{(ku+\omega_c)}{(\omega_R+\omega_c)^2}\omega_R - \frac{(ku+\omega_c)^2}{(\omega_R+\omega_c)^3}\omega_R - \cdots \tag{4.30}$$

and the arguments given above hold for ω_R approaching $-\omega_c$.

The integral I_2 is now conveniently written as two integrals with the appropriate expansion introduced in each. From (4.28), (4.29) and (4.30):

$$I_z = \frac{s^2\omega_p^2}{2k^2}\left\{\int_{-\infty}^{\infty}\left[\frac{-\omega_c}{\omega_R-\omega_c} - \frac{(ku-\omega_c)}{(\omega_R-\omega_c)^2}\omega_R - \frac{(ku-\omega_c)^2}{(\omega_R-\omega_c)^3}\omega_R - \cdots\right]\right.$$

$$\times \exp\left[-\left(\frac{u-\omega_c/k}{\alpha\cos\theta}\right)^2\right]du$$

$$\left.+\int_{-\infty}^{\infty}\left[\frac{\omega_c}{\omega_R+\omega_c} - \frac{(ku+\omega_c)}{(\omega_R+\omega_c)^2}\omega_R - \frac{(ku+\omega_c)}{(\omega_R+\omega_c)^3}\omega_R - \cdots\right]\exp\left[-\left(\frac{u+\omega_c/k}{\alpha\cos\theta}\right)^2\right]du\right\} \tag{4.31}$$

Now making the substitutions $z = ku-\omega_c$ and $z = ku+\omega_c$ in the first and second integrals, respectively, and retaining the first four terms in each series, gives:

$$I_z = \frac{s^2\omega_p^2}{2k^2}\left[\left(\frac{\omega_c}{\omega_R+\omega_c} - \frac{\omega_c}{\omega_R-\omega_c}\right)\int_{-\infty}^{\infty}\exp\left(\frac{-z^2}{k^2\alpha^2\cos^2\theta}\right)\frac{dz}{k}\right.$$

$$\left.-\left(\frac{\omega_R}{(\omega_R-\omega_c)^3} + \frac{\omega_R}{(\omega_R+\omega_c)^3}\right)\int_{-\infty}^{\infty} z^2 \exp\left(\frac{-z^2}{k^2\alpha^2\cos^2\theta}\right)\frac{dz}{k}\right] \tag{4.32}$$

since the integrals involving odd powers of z all vanish. Using the known values of the definite integrals in (4.30), rearranging the coefficients, and replacing s^2 by its definition (4.14) gives finally:

$$I_z = -\frac{\pi^{1/2}\alpha^3\omega_p^2|\cos\theta|\sin^2\theta}{4\omega_c^2}\left[\frac{2\omega_c^2}{\omega_R^2-\omega_c^2} + \frac{\omega_R^2(\omega_R^2+3\omega_c^2)k^2\alpha^2\cos^2\theta}{(\omega_R^2-\omega_c^2)^3}\right] \tag{4.33}$$

Retaining more terms in the Taylor expansions would result in higher order k terms in this result. It is difficult to fix exactly the value of $|\omega_R-\omega_c|$ above which (4.33) can be expected to be valid. It is apparent, however, that since the effective width of the exponential pulses goes to zero for either $k = 0$ or $\cos\theta = 0$, the dispersion relation resulting from using I_2 in the form given by (4.33) should at least be valid in those two limits. For any but these special cases it must suffice at this point to note that roots which give ω_R near ω_c must be regarded with suspicion. A more detailed discussion of this difficulty is included in Chapter 6.

The integral, I_3, is treated in the same way as I_2. In this case the exponential pulses are centered at $u = \pm 2\omega_c/k$, and the necessary restriction on the result is that ω_R not be too near $2\omega_c$. The fact that I_3 contains the multiplier s^4/k^2 where I_2 has s^2/k^2 means that only the first term in the Taylor series need be retained in this case to obtain the result to order k^2. It is

$$I_3 = \frac{\pi^{1/2} k^2 \alpha^5 \omega_p^2 |\cos\theta| \sin^4\theta}{32\,\omega_c^4}\left(\frac{2\omega_c}{2\omega_c+\omega_R} + \frac{2\omega_c}{2\omega_c-\omega_R}\right) \tag{4.34}$$

or:

$$I_3 = \frac{-\pi^{1/2} k^2 \alpha^5 \omega_p^2 |\cos\theta| \sin^4\theta}{4(\omega_R^2 - 4\omega_c^2)} \tag{4.35}$$

The results for I_1, I_2 and I_3 can now be substituted back into the dispersion relation (4.21) to yield, discarding terms of order s^2k^2, s^4, and higher:

$$1 = \frac{\omega_p^2}{\omega_R^2}\cos^2\theta\left(1 - \frac{k^2\alpha^2\sin^2\theta}{2\omega_c^2}\right) + \frac{3}{2}\frac{\omega_p^2}{\omega_R^4}k^2\alpha^2\cos^4\theta$$

$$+ \frac{\omega_p^2\sin^2\theta}{\omega_R^2-\omega_c^2}\left(1 - \frac{k^2\alpha^2\sin^2\theta}{2\omega_c^2}\right) + \frac{1}{2}\frac{\omega_p^2}{\omega_c^2}\frac{k^2\alpha^2\sin^4\theta}{(\omega_R^2-4\omega_c^2)}$$

$$+\frac{1}{2}\left(\frac{\omega_p^2}{\omega_c^2}\right)\frac{\omega_R^2(\omega_R^2+3\omega_c^2)}{(\omega_R^2-\omega_c^2)^3}k^2\alpha^2\sin^2\theta\cos^2\theta \tag{4.36}$$

The neglect of terms of order s^4 requires that

$$s^4 = \frac{k^4\alpha^4\sin^4\theta}{4\omega_c^2} \ll 1 \tag{4.37}$$

The exact number to be placed on this condition is somewhat arbitrary, depending on the accuracy required in the result.

Equation (4.36) represents an enormous simplification of the dispersion relation over its original form. This simplification has been achieved, of course, only at the expense of restrictions on the parameters $k^2\alpha^2\cos^2\theta/\omega_R^2$, $k^2\alpha^2\sin^2\theta/2\omega_c^2$, $|\omega_R - \omega_c|$ and $|\omega_R - 2\omega_c|$.

4.3 Some Limiting Cases of the Simplified Dispersion Relation

The dispersion relation (4.36) is readily evaluated in several limiting cases for comparison with the previously known results. For k = 0, it becomes:

$$1 = \frac{\omega_p^2}{\omega_R^2}\cos^2\theta + \frac{\omega_p^2\sin^2\theta}{\omega_R^2-\omega_c^2} \tag{4.38}$$

and clearing the denominators and rearranging gives:

$$\omega_R^4 - \omega_R^2(\omega_c^2 + \omega_p^2) + \omega_p^2\omega_c^2\cos^2\theta = 0 \qquad (4.39)$$

Equation (4.39) is exactly the result of the macroscopic analysis with k = 0, as can be recognized by setting k = 0 in equation (3.5). The roots are thus the macroscopic roots given for $\omega_p^2 >> \omega_c^2$ and vice versa by equations (3.9) through (3.12) with k set equal to zero.

Particular care must be taken in examining the limits of sin θ or cos θ exactly zero. To take such a limit without introducing extraneous roots, the sines or cosines must be set to zero before clearing the denominators in equation (4.36) or (4.38). Thus in the case of k = 0, sin θ = 0, equation (4.38) becomes simply $\omega_R^2 = \omega_p^2$. The root $\omega_R^2 = \omega_c^2$, obtained by setting sin θ = 0, cos θ = 1 in (4.39) is an actual root of (3.5) but an extraneous one of (4.38), since in the latter equation the term giving rise to the root vanishes in the sin θ = 0 limit. The physical explanation for this difference is not hard to find: it will be remembered from Chapter 3 that the macroscopic root, $\omega^2 = \omega_c^2$, for perfectly parallel propagation, described a particle motion completely uncoupled from the electric field. Since the microscopic calculation carried out here is concerned only with the dispersion relation for the electric field, the root $\omega_R^2 = \omega_c^2$ logically does not appear in the parallel propagation limit.

For finite k, taking the sin θ = 0 limit correctly in the dispersion relation (4.36) gives:

$$1 = \frac{\omega_p^2}{\omega_R^2} + \frac{3}{2}\frac{\omega_p^2}{\omega_R^4}k^2\alpha^2 \tag{4.40}$$

or:

$$\omega_R^4 - \omega_p^2\omega_R^2 - \frac{3}{2}\omega_p^2 k^2\alpha^2 = 0 \tag{4.41}$$

Solving (4.40) for ω_R^2 with the usual quadratic formula and expanding the square root term in powers of $k^2\alpha^2/\omega_p^2$ gives, selecting the positive square root:

$$\omega_R^2 = \omega_p^2 + \frac{3}{2}k^2\alpha^2 + O(k^4\alpha^4/\omega_p^2) \tag{4.42}$$

The negative square root gives:

$$\omega_R^2 = -\frac{3}{2}k^2\alpha^2 + O(k^4\alpha^4/\omega_p^2) \tag{4.43}$$

Equation (4.43) clearly cannot represent a physically meaningful root of the dispersion relation, since the ω_R satisfying (4.43) is pure imaginary instead of pure real. Equation (4.42), however, is just the Landau result (2.20), as expected for propagation parallel to the magnetic field.

Taking the cos Θ = 0 limit of equation (4.36) for finite k gives:

$$1 = \frac{\omega_p^2}{\omega_R^2 - \omega_c^2}\left(1 - \frac{k^2\alpha^2}{2\omega_c^2}\right) + \frac{1}{2}\frac{\omega_p^2 k^2\alpha^2}{\omega_c^2(\omega_R^2 - 4\omega_c^2)} \tag{4.44}$$

which reduces to:

$$\omega_R^4 - \omega_R^2(5\omega_c^2 + \omega_p^2) + 4\omega_c^2\omega_p^2 + 4\omega_c^4 - \frac{3}{2}\omega_p^2 k^2\alpha^2 = 0 \tag{4.45}$$

Solving equation (4.41) by the quadratic formula gives:

$$2\omega_R^2 = 5\omega_c^2 + \omega_p^2 \pm \left[(3\omega_c^2 - \omega_p^2)^2 + 6k^2\alpha^2\omega_p^2 \right]^{1/2} \tag{4.46}$$

which is exactly Bernstein's result for the same limit (see equations (3.14)), except that Bernstein has not mentioned the positive square root possibility. The approximate roots of (4.46) for $\omega_p^2 \gg \omega_c^2$ are Bernstein's:

$$\omega_R^2 = 4\omega_c^2 - \frac{3}{2}k^2\alpha^2 \tag{3.15}$$

and that for the positive square root:

$$\omega_R^2 = \omega_p^2 + \omega_c^2 + \frac{3}{2}k^2\alpha^2 \tag{4.47}$$

which is the cos $\Theta = 0$ limit of Bernstein's arbitrary direction of propagation result, (3.17). For $\omega_c^2 \gg \omega_p^2$, the negative square root gives Bernstein's

$$\omega_R^2 = \omega_c^2 + \omega_p^2 - \frac{1}{2}k^2\alpha^2\frac{\omega_p^2}{\omega_c^2} \tag{3.16}$$

and the positive square root gives:

$$\omega_R^2 = 4\omega_c^2 + \frac{1}{2}k^2\alpha^2\frac{\omega_p^2}{\omega_c^2} \tag{4.48}$$

The cases above exhaust the possibilities for which the dispersion relation (4.36) can be solved in simple form. It is most encouraging that these microscopic results agree perfectly with the macroscopic and previous microscopic work in these cases of parallel and perpendicular propagation, and for k equal to zero. These results are summarized in Table 1.

It should also be noted that the usual Landau result for no imposed magnetic field cannot be recovered by setting

$\omega_c = 0$ in (4.36), since the restriction that $k^2\alpha^2 \sin^2\theta / 2\omega_c^2$ be small cannot possibly then be satisfied.

4.4 General Solution of the Simplified Dispersion Relation

In the general case, that is, k, sin θ and cos θ all non-zero, clearing all the denominators in equation (4.36) yields a sixth-order polynomial in ω_R^2. Such an equation cannot be solved by formula, and attempts made at approximate solution by successive approximations were discouraged by the impossibility of deciding in general which terms are small.

The only recourse is solution by computer. A simple program has been devised in the MAD language to compute the coefficients of the polynomial and call in a routine for its numerical solution. The routine is based on a method due to Muller[12] and is available as IBM SHARE listing 1124. The computations are done in the complex mode and the routine gives both real and imaginary parts for all roots. Since the dispersion relation (4.36) is an equation for the real part of the frequency, it should be expected that the imaginary parts be on the order of 10^6 or more smaller than the real parts, representing cumulative machine error. The inputs to the program are the desired values of θ, the plasma and cyclotron frequencies ω_p and ω_c, and the wave number times the Debye length:

$$kL_D = k\left(\frac{\epsilon_0 KT}{e^2 n_e}\right)^{1/2} = k\alpha / 2^{1/2}\omega_p \qquad (4.49)$$

Cases have been run for representative values of these parameters, and the results are presented and analyzed in Chapter 6. The program itself, named OMEGA, is included as the Appendix.

CHAPTER 5

THE DAMPING DECREMENT

5.1 A Complex Variable Approach to Landau Damping

Given a value for the real part of the frequency, ω_R, and assuming that the imaginary part, ω_I, is very small, ω_I can be obtained relatively easily using a complex variable technique. The quantity $\epsilon(\omega)$ is defined

$$\epsilon(\omega) \equiv 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{F'(u)\,du}{u-\omega/k} - 2\pi i\frac{\omega_p^2}{k^2}F'\left(\frac{\omega}{k}\right) \tag{5.1}$$

so that ϵ vanishing gives the familiar form of the analytically continued dispersion relation, (2.19). Defining the quantity $D(\omega) = k^2\,(1 - \epsilon)$, in general complex, and using (5.1) gives

$$D(\omega) = \omega_p^2\int_{-\infty}^{\infty}\frac{F'(u)\,du}{u-\omega/k} + 2\pi i\,\omega_p^2 F'\left(\frac{\omega}{k}\right) \tag{5.2}$$

Now consider the conformal mapping of the line $\omega = \omega_R$ [the Re(ω) axis] from the complex ω plane into the complex D plane. The result is a closed contour in the D plane, enclosing the conformal map of the complete $\omega_I > 0$ half plane, as shown in Figure 5. Now setting $\epsilon = 0$ gives the point in the D plane corresponding to the solution of the dispersion relation: $D(\omega) = k^2$, and k^2 is real and positive. The merit of the conformal mapping in the present context is simply to suggest from the form of the contour in Figure 5, that $D(\omega) = D(\omega_R + i\omega_I) = k^2$ becomes arbitrarily close to $D(\omega_R)$ as k approaches zero. This point, together with the fact that D is an analytic function of ω so that its derivatives are finite and continuous everywhere, motivates a Taylor series expansion of D about $\omega = \omega_R$:

$$D(\omega) = D(\omega_R) + i\omega_I D'(\omega_R) - \frac{\omega_I^2}{2} D''(\omega_R) + \cdots \tag{5.3}$$

Now with ω_I small, the expansion converges rapidly, and it is sufficient to retain only the first two terms. Separating the resulting expression into real and imaginary parts gives:

$$\mathrm{Re}\left[D(\omega_R)\right] + \omega_I \mathrm{Re}\left[iD'(\omega_R)\right] = k^2 \tag{5.4}$$

$$\mathrm{Im}\left[D(\omega_R)\right] + \omega_I \mathrm{Im}\left[iD'(\omega_R)\right] = 0 \tag{5.5}$$

The second of these equations is the more tractable. In the limit of $\omega_I \longrightarrow 0$, $D(\omega_R)$ is given by:

$$D(\omega_R) = \omega_p^2 P\!\!\int_{-\infty}^{\infty} \frac{F'(u)\,du}{u - \omega_R/k} + \pi i \omega_p^2 F'\left(\frac{\omega_R}{k}\right) \tag{5.6}$$

Equation (5.6) has πi in the last term instead of $2\pi i$ as in (5.1) as a consequence of the Plemelj formulas[9] which apply in this limit. From (5.6):

$$\mathrm{Im}\left[D(\omega_R)\right] = \pi \omega_p^2 F'\left(\frac{\omega_R}{k}\right) \tag{5.7}$$

$$\mathrm{Im}\left[iD(\omega_R)\right] = \omega_p^2 \frac{d}{d\omega_R}\left[P\!\!\int_{-\infty}^{\infty} \frac{F'(u)\,du}{u-\omega_R/k}\right] \tag{5.8}$$

and using equation (5.5) gives for ω_I :

$$\omega_I = \frac{-\pi F'\left(\frac{\omega_R}{k}\right)}{\frac{d}{d\omega_R}\left[P\!\!\int_{-\infty}^{\infty} \frac{F'(u)\,du}{u-\omega_R/k}\right]} \tag{5.9}$$

As a check on this means of obtaining ω_I , it will first be used to recover the familiar result for the Landau damping with no imposed magnetic field. In that case, with F(u) Maxwellian:

$$F(u) = \exp(-u^2/\alpha^2)/\pi^{1/2}\alpha \tag{5.10}$$

$$F'(u) = -2u\exp(-u^2/\alpha^2)/\pi^{1/2}\alpha^3 \tag{5.11}$$

where $\alpha = (2KT/m)^{1/2}$. The denominator of equation (5.9), labelled Y for notational convenience,is given by:

$$Y = \lim_{\delta\to 0} \frac{d}{d\omega_R}\left[\left(\int_{-\infty}^{\omega_R/k-\delta} + \int_{\omega_R/k+\delta}^{\infty}\right)\frac{F'(u)\,du}{u-\omega_R/k}\right] \tag{5.12}$$

which, using the well known formula for differentiation under an integral, gives

$$Y = \lim_{\delta\to 0}\left[\frac{F'(\omega_R/k-\delta)k}{-\delta} - \frac{F'(\omega_R/k+\delta)k}{\delta} + P\!\!\int_{-\infty}^{\infty}\frac{1}{k}\frac{F'(u)\,du}{(u-\omega_R/k)^2}\right] \tag{5.13}$$

Equation (5.13) can be shown to be equivalent to:

$$Y = \frac{k}{\omega_R^2} P\int_{-\infty}^{\infty} \frac{[F'(u) - F'(\omega_R/k)]}{(1 - ku/\omega_R)^2}\, du \tag{5.14}$$

Expanding the denominator and using (5.11) gives:

$$Y = \frac{-2k}{\pi^{1/2}\alpha^3\omega_R^2} \int_{-\infty}^{\infty} \left[u \exp(-u^2/\alpha^2) - \frac{\omega_R}{k} \exp(-\omega_R^2/k^2\alpha^2) \right]$$

$$\times \left[1 + \frac{2ku}{\omega_R} + \frac{3k^2u^2}{\omega_R^2} + \cdots \right] \tag{5.15}$$

For small k, and if u $<< \omega_R/k$ (the range where the significant contribution to the integral occurs), the term $(\omega_R/k) \exp(-\omega_R^2/k^2\alpha^2)$ is negligible compared to u exp $(-u^2/\alpha^2)$. Then noting that the integrals over odd powers of u vanish, discarding terms of order higher than k^2 in the expansion, and using the value of the one resulting nonzero definite integral gives

$$Y = -2k^2/\omega_R^3 \tag{5.16}$$

The numerator of expression (5.9) is

$$X \equiv -\pi F'\left(\frac{\omega_R}{k}\right) = \frac{2\pi\omega_R}{k\alpha^3} \exp(-\omega_R^2/k^2\alpha^2) \tag{5.17}$$

and the imaginary part of the frequency, ω_I, is then given by:

$$\omega_I = \frac{X}{Y} = \frac{-\pi^{1/2}\omega_R^4}{k^3\alpha^3} \exp(-\omega_R^2/k^2\alpha^2) \tag{5.18}$$

Writing down the form obtained by Landau, for comparison

$$\omega_I = \frac{-\pi^{1/2}\omega_p^4}{k^3\alpha^3}\exp\left(-\omega_p^2/k^2\alpha^2\right) \tag{2.21}$$

it is apparent that the complex variable approach automatically yields the correction proposed by Jackson[4], i.e., the replacing of ω_p^2 by $\omega_R^2 = \omega_p^2 + \frac{3}{2}k^2\alpha^2$ in the exponential, as well as the same, less important replacement in the coefficient.

5.2 Application to the Damping in a Magnetic Field

The analysis of the previous section is unchanged, except that now

$$F'(u) = -\frac{2e^{-s^2}}{\pi^{1/2}\alpha^3|\cos\theta|}\sum_{n=-\infty}^{\infty} I_n(s^2)\, u \exp\left[-\left(\frac{ku-n\omega_c}{k\alpha\cos\theta}\right)^2\right] \tag{4.13}$$

The functions of $s^2 = k^2\alpha^2\sin^2\theta/\omega_c^{\,2}$ are expanded just as in Chapter 4. It is now permissible to discard the terms of order s^4 which were retained previously, however, since the coefficient of the integral in which $F'(u)$ appears is now proportional to k instead of $1/k$. Expression (5.14) becomes:

$$\begin{aligned}
\gamma = \frac{-2k(1-s^2)}{\pi^{1/2}\alpha^3\omega_R^2|\cos\theta|}\Bigg(& P\!\!\int_{-\infty}^{\infty} \frac{\left\{u\exp\left[-\left(\frac{u}{\alpha\cos\theta}\right)^2\right] - \frac{\omega_R}{k}\exp\left[-\left(\frac{\omega_R}{k\alpha\cos\theta}\right)^2\right]\right\}du}{(1-ku/\omega_R)^2} \\
&+ \frac{s^2}{2} P\!\!\int_{-\infty}^{\infty}\Bigg\{u\exp\left[-\left(\frac{ku-\omega_c}{k\alpha\cos\theta}\right)^2\right] + u\exp\left[-\left(\frac{ku+\omega_c}{k\alpha\cos\theta}\right)^2\right] \\
&- \frac{\omega_R}{k}\exp\left[-\left(\frac{\omega_R-\omega_c}{k\alpha\cos\theta}\right)^2\right] - \frac{\omega_R}{k}\exp\left[-\left(\frac{\omega_R+\omega_c}{k\alpha\cos\theta}\right)^2\right]\Bigg\}\frac{du}{(1-ku/\omega_R)^2}\Bigg)
\end{aligned} \tag{5.19}$$

For convenience, let the two integrals in equation (5.19) be J_1 and J_2. J_1, with $\alpha\cos\theta$ replaced by α, is the same integral treated in obtaining the Landau damping decrement in the last section. The corresponding result here is

$$J_1 = \pi^{1/2} k\alpha^3 \cos^2\theta\,|\cos\theta|/\omega_R \tag{5.20}$$

The terms in J_2 having the coefficient ω_R/k are negligible only if $|\omega_R - \omega_c| \gg k\alpha\cos\theta$, essentially the same assumption required in obtaining the dispersion relation for the real part of the frequency in Chapter 4. Assuming that, consistent with Chapter 4, this restriction is met, J_2 becomes:

$$J_2 = \frac{s^2}{2}\left\{\frac{\omega_c/k}{(1-\omega_c/\omega_R)^2}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{u-\omega_c/k}{\alpha\cos\theta}\right)^2\right]du\right.$$

$$\left.\frac{-\omega_c/k}{(1+\omega_c/\omega_R)^2}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{u+\omega_c/k}{\alpha\cos\theta}\right)^2\right]du\right\} \tag{5.21}$$

where the term $u/(1 - ku/\omega_R)^2$ has been replaced by the first term of its Taylor series expansion about $u = \pm\,\omega_c/k$. Since the definite integrals are both equal to $\pi^{1/2}\alpha|\cos\theta|$, (5.21) becomes:

$$J_2 = \frac{2\pi^{1/2}\alpha|\cos\theta|s^2}{k}\left[\frac{\omega_c^2/\omega_R}{(1-\omega_c^2/\omega_R^2)^2}\right] \tag{5.22}$$

Then with (5.19), (5.20) and (5.22), and discarding terms of order higher than k^2, the denominator of the expression for ω_I is:

$$Y \approx -\frac{2k^2}{\omega_R^3}\left[\cos^2\theta + \frac{\sin^2\theta}{(1-\omega_c^2/\omega_R^2)^2}\right] \tag{5.23}$$

Now using the expression for F'(u), (4.13), to obtain the numerator in the expression for ω_I :

$$X \equiv -\pi F'\left(\frac{\omega_R}{k}\right) \approx \frac{2\pi^{1/2}}{\alpha^3|\cos\theta|}\frac{\omega_R}{k}\exp\left(\frac{-\omega_R^2}{k^2\alpha^2\cos^2\theta}\right) \tag{5.24}$$

where only the lowest order term in F'(u) has been kept. Then ω_I is approximately

$$\omega_I = \frac{X}{Y} \approx \frac{-\pi^{1/2}\omega_R^4}{k^3\alpha^3|\cos\theta|}\frac{\exp\left(-\frac{\omega_R^2}{k^2\alpha^2\cos^2\theta}\right)}{\left[\cos^2\theta + \frac{\omega_R^4\sin^2\theta}{(\omega_R^2-\omega_c^2)^2}\right]} \tag{5.25}$$

Although it would not be difficult to calculate the next order of k terms in expressions (5.23) and (5.24), this will not be done here. Since the imaginary part of the frequency is already very small compared to ω_R , calculating it to a great many significant figures does not appear warranted. It should be pointed out, however, that were such accuracy desired, several additional terms in k could be kept before it became necessary to use more than two terms in the series (5.3), since (5.25) shows ω_I to be transcendentally small in k.

The approximate result for ω_I given by equation (5.25) has the expected behavior in the easily examined limits, $\sin\theta$ or $\cos\theta = 0$. For $\sin\theta = 0$, or propagation parallel to the magnetic field, $\cos\theta$ is one and the Landau result for no imposed field, (5.18), is recovered. For $\cos\theta = 0$, or perpendicular propagation, the term $\exp(-\omega_R^2/k^2\alpha^2\cos^2\theta)$ vanishes much more rapidly than the $\cos\theta$ term in the denominator, and the damping is zero, as expected for that case.

It should be pointed out that expression (5.25) is not in general agreement with Bernstein's limiting case results for the damping decrement [equations (3.18) and (3.20)], the most significant difference being the $\cos^2\theta$ term in the exponential of the present result. It will be remembered, however, that Bernstein's (3.18) does not have the seemingly required property of vanishing for perpendicular propagation, and should probably be regarded with some suspicion.

The general behavior of the present expression for damping, (5.25), is examined in detail in the next chapter, where numerical values corresponding to the real parts of the characteristic frequencies for representative cases are computed.

CHAPTER 6

NUMERICAL RESULTS AND CONCULSIONS

6.1 Computer Results and Interpretation for the Real Part of the Frequency

As pointed out in Section 4.4, the IBM 7094 computer was used to solve the sixth order polynomial in ω_R^2 resulting from clearing the denominators in the simplified dispersion relation, (4.36). It will be remembered from Section 4.3 that in the limits of parallel and perpendicular propagation with respect to the applied magnetic field, the dispersion relation reduces to a quadratic in ω_R^2 and can be solved analytically. Clearly, there are only two actual roots in these limits—the other four obtained by the computer there are extraneous and result from clearing the denominators of terms with zero coefficients.

For values of the propagation angle between zero and ninety degrees, some roots of the sixth order polynomial may still not be even approximate roots of the exact dispersion relation (4.12). Such extraneous roots have at least two sources: the truncation of the infinite series as it appears in (4.13) could produce a spurious root, and certainly when the approximations of Section 4.2 on $|\omega_R-\omega_c|$ or $|\omega_R-2\omega_c|$

are violated, the polynomial which results from (4.36) cannot be considered a valid approximation of the dispersion relation.

As is well known, a sixth order polynomial with real coefficients may have one, two, three or no pairs of complex conjugate roots. Since the dispersion relation which the present polynomial attempts to approximate is an equation for the real part of the frequency, any complex conjugate pairs of roots of the polynomial cannot be actual roots of the dispersion relation. When such complex roots occur, then, they can be discarded as extraneous. It is not immediately apparent from which possible source these roots come, nor is it certain that all spurious roots will give themselves away, so to speak, by appearing as complex roots.

Tables 2 through 8 summarize the computer results for representative values of the parameters ω_p, ω_c, and kL_D. The spurious negative root obtained in the $\sin\theta = 0$ limit [see equation (4.43)] appears in the computer output as one of the six roots throughout the theta range in almost every case—it has not been tabulated here. At the $\theta = 0$ and 90-degree limits, only the computer roots corresponding to the known actual roots there are given. Where complex conjugate pairs occur, the approximate value of the real part is included beside the notation COMPLEX. The value of $k^2\alpha^2$ corresponding to kL_D and ω_p for each case has been given as an aid to estimating the validity of the approximations on $k^2\alpha^2 \sin^2\theta/2\,\omega_c^2$ and $k^2\alpha^2\cos^2\theta/\,\omega_R^2$. The notation used in the tables is such that, for example, 3.0×10^{12} is written

3.0(12). Tabulated roots recognizable as previously known forms are so labelled. Frequencies are in $\sec^{-1}$.

The values of $\omega_p = 6 \times 10^{12}$ and $\omega_c = 3 \times 10^{11}$ used to obtain Table 2 correspond to a thermonuclear plasma. The value of $k^2\alpha^2$ is sufficiently small that $k^2\alpha^2\sin^2\theta/2\omega_c^2$ and $k^2\alpha^2\cos^2\theta/\omega_R^2$ are always $<< 1$. Since $\omega_p^2 >> \omega_c^2$, Bernstein's expression for that limit can be used for comparison:

$$\omega_R^2 = \omega_p^2 + \omega_c^2\sin^2\theta + \frac{3}{2}k^2\alpha^2 \tag{3.17}$$

The values of ω_R^2 given by equation (3.17) are in agreement with the computer root $\omega_{R_1}^2$ to six significant figures throughout the theta range. The root

$$\omega_R^2 = 4\omega_c^2 - \frac{3}{2}k^2\alpha^2 \tag{3.15}$$

predicted by Bernstein for perpendicular propagation is given exactly by $\omega_{R_2}^2$ (the $k^2\alpha^2$ terms do not appear until the tenth significant figure in this case). The root $\omega_{R_2}^2 = 4\omega_c^2$ appears constant over the theta range, but as has already been pointed out it cannot be an actual root for θ exactly zero. In fact, since it appears to violate the restriction that $|\omega_R - 2\omega_c|$ be large, it must be viewed with suspicion everywhere except at θ = 90 degrees (see Section 4.3). The root $\omega_{R_3}^2$ agrees with the root obtained from the macroscopic analysis in the $\omega_p^2 >> \omega_c^2$ limit,

$$\omega_R^2 = \omega_c^2\cos^2\theta - \omega_c^2\frac{\omega_c^2}{\omega_p^2}\sin^2\theta\cos^2\theta \tag{3.10}$$

to five significant figures throughout its real range. Between

10 and 5 degrees, as $\cos\theta \longrightarrow 1$, $\omega_{R_3}^2$ becomes complex and must be considered a spurious root. The roots $\omega_{R_4}^2$ and $\omega_{R_5}^2$ comprise a complex conjugate pair (whose real part is near ω_c^2) throughout the theta range, and are therefore spurious.

Table 3 is based on the same values of ω_p and ω_c as Table 2, but with kL_D increased to 1.58×10^{-2} (corresponding to decreasing the wavelength of the oscillation component being investigated compared to the Debye length). Assuming that the smallest root is approximately $\omega_c^2\cos^2\theta$, the maximum value of $k^2\alpha^2\cos^2\theta/\omega_R^2$ is then about 1/5, so that the approximate restriction given by equation (4.25) is still met. However, the $k^2\alpha^2$ contributions to the roots are certainly no longer negligible. The root $\omega_{R_1}^2$ from Table 3 agrees with the values given by Bernstein's expression, (3.17), to five or more significant figures throughout the theta range. The root $\omega_{R_2}^2$ agrees to four significant figures with Bernstein's (3.15) in the perpendicular propagation limit, and the agreement can be extended to six places if one more term in the expansion leading to (3.15) is kept. This root is approximated very well over the entire theta range by $\omega_R^2 = 4\omega_c^2 - \frac{3}{2}k^2\alpha^2\sin^4\theta$, but once again this root is suspicious in view of the restriction on $|\omega_R - 2\omega_c|$. The root $\omega_{R_3}^2$ still has basically an $\omega_c^2\cos^2\theta$ variation, but with relatively large deviations due to the now important $k^2\alpha^2$ terms. The form of these terms is inaccessible to the macroscopic approach and to direct calculation from the present microscopic expressions; they might be obtained by curve fitting to the computer data, but this has not yet been done. Finally, the roots $\omega_{R_4}^2$ and $\omega_{R_5}^2$ are again a complex conjugate pair with real part near ω_c^2 throughout the theta range.

Tables 4, 5, and 6 present the results of computer runs done for $\omega_p = \omega_c = 1 \times 10^{11}$. These frequencies are appropriate for a hot plasma in a laboratory magnetic field. The computer roots are tabulated only from 5 to 85 degrees. The correct roots at $\theta = 0$ and $\theta = 90$ degrees are obtained from equations (4.42) and (4.46) more easily than they are sorted out of the spurious computer roots in those limits. The three tables are for $kL_D = .001$, .1, and .3, respectively, corresponding to increasing importance of thermal effects. The computer roots for $\omega_p = \omega_c$ and theta not 0 or 90 degrees are of particular interest because this is exactly the case which is inaccessible to previous approximate microscopic treatments, notably Bernstein's. The k = 0 limit of the macroscopic dispersion relation (3.15) may be of use in identifying one of the roots in Table 4, but with that exception there are no analytical expressions with which to compare the values in Tables 4, 5, and 6. The assumption that at least some of the roots are correct is based on the fact that the expression which yields them gives the correct results for all the known limiting cases in which its restrictions are not violated.

The general form of the roots is not hard to obtain from the Tables: $\omega_{R_1}^2$ varies between ω^2 and $2\omega^2$ (where $\omega^2 = \omega_p^2 = \omega_c^2$) and when thermal effects are unimportant, as in Table 4, the variation is given almost exactly by

$\omega_{R_1}^2 = \omega^2(1 + \sin\theta)$, which is, in fact, just the result obtained if one sets $\omega_p = \omega_c$ and k = 0 in the macroscopic dispersion relation (3.5). The second root is the by now familiar $\omega_{R_2}^2 = 4\omega_c^2$ with thermal corrections taking on considerable importance in Tables 5 and 6. The root $\omega_{R_3}^2$ decreases rapidly from ω^2 with increasing theta, but does not quite fit $\omega^2\cos\theta$ or $\omega^2\cos^2\theta$ even when thermal corrections are unimportant. The point that any of these roots are suspicious when near ω_c or $2\omega_c$ need hardly be belabored further here.

Tables 7 and 8, when used with Table 6, show the variation of the roots as ω_c is varied from slightly less than through slightly greater than ω_p, which is held fixed at 10". kL_D is fixed at 0.3, where thermal effects are relatively important. It appears from these results that $\omega_{R_1}^2$ varies essentially from the larger of ω_p and ω_c to the sum of the two as theta increases from 0 to 90 degrees, with a thermal variation which increases the frequency at small theta and decreases it as perpendicular propagation is approached. The root $\omega_{R_2}^2$ is a constant $4\omega_c^2$ plus thermal variations with theta regardless of the relative size of ω_p^2, and $\omega_{R_3}^2$ seems to be proportional to the smaller of ω_p^2 and ω_c^2 times a decreasing function of θ. Additional runs not tabulated here reinforce these conclusions. The approximate behavior of the roots for ω_p near ω_c is indicated schematically in Figure 6.

Runs were also carried out for $\omega_c^2 \gg \omega_p^2$. In these cases a peculiarity of the polynomial solving computer routine resulted in meaningless data. The difficulty is not, in principle, a serious one, but sufficient time was not available for this work to resolve it by using another routine.

The basic conclusion to be drawn on the basis of the computer results summarized here is that the present, simplified dispersion relation, (4.36), provides accurate values for in the range between parallel and perpendicular propagation, including the case, ω_c near ω_p (which is not accessible to simple expansion of the microscopic dispersion relation), provided the restriction that $|\omega_R-\omega_c|$ and $|\omega_R-2\omega_c|$ not be too small is met. It is surmised from the evidence in the tables that the complex behavior of some roots results from the violation of the $|\omega_R-\omega_c|$ restriction, although it is not at all clear that a root becomes complex as soon as the root violates that restriction. The roots near $4\omega_c^2$ do not, in general, exhibit the same complex behavior, although they appear to make $|\omega_R-2\omega_c|$ small.

Some further comments on the difficulty for ω_R very near ω_c or $2\omega_c$ are appropriate. The successive contributing terms in the Taylor series expansions (4.29) and (4.30) are in the ratio $(ku\mp\omega_c)^2/(\omega_R\mp\omega_c)^2$ for ω_R very near ω_c. Since the half width of the exponential pulse centered at $u = \pm\omega_c/k$ is on the order of $\alpha\cos\theta$, the condition that the series converge rapidly over the nonzero region of the pulse can be written, approximately

$$\frac{k^2\alpha^2\cos^2\theta}{(\omega_R-\omega_c)^2} \ll 1 \qquad (6.1)$$

Now, for example, with $\omega_c^2 >> \omega_p^2$ and ω_R^2 differing from ω_c^2 only by, say, $\omega_p^2 \sin^2\theta + \frac{3}{2} k^2\alpha^2 \sin^2\theta$ as in the macroscopic root (3.11), it is not hard to see that the restriction (6.1) is violated for almost any $\theta \neq 90$ degrees. A more interesting case in the light of the present computer results is the equivalent restriction arising when ω_R is very near $2\omega_c$:

$$\frac{k^2\alpha^2\cos^2\theta}{(\omega_R - 2\omega_c)^2} << 1 \qquad (6.2)$$

The computer solution gives a root at $4\omega_c^2$ plus some theta-varying thermal terms, which corresponds to Bernstein's root for perpendicular propagation, (3.15), at $\sin\theta = 1$. If one writes this computer root in the general form, $\omega_R^2 \approx 4\omega_c^2 + k^2\alpha^2 f(\theta)$ where $f(\theta)$ is nonzero except at $\theta = 0$, (6.2) becomes, approximately:

$$\frac{1}{16}\frac{k^2\alpha^2}{\omega_c^2}\left[f(\theta)\right]^2 >> \cos^2\theta \qquad (6.3)$$

The restriction (6.3) seems certain to be violated for any θ more than a few degrees from perpendicular, since from the form of the computer root $f(\theta)$ is something like $\frac{3}{2}\sin^4\theta$, and $k^2\alpha^2/\omega_c^2$ is restricted by (4.25) to be smaller than 1/2. Thus the computer root near $4\omega_c^2$ must be spurious for all but the immediate region around perpendicular propagation.

6.2 Some Numerical Values for the Damping

It is a simple matter, given any value for the real part of the characteristic frequency of oscillation, ω_R , to calculate the imaginary part or damping decrement from the formula

$$\omega_I = \frac{\pi^{1/2}\omega_R^4}{k^3\alpha^3|\cos\theta|}\,\frac{\exp(-\omega_R^2/k^2\alpha^2\cos^2\theta)}{\left[\cos^2\theta + \dfrac{\omega_R^4}{(\omega_R^2-\omega_c^2)^2}\sin^2\theta\right]} \tag{5.25}$$

It is apparent that the damping is, in general, dominated by the exponential, and that it can only become appreciable when the ratio $\omega_R^2/k^2\alpha^2\cos^2\theta$ is not too large. As pointed out in Section 4.2, however, there is a lower limit of about 4 on this ratio, below which equation (5.25) cannot be considered valid. If the different real roots corresponding to a given set of plasma parameters are widely separated in magnitude, as they are if $\omega_p^2 \gg \omega_c^2$ or vice versa, the above restriction means that appreciable damping can be calculated within the limitations of the present analysis only for the smallest roots.

For example, using the results from Table 3, where $\omega_p = 6 \times 10^{12}$, $\omega_c = 3 \times 10^{11}$, and $kL_D = 1.58 \times 10^{-2}$, and calculating the damping for the largest root, $\omega_{R_1}^2 \approx \omega_p^2$, gives, at $\theta = 0$ degrees: $\omega_I \sim 10^{17} \times \exp(-2000)$, an infinitesimal number indeed. The damping is virtually zero even for parallel propagation, and decreases still further as $\cos\theta$ decreases from 1. For the same plasma parameters the root $\omega_{R_2}^2 \approx 4\omega_c^2$, if in fact it exists for other than perpendicular propagation, is only very slightly damped when $\theta \neq 90$

degrees. At θ = 10 degrees, for example, ω_{R_2} from Table 3 is 6×10^{11} and (5.25) gives $\omega_I = -1.1 \times 10^5$, which is less damping than could be expected from collisions. The smallest characteristic frequency, $\omega_{R_3}^2 \approx \omega_c^2 \cos^2\theta$ is, however, very heavily damped. For this root, Table 3 and (5.22) give:

θ	ω_{R_3}	$-\omega_I$
5°	2.81(11)	.58(11)
30°	2.39(11)	.28(11)
80°	5.76(10)	.14(10)

It is apparent that at the wavelength corresponding to the present value of kL_D , the characteristic frequency near $\omega_c \cos\theta$ is almost completely damped out after on the order of ten oscillations for small propagation angles. As expected, the damping decreases as perpendicular propagation is approached, although relatively slowly because of the cos θ dependence of the real part of this root.

When the cyclotron and plasma frequencies do not differ greatly, all the roots may have appreciably nonzero damping. For example, the results of Table 7, where $\omega_p = 1 \times 10^{11}$, $\omega_c = 8 \times 10^{10}$, $kL_D = 0.3$, give, with (5.25):

θ	ω_{R_1}	$-\omega_{I_1}$	ω_{R_2}	$-\omega_{I_2}$	ω_{R_3}	$-\omega_{I_3}$
5°	1.11(11)	3.2(9)	1.60(11)	9.3(6)	7.35(10)	2.5(10)
30°	1.23(11)	4.3(7)	1.61(11)	7.5(4)	6.08(10)	1.9(10)
60°	1.25(11)	4.8(-3)	1.64(11)	1.7(-13)	3.75(10)	1.3(10)

The root ω_{R_1} is significantly Landau damped near parallel propagation, but for propagation angles greater than about 30 degrees this damping becomes negligible compared to that which would be caused by collisions. For the $\omega_{R_2} \approx 2\omega_c$ root, Landau damping is probably less important than collisional damping even well away from perpendicular propagation (if the root exists there). The third and smallest root, ω_{R_3} , is very heavily Landau damped over most of the theta range, so that it effectively ceases to exist after only a few oscillations. The damping of this root very nearly represents the maximum damping, ω_I/ω_R , which can be calculated within the limits of the present analysis, since $\omega_{R_3}^2/k^2\alpha^2\cos^2\theta \approx 3$ is already stretching the restriction on that ratio established in Section 4.2.

6.3 Conclusions and Suggestions for Further Work

The foregoing analysis has reduced the dispersion relation for plasma oscillations in a uniform external magnetic field to an algebraic equation in the real part of the frequency and an expression for the imaginary part of the frequency, or damping, in terms of the real part. The essential restrictions on the analysis are that $\omega_R^2/k^2\alpha^2\cos^2\theta$, $2\omega_c^2/k^2\alpha^2\sin^2\theta$, $(\omega-\omega_c)^2/k^2\alpha^2\cos^2\theta$ and $(\omega-2\omega_c)^2/k^2\alpha^2\cos^2\theta$ all be large. The results for the real part of the frequency are correct in the previously known limiting cases when the restrictions are not violated, and the expression for the damping has the known correct behavior in the limits of parallel and perpendicular propagation. An important result of the present work is that the previously inaccessible case of arbitrary direction of propagation when the plasma frequency is of the same order of magnitude as the cyclotron frequency can now be handled.

Two important areas for continuation of this work are immediately apparent. First, the expressions for the real and imaginary parts of the frequency should be evaluated at small intervals over a broad range of ω_p/ω_c , with particular attention to the previously unexplored range near $\omega_p/\omega_c = 1$. For detailed information as to the variation of the frequencies with propagation angle and wave number, smaller intervals in the range of these parameters should also be sampled. Such results would be of greatest value if presented in completely nondimensional, graphical form.

Second, an attempt should be made to circumvent the inability of the present analysis to treat frequencies with real parts very near ω_c and $2\omega_c$. At the very least, a more detailed specification of the points where roots near these values must be considered spurious would be useful.

Another worthwhile direction for future work is the extension of the present analysis to non-Maxwellian initial electron distribution functions. An interesting example would be the well known anisotropic distribution function[8], for which the electron temperatures parallel and perpendicular to the magnetic field are not the same. In this case the plasma is no longer necessarily stable with respect to electrostatic oscillations, and roots may be found with either positive or negative imaginary parts.

Appendix

Computer Program for the Real Part of the Frequency:

OMEGA

```
  OMEGA MAD

  DIMENSION THETA(20),C(16),RR(15),RI(15)
  INTEGER N
  INTEGER X
  INTEGER Y
  INTEGER Z
  C(1)=1.
  N=6
REPEAT   PRINT COMMENT $THETA VALUES AND PARAMETERS$
  READ DATA
  PRINT COMMENT $DISPERSION RELATION ROOTS$
  K=2.*KLD.P.2
  R=(WP/WC).P.2
  H=K*R
  X=0
SWITCH   X=X+1
  WHENEVER X.E.Y, TRANSFER TO REPEAT
  THET=THETA(X)
  THT=.017453293*THETA(X)
  COST=COS.(THT)
  COST2=COST.P.2
  COST4=COST2.P.2
  F=-H*(1.-COST2)/2.
  C(2)=-7.-R*F-.5*R*H*(1.-COST2)
  C(3)=15.+R*F*(6.+COST2)+R*H*(1.5-2.5*COST2-.5*COST4)
  C(4)=-13.-R*F*(9.+6.*COST2)+R*H*(-1.5+9.*COST2+3.* COST4)
  C(5)=4.+R*F*(4.+9.*COST2)+R*H*(.5-COST2-22.*COST4)
  C(6)=-4.*R*F*COST2+19.5*R*H*COST4
  C(7)=-6.*R*H*COST4
  EXECUTE MULLER.(C,N,RR,RI)
  WC2=WC.P.2
  THROUGH A, FOR VALUES OF Z=1,2,3,4,5,6
  RR(Z)=WC2*RR(Z)
A RI(Z)=WC2*RI(Z)
  PRINT RESULTS THET
  PRINT RESULTS RR(1)...RR(6),RI(1)...RI(6)
  TRANSFER TO SWITCH
  END OF PROGRAM
```

TABLE 1

Comparison of the Present Microscopic Expression with the Macropscopic Results and Bernstein's Results in Certain Limiting Cases

ω_R^2, Present Result	ω_R^2, Bernstein	ω^2, Macroscopic
	($k=0 \quad \omega_p^2 \gg \omega_c^2$)	
$\omega_p^2 + \omega_c^2 \sin^2\theta$	$\omega_p^2 + \omega_c^2 \sin^2\theta$	$\omega_p^2 + \omega_c^2 \sin^2\theta$
$\omega_c^2 \cos^2\theta$	—	$\omega_c^2 \cos^2\theta$
	($k=0 \quad \omega_c^2 \gg \omega_p^2$)	
$\omega_c^2 + \omega_p^2 \sin^2\theta$	—	$\omega_c^2 + \omega_p^2 \sin^2\theta$
$\omega_p^2 \cos^2\theta$	$\omega_p^2 \cos^2\theta$	$\omega_p^2 \cos^2\theta$
	($\sin\theta = 0$)	
$\omega_p^2 + \frac{3}{2}k^2\alpha^2$	$\omega_p^2 + \frac{3}{2}k^2\alpha^2$	$\omega_p^2 + \frac{3}{2}k^2\alpha^2$
	($\cos\theta = 0 \quad \omega_p^2 \gg \omega_c^2$)	
$\omega_p^2 + \omega_c^2 + \frac{3}{2}k^2\alpha^2$	$\omega_p^2 + \omega_c^2 + \frac{3}{2}k^2\alpha^2$	$\omega_p^2 + \omega_c^2 + \frac{3}{2}k^2\alpha^2$
$4\omega_c^2 - \frac{3}{2}k^2\alpha^2$	$4\omega_c^2 - \frac{3}{2}k^2\alpha^2$	—
	($\cos\theta = 0 \quad \omega_c^2 \gg \omega_p^2$)	
$\omega_c^2 + \omega_p^2 - \frac{1}{2}k^2\alpha^2\frac{\omega_p^2}{\omega_c^2}$	$\omega_c^2 + \omega_p^2 - \frac{1}{2}k^2\alpha^2\frac{\omega_p^2}{\omega_c^2}$	$\omega_c^2 + \omega_p^2 + \frac{3}{2}k^2\alpha^2$
$4\omega_c^2 + \frac{1}{2}k^2\alpha^2\frac{\omega_p^2}{\omega_c^2}$	—	—

TABLE 2

$\omega_p = 6(12)$ $\omega_c = 3(11)$ $kL_D = 3.54(-6)$ $k^2\alpha^2 = 9.02(12)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$	$\omega_{R_5}^2$
0	36.0000(24)	--	--	--	--
5	36.0007(24)	36.0000(22)	8.9(22)COMPLEX*	9.0(22)	COMPLEX
10	36.0027(24)	"	8.72875(22)	"	
30	36.0225(24)	"	6.74578(22)	"	
60	36.0675(24)	"	2.24579(22)	"	
80	36.0873(24)	"	2.70727(21)	"	
85	36.0893(24)	"	6.81959(20)	"	
90	36.0900(24)	"	--	--	--
	$\omega_p^2 + \omega_c^2 \sin^2\theta + \cdots$	$4\omega_c^2 - \cdots$	$\omega_c^2 \cos^2\theta - \cdots$		

*The conjugate to this root replaced the usual spurious negative root (not tabulated) at this one angle.

TABLE 3

$\omega_p = 6(12)$ $\omega_c = 3(11)$ $kL_D = 1.58(-2)$ $k^2\alpha^2 = 1.8(22)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$	$\omega_{R_5}^2$
0	36.02694(24)	--	--	--	--
5	36.02763(24)	35.9998(22)	7.91011(22)	9.0(22)COMPLEX	
10	36.02967(24)	35.9970(22)	7.33861(22)	"	
30	36.04953(24)	35.8086(22)	5.72054(22)	"	
60	36.09463(24)	34.4503(22)	2.57377(22)	"	
80	36.11443(24)	33.4459(22)	3.32675(21)	"	
85	36.11646(24)	33.3265(22)	8.41477(20)	"	
90	36.11714(24)	33.2856(22)	--	--	—
	$\omega_p^2+\omega_c^2\sin^2\theta + \cdots$	$4\omega_c^2 - \cdots$	$\omega_c^2\cos^2\theta - \cdots$		

TABLE 4

$\omega_p = 1(11)$ $\omega_c = 1(11)$ $kL_D = 1(-3)$ $k^2\alpha^2 = 2(16)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2, \omega_{R_5}^2$
0	--	--	--	--
5	1.08719(22)	4.00000(22)	9.12827(21)	1(22)COMPLEX
10	1.17367(22)	"	8.26347(21)	"
30	1.50001(22)	"	5.00001(21)	"
60	1.86603(22)	"	1.33975(21)	"
80	1.98481(22)	"	1.51923(20)	"
85	1.99619(22)	"	3.80532(19)	"
90	--	--	--	--

$4\omega_c^2 + \cdots$

TABLE 5

$\omega_p = 2(11)$ $\omega_c = 1(11)$ $kL_D = 1(-1)$ $k^2\alpha^2 = 2(20)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$	$\omega_{R_5}^2$
0	--	--	--	--	--
5	1.16541(22)	4.00000(22)	8.71206(21)	1(22)COMPLEX	
10	1.25636(22)	4.00001(22)	7.98612(21)	"	
30	1.56522(22)	4.00086(22)	5.08675(21)	"	
60	1.88089(22)	4.00817(22)	1.39979(21)	"	
80	1.97369(22)	4.01396(22)	1.59577(20)	"	
85	1.98226(22)	4.01465(22)	3.99887(19)	"	
90	--	--	--	--	--

$4\omega_c^2 + \cdots$

TABLE 6

$\omega_P = 1(11)$ $\omega_c = 1(11)$ $kL_D = 3(-1)$ $k^2\alpha^2 = 1.8(21)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$	$\omega_{R_5}^2$
0	--	--	--	--	--
5	1.34684(22)	4.00001(22)	8.38157(21)	1(22)COMPLEX	
10	1.47367(22)	4.00011(22)	7.63811(21)	"	
30	1.82902(22)	4.00793(22)	5.41673(21)	"	
60	1.96983(22)	4.07259(22)	1.75365(21)	"	
80	1.89293(22)	4.11973(22)	2.05124(20)	"	
85	1.87837(22)	4.12511(22)	5.14998(19)	"	
90	--	--	--	--	--

$4\omega_c^2 + \cdots$

TABLE 7

$\omega_p = 1(11)$ $\omega_c = 8(10)$ $kL_D = 3(-1)$ $k^2\alpha^2 = 1.8(21)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$	$\omega_{R_5}^2$
0	--	--	--	--	--
5	1.24422(22)	2.56001(22)	5.44577(21)	$64(10^{20})$COMPLEX	
10	1.29854(22)	2.56023(22)	4.98440(21)	"	
30	1.52811(22)	2.57659(22)	3.71060(21)	"	
60	1.54891(22)	2.70323(22)	1.40851(21)	"	
80	1.43093(22)	2.78260(22)	1.69189(20)	"	
85	1.41259(22)	2.79110(22)	4.25395(19)	"	
90	--	--	--	--	--

$4\omega_c^2 + \cdots$

TABLE 8

$\omega_p = 1(11)$ $\omega_c = 1.5(11)$ $kL_D = 3(-1)$ $k^2a^2 = 1.8(21)$

Theta	$\omega_{R_1}^2$	$\omega_{R_2}^2$	$\omega_{R_3}^2$	$\omega_{R_4}^2$		$\omega_{R_5}^2$
0	--	--	--	--		--
5	2.48615(22)	9.00000(22)	1.20168(22)		2.25(22)COMPLEX	
10	2.63241(22)	9.00004(22)	1.14910(22)		"	
30	3.05342(22)	9.00286(22)	7.89515(21)		"	
60	3.28289(22)	9.02610(22)	2.31223(21)		"	
80	3.22429(22)	9.04380(22)	2.67964(20)		"	
85	3.20914(22)	9.04587(22)	6.72487(19)		"	
90	--	--	--	--		--

$4\omega_c^2 + \cdots$

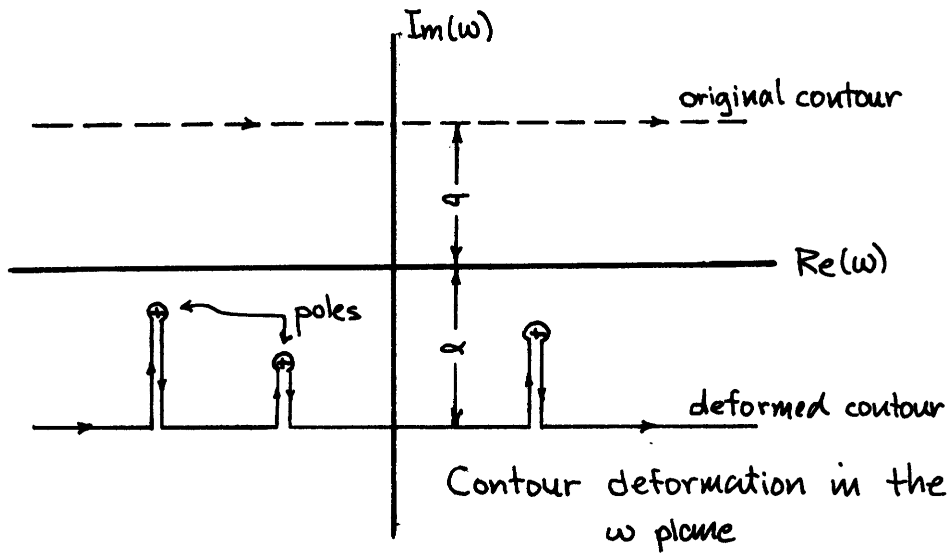

Figure 1.

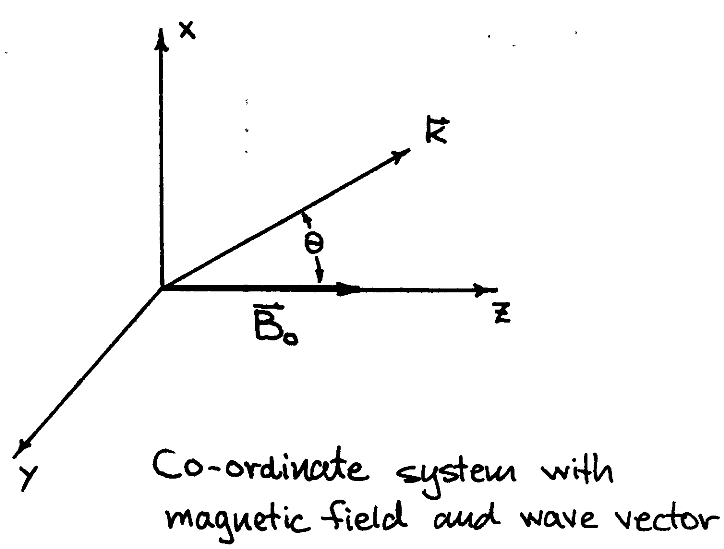

Figure 2.

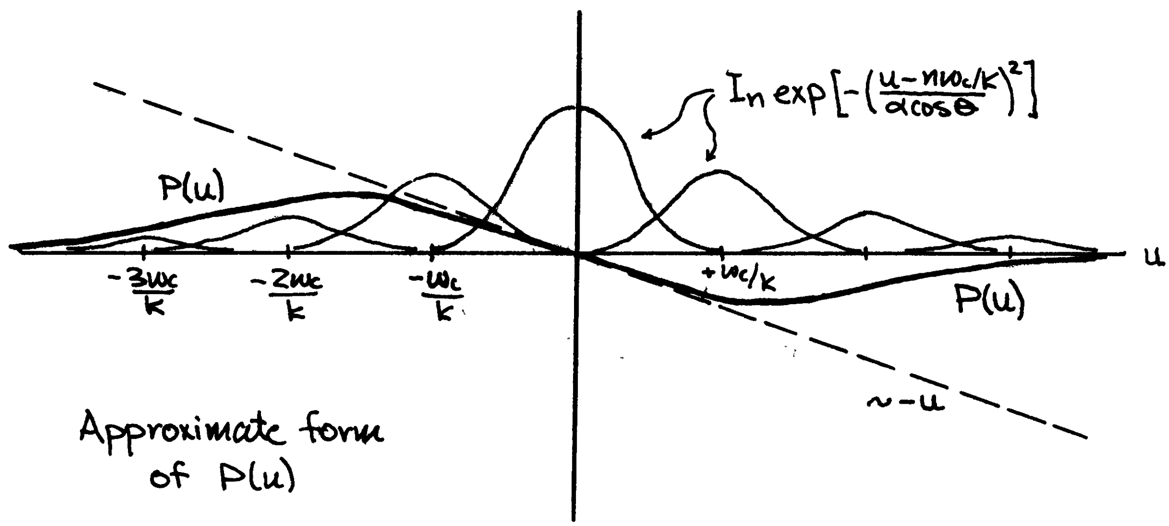

Figure 3.

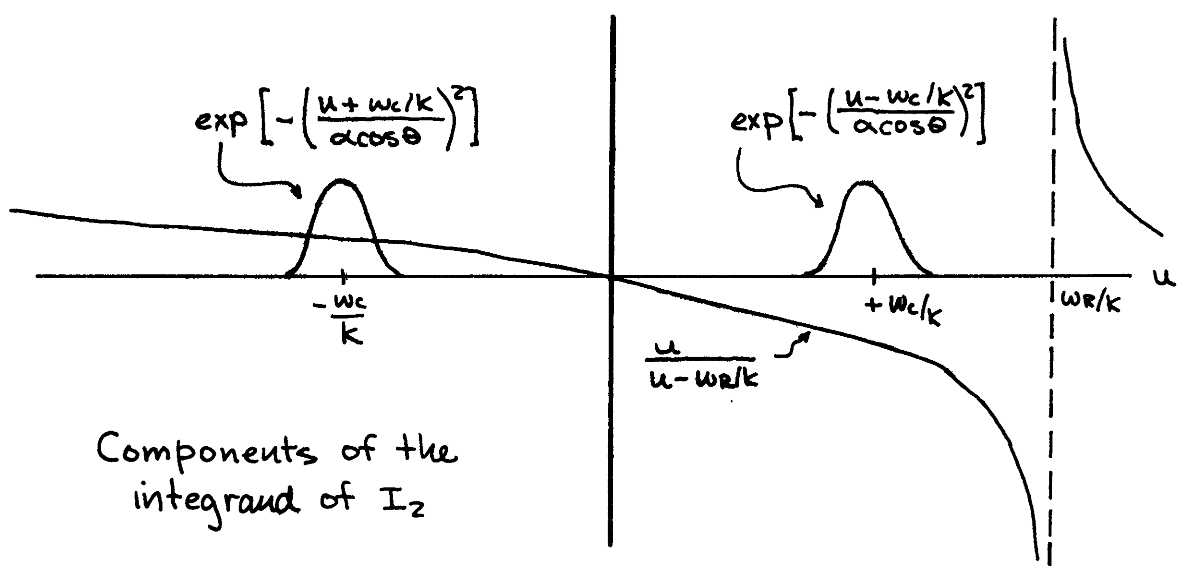

Figure 4.

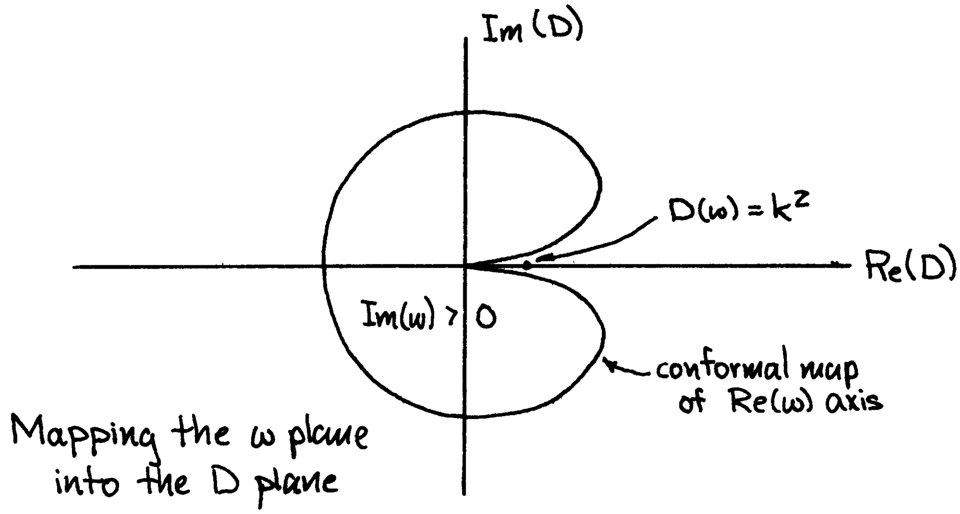

Figure 5.

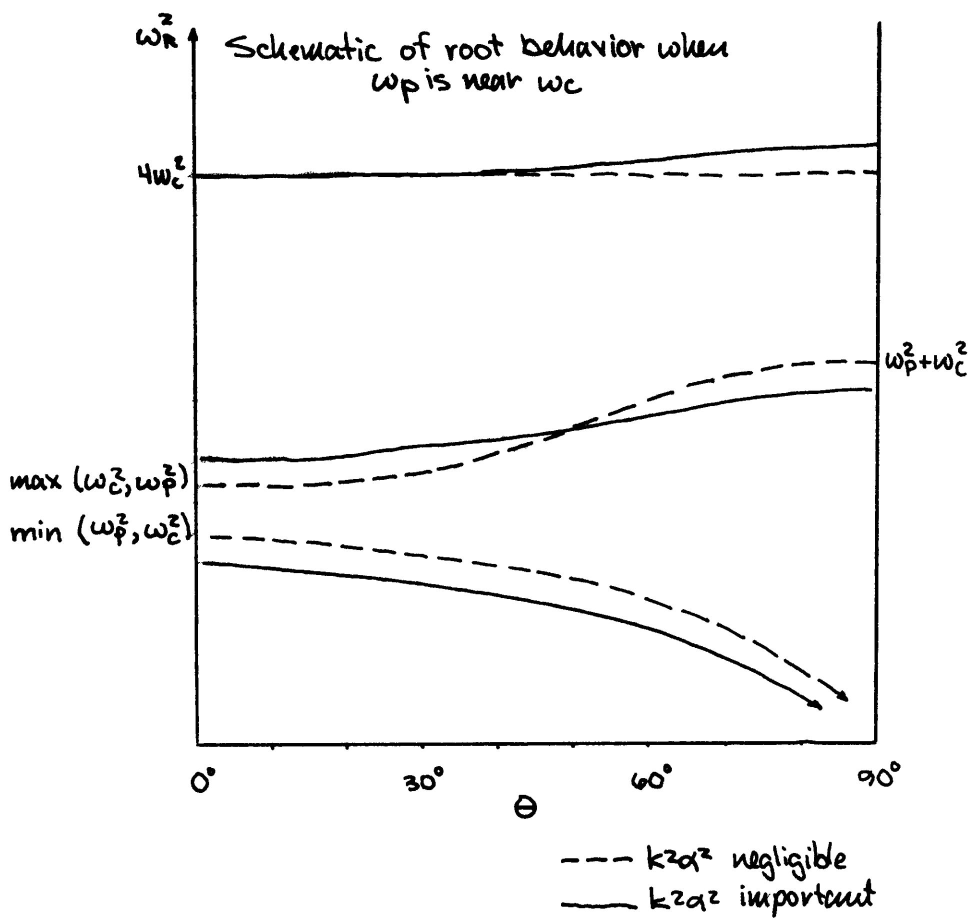

Figure 6.

REFERENCES

1. Landau, L.D., "On the Vibrations of the Electronic Plasma," Journal of Physics (U.S.S.R.), 9, 25, 1945.

2. Dawson, J., "On Landau Damping," Physics of Fluids, 4, 869, 1961.

3. Stix, T.H., The Theory of Plasma Waves, McGraw-Hill New York, 1962.

4. Jackson, J.D., "Longitudinal Plasma Oscillations," Journal of Nuclear Energy, Part C: Plasma Physics, 1, 171, 1960.

5. Malmberg, J.H., Wharton, C.B., and Drummond, W.B., "Landau Damping of Electron Plasma Waves," Paper CN 21/116, IAEA Conference on Controlled Fusion, Culham Laboratory, Abingdon, U.K., Sept. 1965.

6. Bernstein, I.B., "Waves in a Plasma in a Magnetic Field," Physical Review, 109, 10, 1958.

7. Harris, E.G., "Plasma Instabilities Associated with Anisotropic Velocity Distributions," Journal of Nuclear Energy, Part C: Plasma Physics, 2, 133, 1961.

8. McCune, J.E., "General Criterion for Electrostatic Plasma Instabilities with Uniform Magnetic Field," Physical Review Letters, 15, August, 1965.

9. Montgomery, D.C., and Tidman, D.A., Plasma Kinetic Theory, McGraw-Hill, 1964.

10. Watson, G.N., Theory of Bessel Functions, Cambridge University Press, Cambridge, England, 1945.

11. Dwight, H.B., Tables of Integrals and Other Mathematical Data, MacMillan, 1961.

12. Muller, D.E., "A Method for Solving Algebraic Equations Using an Automatic Computer," Mathematical Tables and Other Aids to Computation, 208, Oct., 1956.