APPROXIMATE INTEGRATION METHODS APPLIED TO WAVE PROPAGATION

by

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Donald van Zelm Wadsworth

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Chairman, Departmental Committee
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ABSTRACT

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The standard techniques for handling the integral solutions to geophysical wave propagation problems yield results of limited applicability. Furthermore in attacking a particular problem it is not always clear which techniques should be tried, as the relationships between many of these techniques are not well systematized.

The purpose of this thesis is to explore new techniques based on topological considerations as well as to extend standard techniques. Also attention is given to clarifying the interrelation between the standard techniques and to relating these to the new techniques.

The principal new technique developed is the "cliff" method of integration originated by Dr. M. V. Cerrillo of the Massachusetts Institute of Technology. This method will often yield compact solutions to integrals associated with branch cuts when well-known methods such as quadrature formulas are impractical to apply. The basic idea of the cliff method is the use of rational function approximants to replace branch cuts by chains of poles. Contour integration around the original branch cuts then can be collapsed onto the poles and the solution obtained by the Residue Theorem.

This cliff method is generalized in two ways. First, in the "extended" cliff method the convergence of the cliff method is improved by letting the number of poles in the approximants become infinite. For certain applications the solutions can be given in compact form. Second, the basic ideas of the cliff method are generalized by expanding the argument of a function in rational functions. The branch cuts are then replaced by more complicated singularities than the poles of the (simple) cliff method.

Finally a means is given for extending the standard saddle point methods by combining the topographic features of the saddle point methods with either the cliff methods or with quadrature methods. The solutions

are convergent and reasonably compact.

As is shown by a number of examples, the cliff methods together with the extension of the saddle point method offer a practical means for overcoming the following limitations of standard integration methods: (1) they make it possible to extend saddle point methods to integrands having broad saddles and sharply curved steepest descent paths, (2) the cliff methods offer a simple means of handling many integrals for which quadrature methods are difficult to develop, (3) the cliff methods can handle many singular integral equations which do not readily yield to standard techniques such as Gaussian quadrature.

Thesis Supervisor: Dr. M. V. Cerrillo
Research Associate
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I am indebted to my thesis supervisor, Dr. M. V. Cerrillo of the Research Laboratory of Electronics, Massachusetts Institute of Technology, for suggesting the thesis topic and many of the ideas on which it is based. His guidance during many consultations was an inspiration and invaluable in carrying out the various ramifications of the research.

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Others to whom I am grateful for consultations on various aspects of the thesis investigation include Professor L. V. Ahlfors of the Department of Mathematics, Harvard University; Professor F. B. Hildebrand of the Department of Mathematics, M.I.T.; Professor K. U. Ingard of the Department of Physics, M.I.T.; Dr. J. C. Savage of the Department of Geology and Geophysics, M.I.T.; Dr. S. C. Wang of the Lincoln Laboratories of M.I.T.

I also wish to express my appreciation for the cooperation of various members of the Department of Geology and Geophysics and to the Research Laboratory of Electronics for the use of some of its facilities.

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Mining Co. Much of this research was carried out while I was a National Science Foundation Terminal Fellow.

Mrs. Lena Fried of the Joint Computational Group of M.I.T. carried out most of the numerical computations. Mrs. Joan Grine made it possible to meet the thesis deadline by doing most of the drafting.

BIOGRAPHY

The author, Donald van Zelm Wadsworth, was born in Mamaroneck, New York on July 14, 1931. After attending grammar schools in New York and Connecticut, his family moved to Miami, Florida where he graduated from Ponce de Leon High School in 1949. From then until 1953 he majored in physics at Williams College, Williamstown, Massachusetts where he received a B. A. degree in June, 1953. The title of his thesis was "Survey of the Most Recent Developments in Experimental Physics". From June, 1953 until February, 1958 he was a candidate for a Ph. D. degree in the Department of Geology and Geophysics at M.I.T.

During high school he received an honorable mention in the Westinghouse Science Talent Search for a paper on "Rocket Satellites of the Earth". At Williams College and for three years of graduate study, he was a Tyng Scholar. During his fourth year at M.I.T. he was a National Science Foundation Terminal Fellow and was also awarded a Postdoctoral NSF Fellowship. He is a member of Phi Beta Kappa, Sigma Xi, the American Geophysical Union, the Society of Exploration Geophysicists and the American Physical Society.

INTRODUCTION

Most of the geophysical problems connected with electromagnetic and seismic wave propagation can only be solved approximately. The method of approximate solution will depend on whether the problem is formulated in terms of differential equations, integral equations or a combination of these. In a particular case, it may be easier to deal directly with the differential equation rather than a solution in integral form. Various techniques of approximate solution such as perturbation calculations, variational methods and relaxation methods are described, for instance, by P. M. Morse and H. Feshbach in "Methods of Theoretical Physics" and by F. B. Hildebrand in "Methods of Applied Mathematics". Some of these methods apply especially well to scattering and diffraction problems. However, in this thesis, we shall restrict ourselves to approximate methods which deal with solutions already in integral form--perhaps multiple integrals, but no unknowns in the integrands. Nevertheless an unanticipated fruit of the research is that one of the methods developed--the "cliff" method--has important applications to integral equations, as described in Appendix E.

The various techniques for handling the integrals we are concerned with can be grouped under the two classifications:

(1) Topological Methods. These include the methods of complex analysis which are concerned with: the nature and location of singularities on a surface; the structure of a Riemann surface and mappings from

one Riemann surface to another; the topography of a surface, particularly with respect to saddle points and steepest descent paths. Specific examples are the powerful saddle point methods of integration and the Residue Theorems.

(2) Non-topological Methods. These are the methods which are not primarily concerned with the behavior of an integrand on a surface. In fact the variable of integration is not generalized to a two-dimensial or complex variable. In the case of single integrals, the operations are in one dimension only. Specific examples include the quadrature methods of integration such as Gaussian quadrature and Simpson's rule, expansions in orthogonal functions with term by term integration (Fourier series, Bessel series, orthogonal polynomial series, etc.), power series developments with summation by continued fractions and many others.

The topological methods possess an inherent power which the other methods lack. All the effort in the latter is concentrated on one fixed line in the complex. In the topological methods, we consider the whole scope of the complex plane and can see where to move our line of integration to the best advantage. For instance, the convergence of the non-topological methods may be very poor on part of the fixed line because of the nearness of a singularity. In the topological methods, we can often deform our line of integration to a less sensitive position where the convergence is improved.

Many of the integrals appearing in the solutions to geophysical problems connected with wave propagation can be handled by the topological methods of complex analysis. In general they can be put in the form

$$\int_L f(z)\, e^{w(z)}\, dz$$

where f(z) and w(z) may be multivalued functions and may contain parameters. The exponential behavior of the integrand is concentrated in w(z). L is a prescribed contour in the complex z plane. The two principal techniques for handling these integrals are the deformation of integration contours onto steepest descent paths (which usually pass through saddle points) or onto the singularities of the integrand. In the former case, the solutions are obtained by the saddle point methods of integration, while in the latter case, the solutions are obtained by the Residue Theorem, if the singularities are poles. Of course a given problem may require the use of both techniques.

For the type of integral of interest in wave propagation in dispersive media, these techniques have serious limitations. The saddle point methods are asymptotic, so that the solutions are valid only in a restricted region--usually the far field. In many cases the asymptotic solutions cannot be differentiated. In the second technique, the singularities which contribute to the final solution frequently include branch cuts, besides poles. Often the integrals associated with these branch cuts are as difficult to evaluate as the original integral, or else the available (non-topological) methods of handling them yield solutions which have reasonable convergence only in a restricted region.

This thesis is primarily concerned with exploiting the inherent power of the topological methods to overcome the above limitations. This goal is attained, in part, through two principal developments. First integration processes called "cliff" methods are developed to handle branch cut integrals. Secondly the ordinary saddle point methods are extended through application of the cliff methods and through adaptation of standard quadrature methods.

The basic idea of the cliff methods can be seen by considering the integral

$$I = \int_L g(z) f(z) dz$$

where L is the lancet contour of Figure 0-1. The singularities of g(z) are outside of this contour, whereas f(z) has a branch cut inside the contour. Now by a theorem of Mittag-Lefler or a similar theorem by Runge (see Appendix C) we can replace f(z) by a rational function approximation with poles in the original branch cut position, as indicated in Figure 0-2. We then collapse the contour L onto these poles employing the Residue Theorem to obtain the approximation to the branch cut integral I. If the number of poles is increased indefinitely, we can cause the approximation to converge to the true value of I. In most practical cases only a few poles are needed.

This method of replacing the branch cut by poles or pole-zero chains, since there are always zeros between the poles, and then using the Residue Theorem has been called the cliff method of integration (Cerrillo, 1953). The name comes from the fact that the surface of the function f(z) has a discontinuity like a cliff at the branch cut.

The principal developments of this thesis are based on research carried out since 1950 by Dr. M. V. Cerrillo of the Massachusetts Institute of Technology. His investigations showed the practicality of the developments by obtaining new forms for the solution to an electromagnetic wave propagation problem (Research Laboratory of Electronics Quarterly Progress Report, July 15, 1953). The integrals associated with wave propagation problems appeared to be well suited to the mathematical approach of these investigations. Since this coincided with my interest in geophysics, I decided to make this my thesis area.

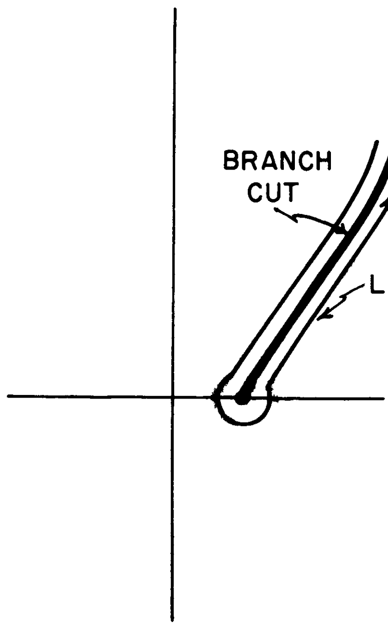


FIGURE O-1

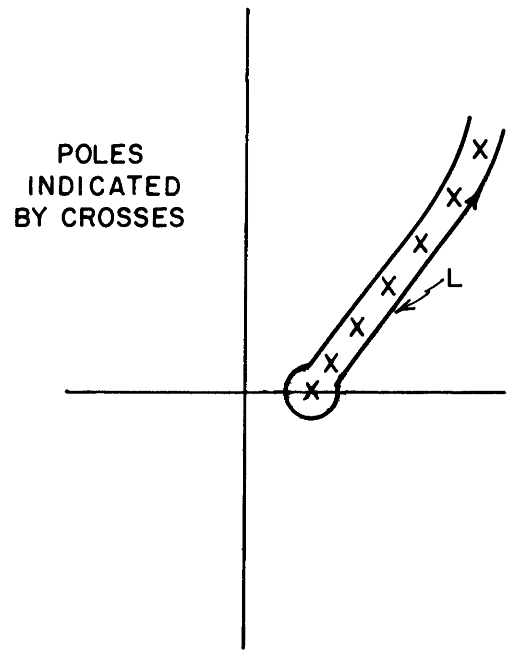


FIGURE O-2

THESIS ORGANIZATION

Chapter I is concerned with the evaluation of branch cut integrals by the cliff methods of integration. These methods are developed in detail and compared with the non-topological methods of numerical analysis.

Chapters II and III are devoted to integration methods which are primarily concerned with saddle points and steepest descent paths. Chapter II demonstrates how the cliff methods of integration can be advantageously combined with the topographic features of saddle points. Chapter III demonstrates how non-topological methods--quadrature methods--can be used to extend the range of the well-known saddle point methods of integration. The first part of Chapter III is devoted to a review of the saddle point methods and their various modifications in order to provide a basis for evaluating the quadrature method extension.

In Chapter IV the Sommerfeld dipole radiation problem is used to illustrate the analytical steps which must be taken before applying the approximate integration methods to a wave propagation problem.

A perusal of the Table of Contents will give a more detailed picture of the organization.

Chapter I

CLIFF METHODS AND BRANCH CUTS

In this chapter the application of cliff methods of integration to branch cut integrals will be developed. The results will then be compared with the standard quadrature methods for handling these integrals. In order to employ the cliff methods, three basic steps must be taken.

First the integral to be evaluated must be put in the form $\int_C g(z)f(z)dz$ where C is a lancet contour about the branch cut(s) of f(z) and g(z) contains no singularities inside this lancet contour. Later on these conditions will be relaxed somewhat.

Second, the function f(z) which is generated by branch cuts and perhaps additional singularities must be approximated by rational functions. The conditions under which this can be done and the mechanism for finding the appropriate rational functions are given in the section on Representation by Rational Functions. One method for generating the approximations is given by the branch of analysis called continued fraction analysis. It is a logical starting point, as it is a well developed field. However for our purposes, a more general approach comes directly from the Cauchy integral, and it is this latter method which will be developed in detail.

The third step is to replace f(z) by its rational function approximant. The poles of this approximant will be in the same position as the original branch cut, so we can collapse the lancet contour C onto these poles. For the (simple) cliff method the integration around the poles is accomplished by the Residue Theorem. The same is true for the "extended cliff" method to be developed in this chapter. In the section on the "general cliff" method, the integrations are performed in an entirely different manner due to the fact that the approximant to f(z) is no longer a rational function with simple poles but is a function of a rational function.

The formal basis for what follows in this chapter is to be found in the work of Borel, Hadamard, Mittag-Lefler, Weierstrass and others. For instance the Weierstrass factorization theorem for entire functions, a similar theorem for meromorphic functions by Hadamard and the Mittag-Lefler theorem on partial fraction expansions are basic. However, only a theorem by Runge (which includes the Mittag-Lefler theorem) will be necessary for an orderly development of what follows. Rather than couch the ideas in a great deal of mathematical rigor, I have decided to make the presentation simpler by including a minimum of general theorems, as these can be found in the references. This does not affect the methods developed or the conclusions obtained. Also much of the conciseness of formal mathematics has been sacrificed in order to make the ideas accessible to a wider audience.

REPRESENTATION BY RATIONAL FUNCTIONS

By the theorem of Mittag-Lefler or of Runge, a function f(z) which is generated by poles, branch cuts and essential singularities can be approximated by rational functions which, of course, have only pole singularities. Furthermore the rational functions can be made to converge uniformly to the original function. In general, we have

$$f(z) = \lim \sum_j \frac{a_j}{z - z_j} + h(z),$$

where h(z) is a polynomial, as the expansion in rational functions. If the sum is truncated after a finite number of terms, it is called a partial fraction expansion. The partial fraction expansion together with the polynomial form a rational function approximant to f(z). The methods for locating the poles z_j and determining the coefficients a_j will now be given.

The powerful methods of continued fraction analysis enable us to obtain the coefficients a_j and the poles z_j for the rational function approximants to a large class of functions. There are quite general theorems which show when the approximants obtained by these methods converge uniformly to the original function. If the singularities of a function are branch cuts, in general the poles and zeros of the approximants will be in the position of the branch cuts. The theorems and details are given by Perron and Wall.

This approach to finding the rational function approximants to the original function is limited by a certain rigidity as to the shape and position of the branch cuts involved. A more objectionable limitation for our applications is that the positions of the poles of the approximants are predetermined by the method, so that in general the poles are not optimally located with respect to the integration around the branch cut. Also there is no simple method for obtaining the numbers a_j and z_j.

Now it can be shown (see Perron) that the representation given by continued fraction analysis is equivalent to a representation of the original function by Stieltjes integrals. This representation in turn is, for our purposes, a special case of a more flexible method which employs the Cauchy integral and is developed in what follows.

Since it is basic to the discussion, the Cauchy Integral Formula also known as the Cauchy Integral Theorem is repeated here. If F(z) is continuous on C and analytic interior to C then

$$\frac{1}{2\pi i}\int_C \frac{F(t)}{t-z}\,dt = \begin{cases} F(z) & z \text{ interior to } C \\ 0 & z \text{ exterior to } C \end{cases}$$

where C is the smooth boundary of a finite, finitely connected region. Generalizations and rigor are given in the references by Muskhelishvili, Plemelj and Privalov among others.

We shall now illustrate how the Cauchy Formula is applied to obtain rational approximants to a function f(z). We shall take the specific case $f(z) = (1-z^2)^{-\frac{1}{2}}$ where we take the branch for which the real part of this function is positive in the upper half plane when the branch cut is chosen as in Figure 1-1. If $C_1 + C_2$ is the contour of Figure 1-1 then

$$(1\text{-}1)\qquad (1-z^2)^{-\frac{1}{2}} = \frac{1}{2\pi i}\int_{C_1}\frac{(1-t^2)^{-\frac{1}{2}}}{t-z}\,dt + \frac{1}{2\pi i}\int_{C_2}\frac{(1-t^2)^{-\frac{1}{2}}}{t-z}\,dt$$

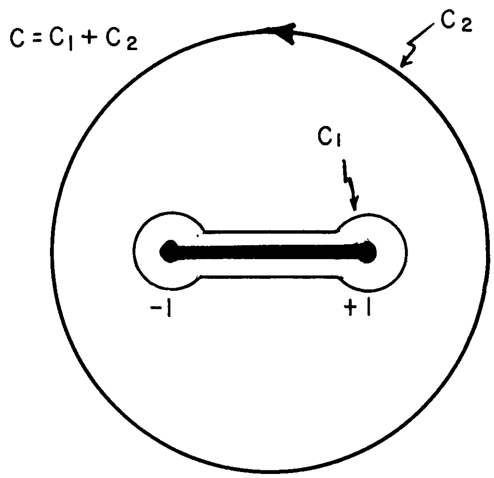


FIGURE 1-1

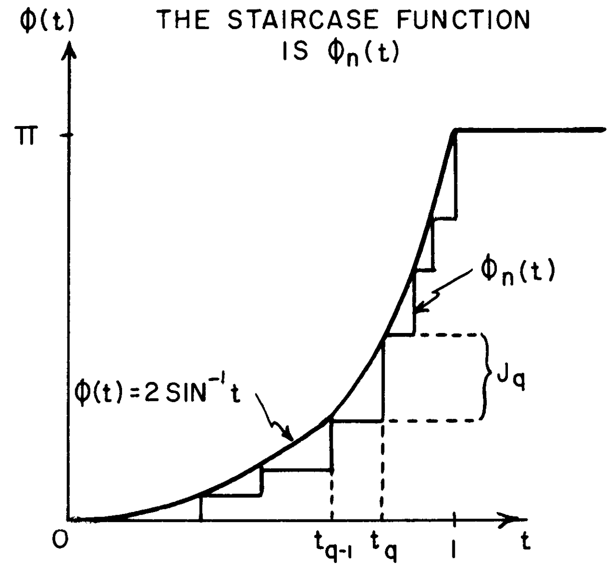


FIGURE 1-2

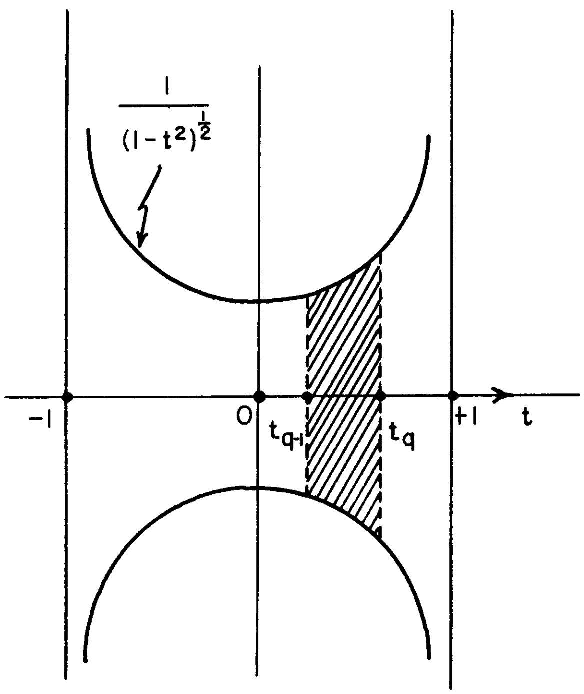


FIGURE 1-3

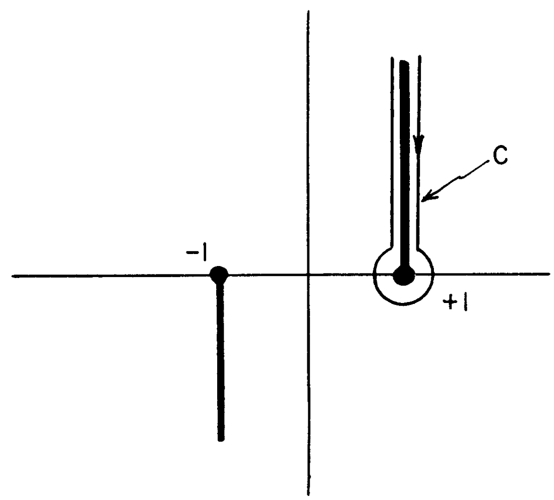


FIGURE 1-4

The integral on C_2 vanishes as the radius of the circle is extended to infinity, so that upon collapsing C_1 onto the cut we are left with

$$(1-2) \quad (1-z^2)^{-\frac{1}{2}} = \frac{1}{2\pi i}\int_{-1}^{+1} \frac{2(1-t^2)^{-\frac{1}{2}}}{t-z}\,dt = \frac{1}{2\pi i}\int_0^1 \frac{2z}{t^2-z^2}\,2(1-t^2)^{-\frac{1}{2}}dt$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{2z}{t^2-z^2}\,d\varphi(t)$$

where $\varphi(t) = 2\sin^{-1}t$. In this particular example, due to the symmetry, we find it simpler to deal with the integration over (0,1) instead of over (-1,1). The final integral above is in the form of a Stieltjes integral where φ (t) is called the distribution function. We require the Stieltjes form because the method of approximation we shall employ cannot be developed with just the Riemann integral. In general the distribution function is obtained by integration:

$$\varphi(t) = 2\int_0^t (1-t^2)^{-\frac{1}{2}}dt = 2\sin^{-1}t \; .$$

The properties of the Stieltjes integrals are described, for instance, by Widder and will not be repeated here.

In order to obtain a rational function approximant from 1-2 we approximate the distribution function $\varphi(t)$ by the staircase function $\varphi_n(t)$ shown in Figure 1-2. Then by the definition of the Stieltjes integral we have on substituting $\varphi_n(t)$ into the final integral of 1-2

$$(1-3) \quad (1-z^2)^{-\frac{1}{2}} = \frac{1}{2\pi i}\lim \sum_{q=1}^{N} \frac{2z}{t_q^2-z^2}\,J_q$$

where $J_q = \varphi(t_q)-\varphi(t_{q-1})$ are called jumps for obvious reasons. The exact meaning of the J_q is clearer if we use the definition of the distribution functions to write

$$J_q = 2 \int_{t_{q-1}}^{t_q} \frac{dt}{(1-t^2)^{\frac{1}{2}}}$$

Now the function $(1-z^2)^{-\frac{1}{2}}$ has a cliff-like discontinuity at the branch cut from -1 to +1 on the real axis. The face of this cliff is shown in Figure 1-3. It is evident from the above integral representation that the jumps J_q are just the areas of the cliff face between pairs of poles t_q and t_{q-1}. In the general case when the branch cuts are not on the real or imaginary axes, the cliff will have a complex area so that the jumps J_q will also be complex.

The position and number of poles in the right side of 1-3 will depend on the particular application. Suppose we want to evaluate $\int_{C_1} g(z)(1-z^2)^{-\frac{1}{2}}dz$ where C_1 is the contour of Figure 1-1. Then the optimum position of the poles is determined by the weighting factor $g(z)$ and the accuracy desired. By the method of construction it is clear that we are free to place the poles where we want them as long as they are in the position of the branch cut. This is is contrast to the continued fraction analysis approach in which the pole positions are predetermined (see Perron or Wall).

Suppose that the weighting factor $g(z)$ is such that we can take the jumps J_q of Figure 1-2 to be equally spaced so that $J_q = \pi/N$. Then $\varphi(t_q) = q\pi/N$ so that $t_q = \sin(q\pi/2N)$. Consequently

$$(1\text{-}4)\quad (1-z^2)^{-\frac{1}{2}} = \frac{1}{2\pi i} \lim_{N\to\infty} \frac{\pi}{N} \sum_{q=1}^{N} \frac{2z}{\sin^2\left(\frac{q\pi}{2N}\right)-z^2} \equiv \frac{-1}{2\pi i} \lim \frac{\pi}{N} \sum_{q=1}^{N} \left[\frac{1}{z-\sin\frac{q\pi}{2N}} + \frac{1}{z+\sin\frac{q\pi}{2N}}\right]$$

If N is finite, the right side of 1-4 is a rational function approximant to $(1-z^2)^{-\frac{1}{2}}$. If the limit process is carried out we obtain the function on the left as expected. In this example the jumps are equally spaced. In some problems it might be preferable to choose some other spacing,or we could choose the poles t_q to be equally spaced. Various theorems on convergence and estimates on the error for a given number of terms can be derived from the results of Perron.

CLIFF METHOD

The preceding section gave the mechanism for expanding a function f(z) in terms of rational functions which we shall denote by $R_n(z)$. The theorems given Appendix C show that the $R_n(z)$ can be uniformly convergent. Even though these functions converge uniformly to f(z), this is no guarantee that the right hand integral of

$$(1\text{-}5)\quad I \equiv \int_C g(z) f(z)\, dz = \lim \int_C g(z) R_n(z)\, dz$$

where $R_n(z)$ is the rational approximant to f(z) will also converge uniformly to the limit. In fact the limit may well be divergent. Sufficient conditions for convergence can be obtained from the theorems of Appendix C. If these conditions are met, then we can make some very definite observations about the whole structure of the cliff method.

In 1-5 let us substitute for $R_n(z)$ its general form

$$\sum_j \frac{a_j}{z-z_j} + h(z)$$

where h(z) is a polynomial. Then we obtain

$$(1\text{-}6) \qquad I = \lim \sum_j g(t_j)\, 2\pi i a_j = \lim \sum_j g(t_j)\, J_j$$

where J_j are the jumps derived from the distribution function $\varphi(t)$ for f(t). We have assumed C to be an appropriate contour such as that of Figure 1-4, so that the Residue Theorem applies at the poles of $R_n(z)$.

That the a_j can be replaced by $J_j/2\pi i$ in the right side of 1-6 can be seen for the case $f(z) = (1-z^2)^{-\frac{1}{2}}$ if we write the sum on the right side of 1-3 in the form

$$(1\text{-}7) \qquad f(z) = \lim \sum_{q=1}^{N} \frac{2z}{t_q^2-z^2}\cdot\frac{J_q}{2\pi i} = \lim \sum_{j=-N}^{j=+N} \frac{1}{t_j-z}\cdot\frac{J_j}{2\pi i}$$

where $J_j = \pm J_q$ and $t_j = \pm t_q$, the upper sign corresponding to j being a positive number. Then since $f(z) = \lim R_n(z) = \lim \sum_j a_j/(z-t_j) + h(z)$, we have $a_j = J_j/2\pi i$ on comparing coefficients. It is not hard to generalize this, but we shall omit the proof here.

Now if we take only a few terms of the sum on the right of 1-6 we have the cliff method approximation to the integral I. The relationship of this approximation to the Stieltjes definition of the original integral can be seen if we replace f(z)dz by $d[\varphi(z)/2]$ in the left hand integral of 1-5 and collapse the contour C onto the cut. We then have by a definition of the Stieltjes integral

$$(1\text{-}8)\quad I = \int_C g(z)\, d\left[\frac{\varphi(z)}{2}\right] = \lim \sum_j g(t_j)\left[\varphi(t_j) - \varphi(t_{j-1})\right]$$

where $\varphi(t)$ is the distribution function for 2f(t). The sum on the right has the same form as the sum on the right of 1-6 if we remember that the jumps J_j were defined in terms of the distribution function for f(t) by $J_j = \varphi(t_j) - \varphi(t_{j-1})$. It is clear now that the cliff method approximation is just a partial sum of the Stieltjes definition of the original integral along the banks of a branch cut.

Our method of approximate integration can viewed from these two standpoints. In one we replace a branch cut by a chain of poles and deform our contour of integration onto these poles. In the other, we deform our contour of integration onto the banks of the branch cut and simply use the Stieltjes definition of the resultant integrals. Since we are dealing with contour deformations and singular lines (branch cuts), the former viewpoint is probably more natural. In the general cliff method described later, this viewpoint will be imperative. For the cliff method and the extended cliff method of the next section, it will be helpful to think in terms of the Stieltjes integral approach as well as the purely topological approach of rational function approximants.

Before developing the extended cliff method, let us see how to apply the cliff method when we do not have a lancet contour around a cut. Suppose the integral to be evaluated is of the form 1-5, but that the contour C only extends along one bank of the cut. As before, expand f(z) in a rational approximant with poles along the cut position. We then deform C onto these poles so that C consists of semicircles around the poles plus straight line segments between the poles. Our cliff method solution to the integral I is then given by taking one-half of the values of the residues at these poles. (If we are dealing with other than double valued functions, then the residues would be weighted differently). We neglect the contribution from the straight line segments in our approximation. That this should be done can be seen by an appeal to the Cauchy Integral Theorem as explained in Appendix G. Another way to see this is to employ the Stieltjes integral along the bank of the cut and show that the approximation obtained in this manner gives the same result as taking half residues.

EXTENDED CLIFF METHOD

It sometimes becomes necessary, in order to obtain a good approximation, to use a large number of steps in the staircase approximation to the distribution function. Then the cliff method becomes impractical. To see how this situation can be remedied, consider the general form of the cliff method solution given in 1-6. If the sum index appears in $g(t_j)$ and $J_j = \phi(t_j) - \phi(t_{j-1})$ in certain ways, it is possible to perform the finite summations and take the limit as the number of terms, and hence poles, become infinite. But this is redundant if, in the summation and limit, the poles become dense along the branch cut position. All we succeed in doing is to obtain the original Stieltjes integral or an equivalent form, since we are actually dealing with a definition of the Stieltjes integral. The important point is that if the original integral is unknown, we may be able to evaluate it approximately by replacing g(z) and the distribution function for f(z) by approximate forms whose Stieltjes integrals are tabulated functions.

A simple example will make the ideas clearer and at the same time show the basic difference between the extended cliff method and quadrature methods. We shall evaluate the integral representation for the Bessel function

$$J_0(a) = \frac{1}{2\pi i} \int_{C_1} \frac{e^{iaz}}{(1-z^2)^{\frac{1}{2}}}\, dz \tag{1-10}$$

where C_1 is the contour of Figure 1-1 and a is real.

The first step is to obtain a rational approximant to $(1-z^2)^{-\frac{1}{2}}$. We start with the right hand integral of 1-2 and replace the distribution $\varphi(t) = 2\sin^{-1}t$ by the straight lines shown in Figure 1-5. These two straight lines, $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$ form an approximate distribution function $\varphi^{(1)}(t) + \varphi^{(2)}(t) \simeq \varphi(t)$. The next step is to approximate $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$ by the staircase functions $\varphi_n^{(1)}(t)$ and $\varphi_n^{(2)}(t)$ indicated in Figure 1-5.

If we take the jumps of $\varphi_n^{(1)}(t)$ and $\varphi_n^{(2)}(t)$ to be equally spaced, we have $J_q^{(1)} = 3/2N$ and $J_q^{(2)} = 3/2N$ so that $\varphi_n^{(1)}(t_q) = 3q/2N$ and $\varphi_n^{(2)}(t_q) = 3/2 + 3q/2N$. Then we can solve for t_q obtaining $t_q^{(1)} = 3q/4N$ and $t_q^{(2)} = 3/4 + q/4N$. Next substitute $\varphi_n^{(1)}(t)$ and $\varphi_n^{(2)}(t)$ into 1-2 obtaining

$$(1-z^2)^{-\frac{1}{2}} \simeq \frac{1}{2\pi i}\left[\sum_{q=1}^{N} \frac{2z}{(t_q^{(1)})^2-z^2} J_q^{(1)} + \sum_{q=1}^{N} \frac{2z}{(t_q^{(2)})^2-z^2} J_q^{(2)}\right]$$

If we now substitute the rational approximant on the right into 1-10 and employ the Residue Theorem, we obtain finally

(1-11) $$J_0(a) \simeq \frac{3}{2\pi N}\sum_{q=1}^{N}\left[\cos\left(\frac{3aq}{4N}\right) + \cos\left(\frac{3a}{4} + \frac{aq}{4N}\right)\right]$$

The extended cliff solution is obtained by letting N become infinite in 1-11. We first perform the finite summations and then take the limit obtaining

(1-12) $$J_0(a) \simeq \frac{6}{\pi a}\sin a - \frac{4}{\pi a}\sin\frac{3a}{4}$$

When we let N become infinite, the stair case functions $\varphi_n^{(1)}(t)$ and $\varphi_n^{(2)}(t)$ became identical with the functions $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$. Then the only error introduced in our approximation is due to the difference between $\varphi^{(1)}(t) + \varphi^{(2)}(t)$ and $\varphi(t)$. In other words the extended cliff method handled the function $\exp(iaz)$ of 1-10 exactly but approximated the integral of $(1-z^2)^{-\frac{1}{2}}$ --that is, the distribution function. The actual error of the above approximation is a few per cent for small z.

The accuracy could be increased by choosing the ordinates in an optimum fashion or using three or four straight lines to approximate the arc sine. We could also have used the first few terms of a fourier expansion or a higher order polynomial as an approximation to the arc

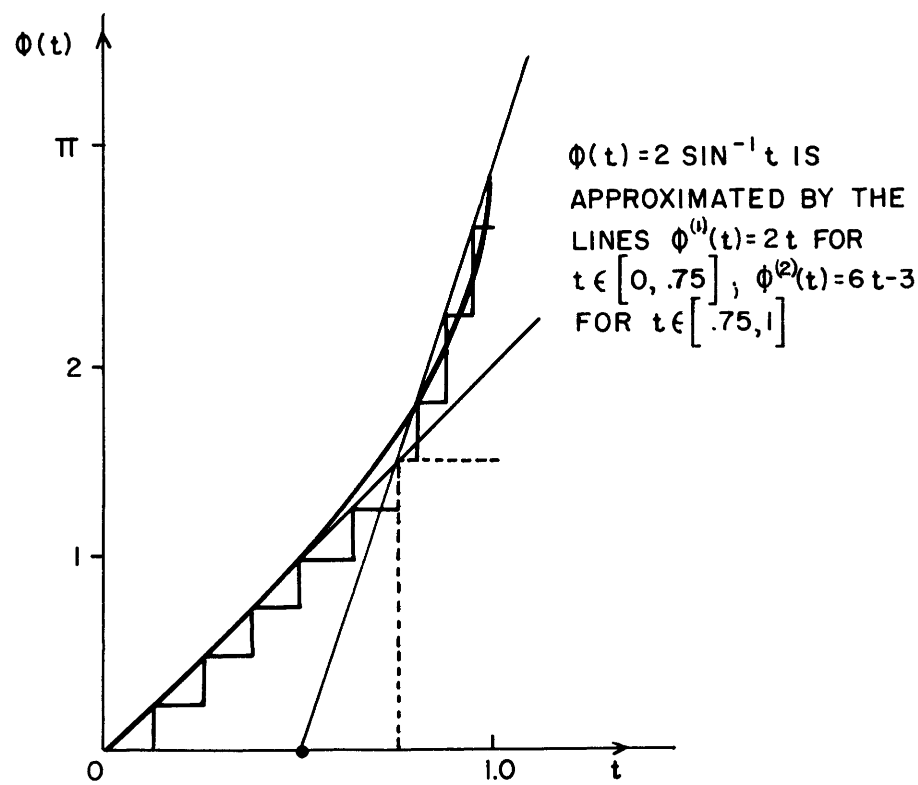

Φ(t)
π
2
1
0
1.0
t
Φ(t) = 2 SIN⁻¹ t IS APPROXIMATED BY THE LINES Φ⁽¹⁾(t) = 2 t FOR t ∈ [0, .75] ; Φ⁽²⁾(t) = 6 t - 3 FOR t ∈ [.75, 1]

FIGURE 1-5

sine.

In Appendix E, Figure E-3, the approximation 1-12 is compared to a four point Chebyshev-Gauss quadrature for the integral obtained by collapsing the contour of 1-10 onto the branch cut:

$$J_0(a) = \frac{1}{\pi}\int_{-1}^{+1} \frac{\cos at}{(1-t^2)^{\frac{1}{2}}}\,dt \tag{1-13}$$

The interesting fact is that for this example, the extended cliff solution compares very favorably with the quadrature solution--in fact the extended cliff solution stays with the Bessel function longer than does the quadrature.

If we had started with the integral 1-13 along the banks of the branch cut, and put it into the Stieltjes form

$$J_0(a) = \frac{1}{\pi}\int_0^1 \cos at \, d\left[2 \sin^{-1} t\right]$$

we could work directly with the distribution function. We simply make the straight line approximation to the distribution function and obtain the solution 1-12. This way we do not have to consider rational function expansions or take limits as the poles become dense along the branch cut position.

We see that the extended cliff method is equivalent to dealing with the Stieltjes form of an integral along the banks of a branch cut. The distribution function is approximated by simpler functions for which the integrations can be carried out. In cases where it is difficult to apply a quadrature rule, the extended cliff method has the advantage of being easy to apply.

The past two sections have brought out the relation between rational function approximants and approximations made directly to the distribution function of a Stieltjes integral. In the general cliff method section, the viewpoint of expansion in rational functions will not be equivalent to a Stieltjes integral representation.

ERROR ANALYSIS

We shall now consider the errors introduced by the cliff and extended cliff methods. First we shall examine a specific function to illustrate what happens in the cliff method from the geometric standpoint.

Let $f(z) = (1-z^2)^{-\frac{1}{2}}$ in the left hand integral of 1-5 and let C be the contour designated by C_1 in Figure 1-1. We shall employ the expansion 1-4 and keep the number of poles finite. We have on substituting 1-4 into the above integral and using the Residue Theorem:

$$I \simeq \sum_{q=1}^{N} \left[g(t_q) + g(-t_q) \right] J_q$$

where $J_q = \pi/N$. Now as shown in Figure 1-3, the jumps J_q are the areas of the face of the cliff for $(1-z^2)^{-\frac{1}{2}}$ between the poles t_q and t_{q-1}. If the weight factor is unity, that is, $g(z) = 1$, then, exactly,

$$I = \sum_{q=1}^{N} 2 J_q$$

Since the integral is just the total area of the cliff face, we need only a finite number of poles for an exact answer. In fact two poles (at $z = \pm 1$) would be sufficient.

Now if $g(z)$ is not unity it is clear that the cliff method weights the areas J_q by the value of $g(z)$ at $z = t_q$ so that $g(z)$ is approximated by a constant between each pair of poles.

Next compare this cliff method solution with a quadrature for which $f(z) = (1-z^2)^{-\frac{1}{2}}$ is a weight factor--the Chebyshev-Gauss formula. In place of the crude step-like approximation to $g(z)$ of the cliff method, the quadrature rule with, say, m points approximates $g(z)$ by a $2m-1$ degree polynomial. This is generally a considerable improvement on the cliff method solution. However if $f(z)$ does not have the form of a weighting function of known orthogonal polynomials, then the cliff method may be the most practical means for obtaining the approximate solution. Moreover the method is straight forward and easy to apply. In fact as the examples of Appendix E show, the cliff method solution is not so crude as might be thought from the above comparison.

A tight error analysis for the cliff method is quite difficult to develop. However a conservative analysis can be obtained by considering the integral of 1-5. We have

$$I = \int_{C} g(z)\, d[\varphi(z)]$$

where $\varphi(z)$ is the distribution function for $f(z)$. The approximation to I is given by

$$I_n = \int_C g(z)\, d[\varphi_n(z)]$$

where for the cliff method $\varphi_n(z)$ is the staircase approximation to $\varphi(z)$ as indicated in Figure 1-2 for the case $\varphi(t) = 2\sin^{-1}t$. The error is then

$$I - I_n = \int_C g(z)\, d[\varphi(z) - \varphi_n(z)]$$

By collapsing C onto the branch cut we can consider this to be a line integral with limits a and b. For this line integral we shall let z = t. By the partial integration formula for Stieltjes integrals we can then express the error as

$$\begin{aligned} I - I_n &= \int_a^b g(t)\, d[\varphi(t) - \varphi_n(t)] \\ &= g(b)[\varphi(b) - \varphi_n(b)] - g(a)[\varphi(a) - \varphi_n(a)] - \int_a^b [\varphi(t) - \varphi_n(t)]\, dg(t) . \end{aligned}$$

For the cliff method the set of points at which $\varphi(t) - \varphi_n(t)$ is discontinuous has measure zero. Also we can assume $\varphi(t) - \varphi_n(t)$ is bounded on $[a,b]$. Then if g(t) is continuous and monotonic, it can be shown by the methods of functional analysis that

$$\left| \int_a^b [\varphi(t) - \varphi_n(t)]\, dg(t) \right| < \mathrm{lub}[\varphi(t) - \varphi_n(t)] \cdot \left| g(b) - g(a) \right|$$

Also if g(t) is merely of bounded variation on $[a,b]$ and $\varphi(t) - \varphi_n(t)$ is continuous (as it is in the extended cliff method) then we have

$$\left| \int_a^b [\varphi(t) - \varphi_n(t)] \, dg(t) \right| < \text{lub} [\varphi(t) - \varphi_n(t)] \cdot V g(t)$$

where lub means least upper bound and V denotes the total variation.

These inequalities enable us to set a bound on the error $I-I_n$. However these bounds are very conservative. Until a better error analysis is developed the best that can be done is to give some numerical examples to show that the cliff method can have high accuracy with only a few terms. These examples are relegated to Appendix E and show that the cliff method compares quite favorably with non-topological methods such as quadrature rules.

GENERAL CLIFF METHOD

Suppose the integral taken on the contour of Figure 1-4 has the form

$$I = \int_C g(z) \, h[f(z)] \, dz \tag{1-14}$$

where the branch cut surrounded by C belongs to f(z). Neither g(z) nor $h[f(z)]$ have any other singularities inside this contour. We shall replace f(z) by a rational function approximant with poles in the position of the original branch cut as before. However there is now a basic difference in our method of approximate integration from that of the (simple) cliff methods. We are now expanding the argument of a function instead of the complete function in rational functions. The previous methods are the special case for h being the identity operator. In general, the singularities of the approximant $h[R_n(z)]$ to $h[f(z)]$ will be more complicated than the simple poles we encountered previously.

To illustrate why this generalization has a practical motivation, let h be the exponential operator so that 1-14 becomes

$$I = \int_L g(z)\, e^{f(z)} dz \tag{1-15}$$

Our first thought might be to apply the (simple) cliff methods after removing f(z) from the exponential by an appropriate transformation or to expand exp f(z) itself in rational functions. In many cases the first alternative is not feasible because of the complicated form of the integrand. In most cases the latter alternative is impractical because the methods available for obtaining a rational function expansion of an exponential of this sort are very awkward. For these reasons it is desireable to develop the ideas of the general cliff method.

In our example we replace exp f(z) by the approximant exp $R_n(z)$ which has essential singularities at the poles of $R_n(z) \equiv \sum_j a_j/(z-z_j) + h(z)$. We have now replaced the branch cut by a chain of essential singularities. When the contour L is collapsed onto these isolated essential singularities, we have the approximation

$$I = \lim \int_L g(z) e^{\sum_j a_j/(z-z_j) + h(z)} dz = \lim \sum_j \oint_{z_j} g(z) e^{a_j/(z-z_j) + \sum_{m\neq j} a_m/(z-z_m) + h(z)} dz \tag{1-16}$$

where near any of the poles z_j the functions g(z) and $\sum_{m\neq j} a_m/(z-z_m) + h(z)$ are nearly constant. These integrals can be evaluated by the method of Appendix F. If only a few terms are needed, then we have a practical solution. Unlike the cliff methods developed previously, there is no simple relation between our approximate solution and the partial sums of the definition of a Stieltjes integral.

To illustrate the application of the general cliff method and some of its limitations, let us consider a typical integral appearing in wave propagation problems:

$$(1\text{-}17) \quad I = \int_C H_0\left[\rho\,(1-z^2)^{\frac{1}{2}}\right] g(z)\,dz$$

H_0 is the Hankel function of the first kind, C is the lancet contour on the left side of Figure 1-6 and we assume the singularities of g(z) are exterior to this contour. Suppose that ρ varies between .1 and 10 so that quadrature methods are awkward to apply.

The first step in the general cliff method is to expand $(1-z^2)^{\frac{1}{2}}$ in a rational function approximant $R_n(z)$. At the zeros of this rational function, the argument of the Hankel function is zero so that $H_0[\rho R_n(z)]$ has logarithmic singularities (logarithmic branch points) at these points. At the poles of $R_n(z)$ the Hankel function has branch points which we shall call essential singularities. The contour C can then be collapsed onto the singularities of $H_0[\rho R_n(z)]$ as shown in the right hand side of Figure 1-6. The branch cutting is that for $H_0[\rho R_n(z)]$ and does not come from the function being approximated. It seems reasonable that the whole effect of the original branch cut which generated $H_0[\rho(1-z^2)^{\frac{1}{2}}]$ is approximated by the singularities of $H_0[\rho R_n(z)]$ which lie in the position

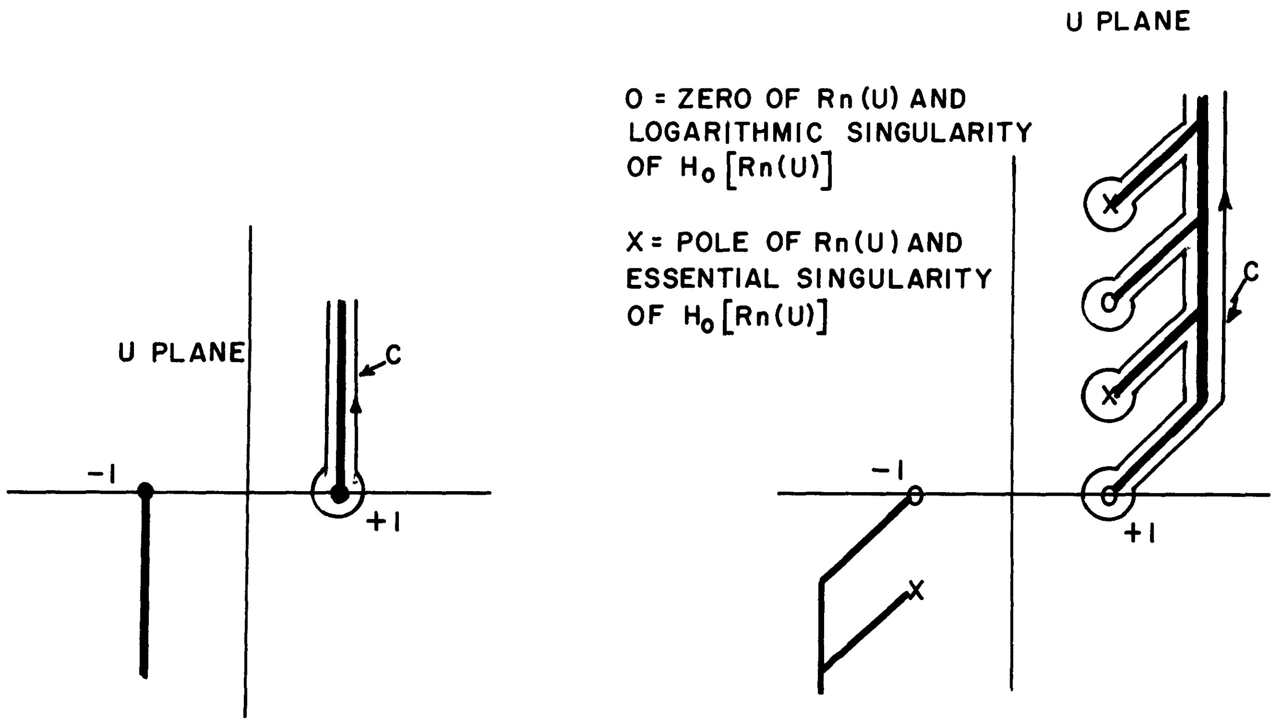


FIGURE 1-6

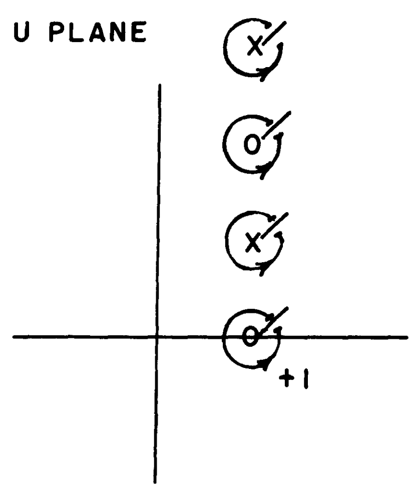


FIGURE 1-7

of the branch cut and not by the new branch cuts introduced for $H_0[\rho R_n(z)]$. Nevertheless when we collapse the contour C onto the singularities of $H_0[\rho R_n(z)]$, we must consider the integrations along the banks of the cuts for this function, as is demonstrated in Appendix F.

It now would appear that all we have succeeded in doing is to replace a single branch cut integral by a number of new ones and have therefor multiplied our difficulties. We shall return to this point later, but for the time being assume that we can surmount these difficulties.

We still have to consider the contributions at the branch points of $H_0[\rho R_n(z)]$ designated by O and X in the right side of Figure 1-6. Near these branch points $H_0[\rho R_n(z)]$ can be replaced by its logarithmic or asymptotic forms. Then if $\overset{x}{z}_j$ are the poles and $\overset{o}{z}_j$ are the zeros of $R_n(z)$, we have the approximation

$$
(1\text{-}18)\quad I = \begin{cases} \lim \sum\limits_j \displaystyle\int_{\circlearrowleft \overset{o}{z}_j} \frac{i\pi}{2} \ln\left[\frac{\gamma}{2}\rho R_n(z)\right] g(z)\, dz \\ + \lim \sum\limits_j \displaystyle\int_{\circlearrowleft \overset{x}{z}_j} \left[\frac{2}{\pi\rho R_n(z)}\right]^{\frac{1}{2}} e^{i\rho R_n(z)} g(z)\, dz \\ + \text{contribution of branch cutting} \end{cases}
$$

where the loops about $\overset{x}{z}_j$ and $\overset{o}{z}_j$ are placed as shown in Figure 1-7. For integrals of the type appearing in the Sommerfeld problem the integrations about $\overset{o}{z}_j$ vanish. The proof is straight forward. The integrations about the $\overset{x}{z}_j$ can be handled by the method shown in Appendix F if the angles of the branch cuts issuing from the $\overset{x}{z}_j$ are adjusted so that the integrals converge and the phase requirements of the asymptotic forms are satisfied.

The problem remains of evaluating the integrals along the banks of the branch cutting for $H_0[\rho R_n(z)]$. Theoretically, though not practically, it is possible to expand the original function $H_0[\rho(1-z^2)^{\frac{1}{2}}]$ in a rational function approximant $S_n(z)$. Now compare this with the approximant $H_0[\rho R_n(z)]$. Both these approximants are generated by their singularities. Since both are approximants to the same function, there must be a relation between the poles of $S_n(z)$ and the branch cuts and branch points of $H_0[\rho R_n(z)]$. If we can find a practical relation between the integrations around the poles of $S_n(z)$ and the integrations along the banks of the cuts for $H_0[\rho R_n(z)]$, it may be possible to evaluate the integrals along the banks of the cuts by applying the Residue Theorem to the poles of $S_n(z)$. It also may not be necessary to have obtained the exact form of $S_n(z)$ first. These possibilities require careful investigation, but were considered to be beyond the scope of the present thesis and are left for future work.

In the present section we have considered two examples of the general cliff method. In the first example, the operator h of equation 1-14 was the exponential function, so that the general cliff method could be carried out. We did not give an actual numerical example, as this will be done in Appendix E. In the second example of this section, the operator h was the Hankel function. As we have seen, in this case we run into difficulties in applying the general cliff method, although more study is necessary before we can make definite conclusions.

SUMMARY

The present chapter has shown that both the cliff and extended cliff methods have practical applications to branch cut integrals. An error analysis was given, but as shown in Appendix E, it is much too conservative. A tight error analysis for the cliff methods is difficult to develop. The examples of Appendix E do show that the error can be quite small-in fact the cliff methods compare quite favorably with quadrature methods. The important observation is that the cliff methods can be applied to integrals which do not readily yield to quadrature methods either because the weight factor is not the right form or because of singular behavior of a factor of the integrand.

The general cliff method was carried to a point where it did not appear too promising for functions such as the Hankel function. The simpler example worked out in Appendix E also shows serious limitations. However more work is necessary before definite conclusions are obtained.

As explained in the example of Appendix E, the cliff method also can be applied to singular integral equations which are not readily adaptible to methods such as Gaussian quadrature.

Chapter II

CLIFF METHODS AND SADDLE POINTS

The previous chapter was concerned with the application of cliff methods of integration to branch cut integrals without any special regard to whether the integral had exponential behavior in its integrand. For many wave propagation integrals, the integrands do have a dominant exponential factor so that the main contribution to the integral comes in the vicinity of saddle points. We shall show how it is possible to combine the cliff methods with the properties of saddle points to obtain a powerful extension to the ordinary saddle point methods. The reader unfamiliar with saddle point methods will find this chapter clearer if he first reads Chapter III.

The general type of integral we shall consider has the form

$$(2\text{-}1) \qquad \int_L f(z)\, e^{w(z)}\, dz$$

where the contour L may be of several types as discussed in what follows. We shall assume that f(z) does not contain terms of exponential order.

CLIFF METHOD--SEPARATED INTEGRAND

We shall apply the cliff method to evaluate 2-1 where we assume this integral is of the "separated" form--that is, f(z) and w(z) do not contain the same multivalued functions. For simplicity assume w(z) has one saddle point as indicated in Figure 2-1 and that L is a lancet contour about a cut which belongs to f(z). The steepest descent line passing through the saddle point is indicated in the figure.

There are three ways in which we can employ the cliff method: (1) we can deform the cut together with the lancet contour L onto the steepest descent path. Then we place the poles of the rational approximant to f(z) in the position of this deformed cut, collapse L onto the poles and employ the Residue Theorem. Since we are on a steepest descent path only a few poles near the saddle point are needed for our approximation. (2) we can first deform L onto the steepest descent path so that it is an open contour. Then we deform the cut onto the steepest descent path as indicated in Figure 2-2. Next we replace the cut by the poles of the rational approximant to f(z). The approximate solution is given by taking weighted residues at the poles.

The proper weighting and necessary assumptions are developed in Appendix G. For double valued functions we take half residues at the poles. Again we only need a few poles near the saddle point since we are on a steepest descent line. If there are no other singularities (such as branch points of g(z)) near the saddle point, then this is a practical method.

(3) we can first deform L as an open contour onto the steepest descent path. Then we expand f(z) in a rational approximant $R_n(z)$ which approximates f(z) in the unshaded region of Figure 2-3. Interior to the shaded region, $R_n(z)$ becomes vanishingly small by Cauchy's Integral Theorem. The poles of $R_n(z)$ lie along the boundary of the two regions as

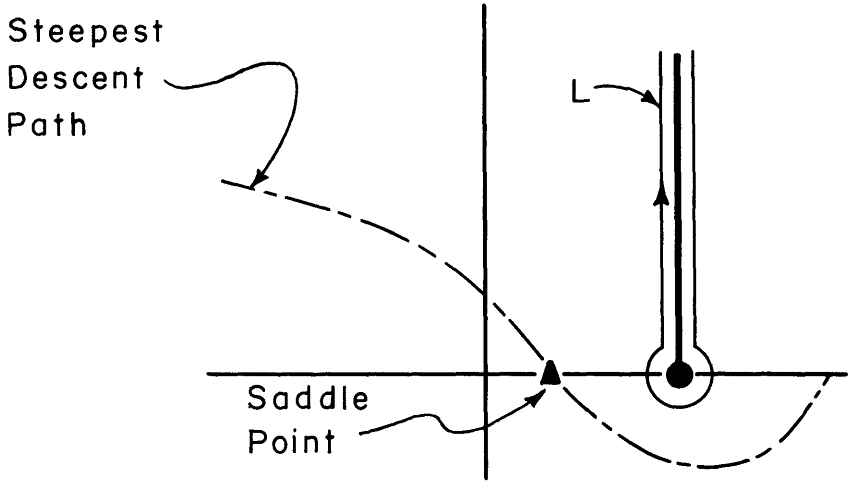


FIGURE 2-1

Branch Cut Deformed Onto
Steepest Descent Path

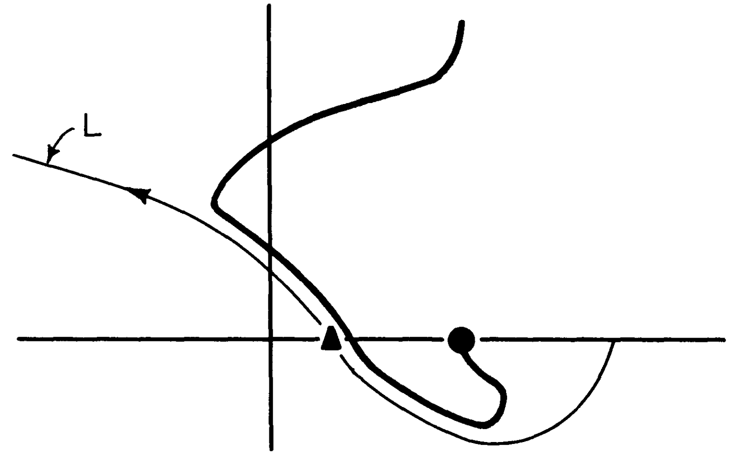


FIGURE 2-2

L is Equivalent To Loops Plus L'

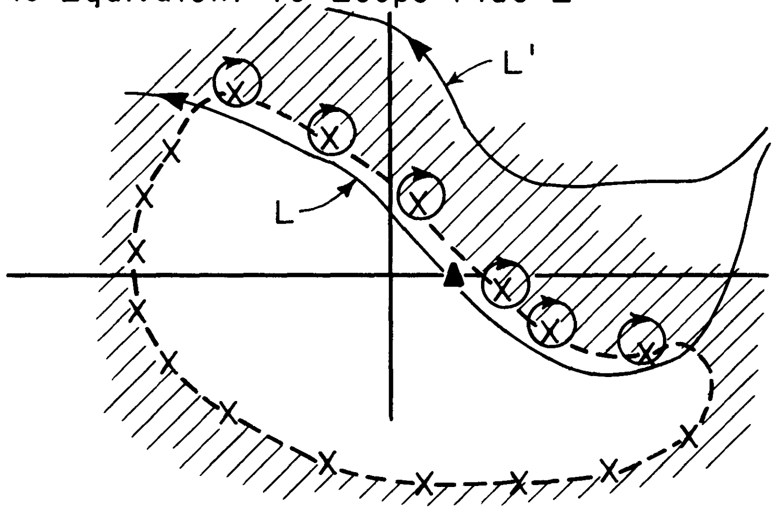


FIGURE 2-3

Branch Cuts For $(1-Z^2)^{\frac{1}{2}}$

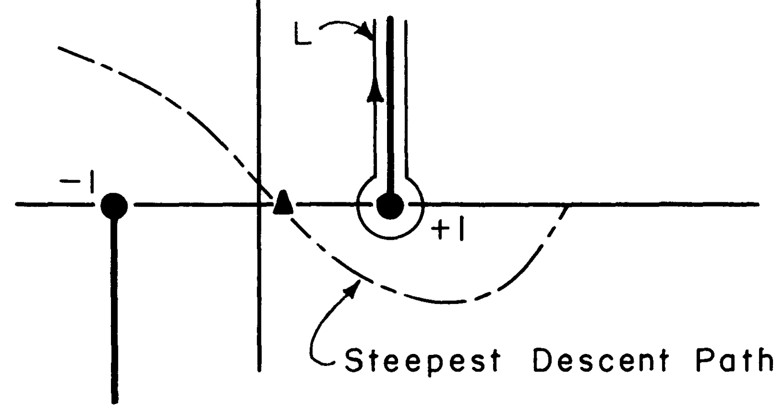


FIGURE 2-4

indicated. The next step is to deform L back onto the shaded region in a position such as L'. We are left with the loops around the poles on the steepest descent path (again only the poles near the saddle point are important) plus the integral on L'. In some cases, depending on w(z), L' can be shown to vanish as we deform it toward the point at infinity. Otherwise we can make the contribution from L' arbitrarily small by taking enourh poles along the steepest descent path. This follows from the well-known properties of the Cauchy Integral Formula. From the form of $R_n(z)$ we can set bounds on the value of the integral along L', if necessary.

These approaches to the integration problem are primarily topological. In practice it is easier to deform L onto the steepest descent path and then throw the integral into Stieltjes form in terms of the distribution function for f(z). We then approximate this distribution function by a stair case function as explained in Chapter I. The result will be the same as would be obtained by (2) or (3) above.

Since we can place our poles as we wish, it is possible to follow curved steepest descent paths and broad saddles.

CLIFF METHOD--MIXED INTEGRAND

Suppose that w(z) and f(z) both contain the same multivalued function--say, $(1-z^2)^{\frac{1}{2}}$ with cuts as indicated in Figure 2-4. If we deform the cut (from +1) together with the lancet contour L onto the steepest descent path, we must expand both w(z) and f(z) before we collapse L onto the singularities of the approximant. Since a rational function expansion in an exponent leads to essential singularities, we cannot employ the (simple) cliff method. However, if we deform L as an open contour on the steepest descent path, we can use the cliff method as follows.

Suppose that the integrand of 2-1 contains a number of multivalued functions. We can consider the Riemann surface rendering this integrand single valued to consist of sheets each of which is subdivided into two leaves corresponding to the two branches of $(1-z^2)^{\frac{1}{2}}$. Now we can expand this subdivision into four sheets, two of which correspond to $(1-z^2)^{\frac{1}{2}}$ in f(z) and two of which correspond to the $(1-z^2)^{\frac{1}{2}}$ in w(z). In other words, we consider these as different functions, although they have the same branch points.

Then to apply the cliff method, we remember that we are on one sheet of our four sheeted subdivision. We deform the cut for the $(1-z^2)^{\frac{1}{2}}$ belonging to f(z) onto the steepest descent path. Then we expand the $(1-z^2)^{\frac{1}{2}}$ of f(z) in a rational function which approximates $(1-z^2)^{\frac{1}{2}}$ in the unshaded region of Figure 2-5. We next deform L back into the shaded region. We are left with the residues at the poles on the steepest descent path plus the contour L' which has wrapped around the cut belonging to the $(1-z^2)^{\frac{1}{2}}$ in the exponent.

Now note that we could have expressed our integral in Stieltjes form along the steepest descent path, so that the distribution function would be generated by f(z). If we then approximate this distribution function in the usual manner with a stair case-like function, we obtain the same approximate solution as we would from the rational function

The Contour L is Deformed Onto The Poles Plus The Branch Cut

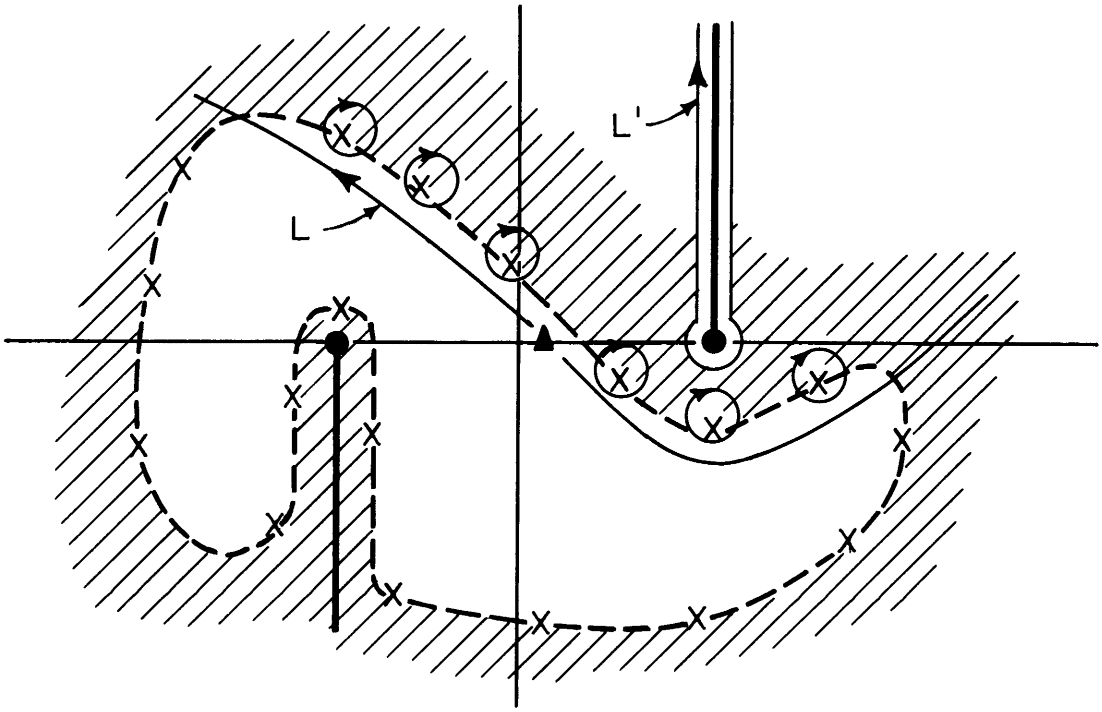


FIGURE 2-5

Steepest Descent Paths For $J_0(Z)$

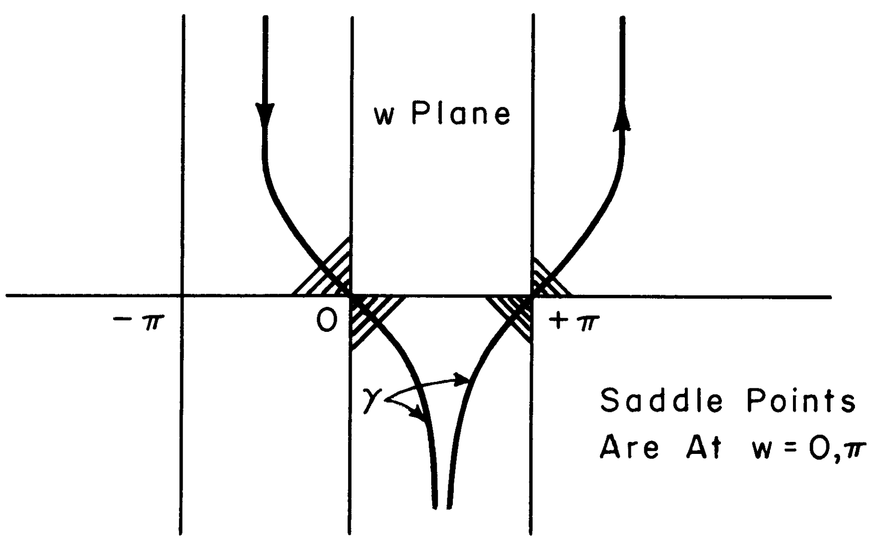


FIGURE 2-6

approach described above. The only difference between the two procedures is that the more topological approach gives some idea of the error that the integral on L' will introduce whereas the Stieltjes integral approach does not consider this.

EXAMPLE OF CLIFF METHOD

As a numerical example, consider the representation of the Bessel function

$$J_0(z) = \frac{1}{2\pi}\int_{\gamma} e^{iz\cos w}\, dw \tag{2-2}$$

where γ is the contour of Figure 2-6 and is already on the steepest descent paths for the two saddle points. In this case we can transform our integral into a line integral along the steepest descent paths. The result is the integral ${}^{1}I$ of equation E-1 of Appendix E. In Table E the result of approximating this integral by the cliff method is compared with the solution obtained by the ordinary saddle point method. For reference, the saddle point method solution is

$$J_0(z) \simeq \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{\pi}{4}\right)$$

At least up to $z = 4\pi$ the cliff method with five poles gives more accurate results. For $z < \pi/4$ the cliff method is not too good (although it does not blow up as does the saddle point method solution). For small z, if it were desireable, the cliff method solution could be easily improved by choosing a different pole spacing.

GENERAL CLIFF METHOD

Suppose that $w(z)$ contains the function $(1-z^2)^{\frac{1}{2}}$ with the cuts as shown in Figure 2-4. Then we have the choices: (1) we can deform the cut from +1 together with its lancet contour L onto the steepest descent path and employ the general cliff method. (2) we can deform L as an open contour onto the steepest descent path and expand $w(z)$ in a rational function $R_n(z)$ which approaches $w(z)$ in a region as indicated by the unshaded area of Figure 2-3. Then $\exp R_n(z)$ has a ring of essential singularities around this unshaded region. $\exp R_n(z)$ approaches $\exp w(z)$ interior to the unshaded region and approaches unity in the shaded region. This can be proven from Cauchy's Integral Theorem and the theorems of Appendix C. Next we deform L onto the singularities along the steepest descent path--again we need only those near the saddle point--plus a contour L' in the shaded region. If the poles of $R_n(z)$ are close enough then $\int_{L'} \exp R_n(z)\, f(z)dz$ approaches $\int_{L'} f(z)dz$ which may be easier to evaluate.

SUMMARY

The cliff method provides an important extension to the saddle point methods because, as shown in this chapter, the method can handle broad saddles and curved steepest descent paths. These are just the cases in which the ordinary (asymptotic) saddle point methods break down.

Because of the difficulties described in the section on the general cliff method of Chapter I and in Appendix E, the application of this method to steepest descent paths is not yet very practical.

Chapter III

SADDLE POINT METHODS

The preceding chapter brought out the power of the cliff methods of integration when applied to steepest descent paths. In this way the cliff methods provide a powerful extension to the popular saddle point method of integration, known also as the method of steepest descents and the stationary phase or col method, depending on the application. In this chapter another means of extending the ordinary saddle point method will be given. Basically this extension is the application of quadrature methods to steepest descent paths and will be called the quadrature saddle point method. A review of the ordinary saddle point method and some of its variations will first be given in order to make the presentation clearer.

The general type of integral handled by the saddle point method is

$$f(t) = \int_L F(s) e^{W(s,t)} ds$$

where L is the contour of integration in the complex s plane, F(s) and W(s,t) are analytic functions on this contour and t denotes any parameters. If F(s) is slowly varying, the exponential factor will dominate the integrand. The goal of the saddle point methods is twofold: to "bunch" the integrand about a point s_c and to deform L through s_c in such a way that the exponential factor does not oscillate on the deformed contour L'. For functions which are analytic on L', these twin goals are compatible.

First assume we can deform L through a point s_c with the property that $\mathrm{Re}\left[W(s,t)-W(s_c,t)\right]$ becomes increasingly negative on both sides of s_c as we move along the deformed contour L'. If the rate of increase is sufficiently rapid, then most of the integral comes from the vicinity of s_c.

Furthermore the rate of increase of $|Re[W(s,t)-W(s_c,t)]|$ as we move away from s_c will be maximized if we choose L' so that $Im[W(s,t)-W(s_c,t)]$ is constant on this contour. This follows from the property of analytic functions that keeping the real or imaginary part constant causes the conjugate part to vary at its maximum rate. We have thus achieved our goal of bunching the integral about a point s_c on a contour for which the exponential factor does not oscillate.

There are many points which will satisfy our requirements on s_c. Generally, the most useful are the solutions of $dW(s,t)/ds = 0$ which are also saddle points. For these points, the first derivative term in the Taylor expansion of the exponent will vanish for all t, so that we have a simpler form. The term saddle point comes from the fact that the Re W(s,t) or Im W(s,t) when plotted over the s plane often has the appearance of a saddle near this point. On the contour L' we come up one side of the saddle, pass the midpoint at s_c and then descend the other side. These are lines of steepest descent from the midpoint at s_c. If these lines are steep enough, then the whole contribution to the integral comes from a small region about the saddle point s_c. We shall denote the segment of L' which lies in this region by L''.

For convenience in what follows, set $W(s,t)-W(s_c,t) = P + iQ$ so that P and Q are the real and imaginary parts of this function. In this notation the contour L' passing through s_c must satisfy the conditions that P be nonpositive and Q be constant. In fact the latter condition must be $Q = 0$ since $W(s,t)-W(s_c,t)$ vanishes at the saddle point. We can say that the contour L' is a $Q = 0$ line in the s plane. On this contour the oscillatory part of the exponential factor is eliminated. Also function theory shows that P will decrease monotonically as we move along L' away from the saddle point, unless there are other saddle points nearby. Our original integral can now be replaced by the following integral on the finite segment L":

$$f(t) \simeq e^{W(s_c,t)} \int_{L''} F(s) e^{P} ds$$

Any error introduced in this step can be made as small as we like by increasing L".

The power of the saddle point method comes from the fact that we can often approximate our integral on the segment L" by certain standard integrals which are well known. It will not matter if the approximation breaks down outside of L" (assuming F(s) is slowly varying) because of the behavior of the exponential factor in both the given integral and in the standard forms.

FIRST ORDER SADDLE POINT METHOD

In the first order saddle point method we assume that on the segment L" we can approximate F(s) by a rational function and W(s,t) by a Taylor series truncated after the second derivative term. As an illustration the cases when F(s) can be approximated by a polynomial or a simple pole are worked out here. The term "first order" as used here corresponds to "second order" in the reference by Cerrillo (1950).

In the Taylor series expansion $W(s,t) = W(s_c,t) + W^{I}(s_c,t)(s-s_c) + W^{II}(s_c,t)(s-s_c)^2/2!$ the second term vanishes if s_c is the saddle point.

Then $P + iQ = W^{II}(s_c,t)(s-s_c)^2/2$. Next let $W^{II}(s_c,t)/2 = he^{i\alpha}$ and $s-s_c = re^{i\beta}$ so that $P = hr^2\cos(\alpha + 2\beta)$ and $Q = hr^2\sin(\alpha + 2\beta)$. From the condition $Q = 0$, we have $\beta = \beta_n = n\pi/2 - \alpha/2$ $(n = 0,1,2,3)$ so that $P = hr^2\cos n\pi$. Our contour must correspond to the $Q = 0$ lines for $n = 1,3$. The situation is shown in Figure 3-1 where the regions of negative P are shaded and the direction of the integration is indicated by the arrows. The $P = 0$ lines bisect the angles between the $Q = 0$ lines as shown. In general the contour L" will be curved near the saddle point. The three term Taylor expansion implies that L" can be approximated by a straight line of length $2r_0$ as shown in Figure 3-2. In fact the two lines must be tangent at the saddle point. Our integral now has the form

$$f(t) \simeq e^{W(s_c,t)}\left\{e^{i\beta_1}\int_0^{r_0} F(re^{i\beta_1})e^{-hr^2}dr + e^{i\beta_3}\int_0^{r_0} F(re^{i\beta_3})e^{-hr^2}dr\right\}$$
$$= e^{W(s_c,t)}e^{i\beta_1}\int_0^{r_0}\left[F(re^{i\beta_1}) + F(re^{i\beta_1+i\pi})\right]e^{-hr^2}dr$$

where $\beta_1 = \pi/2 - \alpha/2$. If hr^2 is large enough at r_0 we can take the range of integration to be infinite.

If F(s) has the form s^m where m is a positive integer, we have

$$f(t) \simeq e^{W(s_c,t)}e^{i\beta_1(m+1)}\left[1+(-1)^m\right]\int_0^{\infty} r^m e^{-hr^2}dr \tag{3-1}$$
$$= e^{W(s_c,t)}\left[1+(-1)^m\right]\Gamma\left(\tfrac{m+1}{2}\right) i^{m+1}/2\left(\frac{W^{II}(s_c,t)}{2}\right)^{\frac{m+1}{2}}$$

where we have employed the standard form

$$\int_0^{\infty} s^{\nu}e^{-hs^2}ds = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2\,h^{\frac{\nu+1}{2}}} \qquad (\nu > -1) .$$

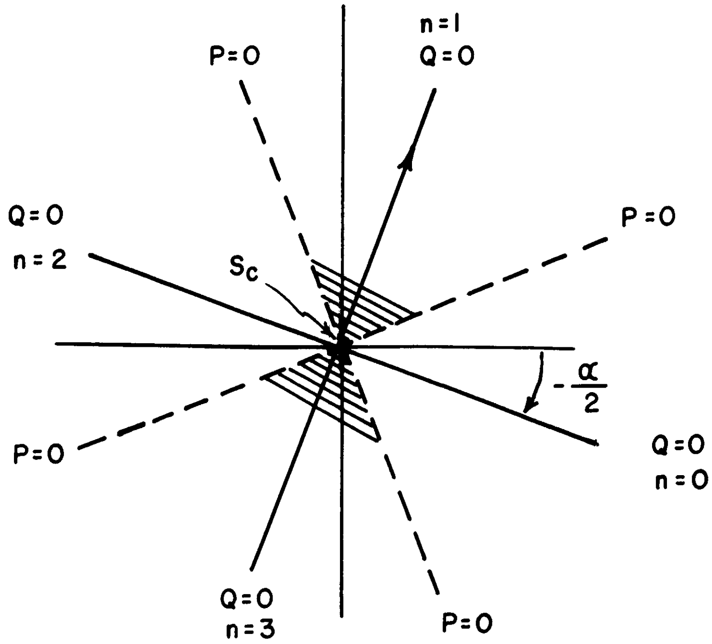


FIGURE 3-1

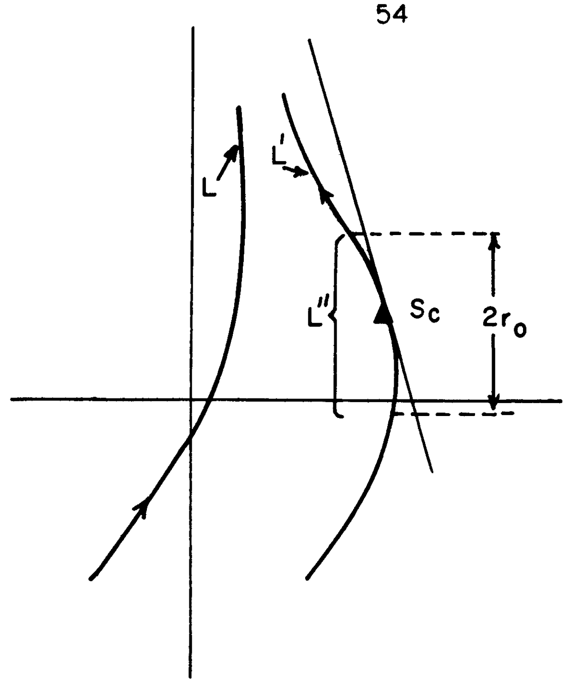


FIGURE 3-2

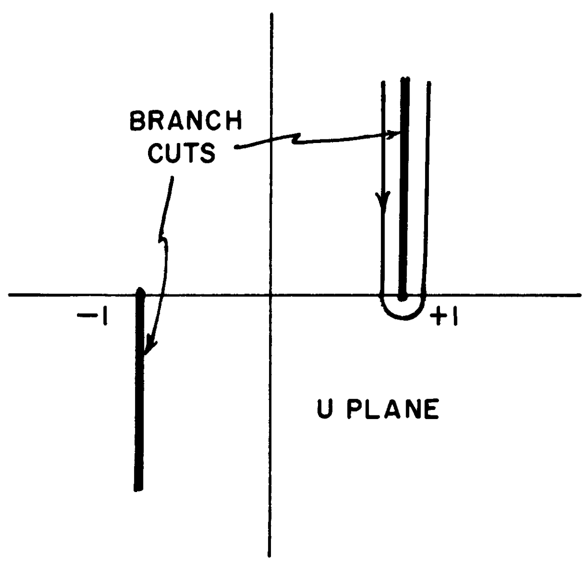


FIGURE 3-3

Evidently if the order m is even the solution vanishes. The extension to polynomials multiplied by a factor s^{ν} where $\nu > -1$ follows directly. We can also handle the case of a pole when $F(s) = 1/(s-s_p)$. Then we have

$$f(t) \simeq e^{W(s_c,t)} \int_0^{\infty} \left(\frac{1}{r-v} + \frac{1}{e^{i\pi} r - v} \right) e^{-hr^2} dr$$

$$= e^{W(s_c,t)} 2v \int_0^{\infty} \frac{e^{-hr^2}}{r^2 + e^{-i\pi} v^2} dr$$

where $e^{-i\beta}(s_p - s_c) = v$. If we let $r^2 = u$, we can put this integral in a standard form appearing in a table of Laplace transforms. The final result is

$$f(t) \simeq i\pi \, e^{W(s_c,t) + W^{II}(s_c,t)(s_p - s_c)^2/2} \, Erfc\left[e^{-i\pi} \left(W^{II}(s_c,t)/2 \right)^{\frac{1}{2}} (s_p - s_c) \right] \tag{3-2}$$

where Erfc is the error function complement defined by

$$Erfc(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_x^{\infty} e^{-t^2} dt = 1 - \frac{2}{\pi^{\frac{1}{2}}} \int_0^{x} e^{-t^2} dt \,.$$

The finite integral on the right is the tabulated error function. Notice there is no restriction on how far the pole is displaced from the saddle point. The only restriction is that $\left| \arg\left[W^{II}(s_c,t)(s_p - s_c)^2 e^{-2\pi i} \right] \right| < \pi$. If $s_p = 0$ the pole coincides with the saddle point and we have simply

$$f(t) \simeq i\pi \, e^{W(s_c,t)}$$

Although a different approach is employed, the method of van der Waerden (1951) for handling poles is essentially equivalent to that presented here.

Just what is meant by saying that F(s) is slowly varying, how to handle the case of multiple saddle points, the proof that the solutions are asymptotic and other considerations are given in the more detailed analysis in the references.

EXTENDED SADDLE POINT METHODS

When we require more than three terms of the Taylor series to approximate W(s,t) on the segment L", the first order method fails and the analysis becomes more involved. The case when we truncate the Taylor series after the third derivative term is called the second order saddle point method. The standard integrals which approximate our given integral are extended Airy-Hardy integrals. These are discussed, for example, by Cerrillo (1950) and tables are given in Cerrillo (1951). This method will sometimes work when there is a definite curvature in L" at the saddle point. However if our approximation requires still higher order terms of the Taylor series, the second order method also fails. In special cases the solutions for higher order terms have been worked out, but if we require too many terms of the Taylor series the method becomes unworkable. In some cases a power series expansion other than the Taylor expansion will produce faster convergence.

If the segment L" is sharply curved at the saddle point, it is usually possible to transform the integrand so that the $Q = 0$ lines become straight. The price we pay is that W(s,t) and F(s) may be more complex in form and the transformed F(s) may be dependent on the parameter t. (For some applications this obscures the physical interpretation of the solution.) If in the transformed plane, e^P can be represented along L" by the expansion $e^P = \sum_j c_j e^{-h_j r^2}$, we can then employ the first order saddle point method. The saddle points for each of the terms of the sum will be displaced from the saddle point of the original function and we can speak of these as satellite saddle points. The details and some examples are given by Wernlein (1957).

A method of extending the range of the saddle point method in certain cases by the use of partial asymptotic expansions is given by Clemmow (1950) and an example of the applications is given by Pearson (1953).

LIMITATIONS AND STEEPEST DESCENT PATHS

In order to illustrate the limitations of the saddle point method as well as to compare it with other methods, we shall evaluate an integral appearing in the Sommerfield problem. This integral has the form

$$(3\text{-}3) \qquad I = \int_{\mathfrak{B}_{+1}} e^{iau} \frac{H_0^{(1)}[\rho(1-u^2)^{\frac{1}{2}}]}{u-u_p}\, du$$

where a and ρ are positive real numbers and $|u_p| < 1/2^{\frac{1}{2}}$, $3\pi/2 \geqslant \arg u_p \geqslant 3\pi/4$. The argument of the Hankel function is defined on the Riemann sheet B given in Figure D-1. The branch cuts and contour of integration are shown in Figure 3-3.

As shown in Appendix A, if $|\rho(1-u^2)^{\frac{1}{2}}| > \sim 3$ we can approximate the Hankel function by

$$H_0^{(1)}[\rho(1-u^2)^{\frac{1}{2}}] \simeq \sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}}\, e^{i\rho(1-u^2)^{\frac{1}{2}} - \frac{1}{4}\ln(1-u^2)}$$

Then the exponentially varying part of the integrand has the exponent $W(u) = iau + i\rho(1-u^2)^{\frac{1}{2}} - \ln(1-u^2)$. If u_c is the saddle point then we have

$$I \simeq \sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}} e^{W(u_c)} \int_{\mathcal{B}_{+1}} \frac{e^{W(u)-W(u_c)}}{u-u_p}\, du \tag{3-4}$$

The saddle points are found from

$$\frac{dW(u)}{du} = W^I(u) = ia - i\rho u(1-u^2)^{-\frac{1}{2}} + u/[2(1-u^2)] .$$

This can be put in the form of a fourth degree equation

$$u^4 + i\frac{a}{\rho^2+a^2}u^3 - \left(\frac{\frac{1}{4}+a^2}{\rho^2+a^2} + 1\right)u^2 - i\frac{a}{\rho^2+a^2}u + \frac{a^2}{\rho^2+a^2} = 0$$

If we make the approximation $\frac{1}{4}/(\rho^2 + a^2) \ll 1$, we can factor this equation into

$$(u^2-1)\left(u^2 + i\frac{a}{\rho^2+a^2}u - \frac{a^2}{\rho^2+a^2}\right) = 0$$

Since $u = \pm 1$ are branch points they cannot be saddlepoints. The other roots are

$$u = -i\frac{a}{\rho^2+a^2} \pm \frac{a}{(\rho^2+a^2)^{\frac{1}{2}}} .$$

To see which Riemann sheet these points lie on, we can examine $W^I(u)$ on sheet B for small values of u. We have $W^I(u) \simeq ia - iu + u/2 = 0$. If $\rho \gg \frac{1}{2}$ then $u = a/\rho$ so that the saddle point on sheet B is

$$u_c = -i\frac{a}{\rho^2+a^2} + \frac{a}{(\rho^2+a^2)^{\frac{1}{2}}} .$$

The other root corresponds to a saddle point on sheet A.

When a = 0 we can find the saddle points exactly. The analysis shows that there are four saddle points -- one at the origin on sheet B, one at the origin on sheet A and two more on sheet A at the points $u = \pm (1 + 1/(4\rho^2))^{\frac{1}{2}}$. For the case $a \neq 0$ the exact analysis shows that the roots at ± 1 in the approximate analysis correspond to two saddle points which are near these branch points and lie on sheet A. Consequently we have only one saddle point to consider on sheet B. The analysis that follows will be simplified if we approximate the saddle point by $u_c = a/(\rho^2 + a^2)^{\frac{1}{2}}$, though this is not a necessary restriction. Then our saddle point lies on the real axis in the interval (0,+1) so that it is no problem to deform our contour of integration through the saddle point.

In order to apply the first order saddle point method we need the second derivative in the Taylor expansion of $W(u)$. This is

$$W^{II}(u) = -i\frac{\rho}{(1-u^2)^{\frac{3}{2}}} + \frac{1}{2}\,\frac{1+u^2}{(1-u^2)^2}$$

so that

$$W^{II}(u_c) = -i\rho\left(1+\frac{a^2}{\rho^2}\right)^{\frac{3}{2}} + \frac{1}{2}\left(1+\frac{a^2}{\rho^2}\right)\left(1+\frac{2a^2}{\rho^2}\right) \simeq -i\rho$$

In the notation of the first section $\arg W^{II}(u_c) = \alpha \simeq -\pi/2$. As a glance at Figure 3-1 shows, our contour crosses the saddle point in the opposite sense to that indicated by the arrows so that we must prefix a minus sign to our solution. Then from the standard form for the first order method with a pole we have the solution

$$(3\text{-}5)\quad I \simeq -(2\pi)^{\frac{1}{2}} e^{i\frac{\pi}{4}} \frac{(\rho^2+a^2)^{\frac{1}{4}}}{\rho} e^{i(\rho^2+a^2)^{\frac{1}{2}}+B}\, Erfc(B^{\frac{1}{2}}) \quad \text{where } B^{\frac{1}{2}} = \left(\frac{W^{II}(u_c)}{2}\right)^{\frac{1}{2}}(u_p-u_c)e^{-i\pi} \simeq i\left(\frac{i\rho}{2}\right)^{\frac{1}{2}}(u_p-u_c)$$

Now this solution is only valid when the second derivative in the Taylor expansion is much larger in magnitude than the higher order derivatives. In order to get a quantitative figure on the range of validity of our results we first form the derivatives

$$W^{III} = -i\,\frac{3\rho u}{(1-u^2)^{5/2}} - u\,\frac{(1+2u^2)}{(1-u^2)}$$

$$W^{IV} = -i\,3\rho\,\frac{1+4u^2}{(1-u^2)^{7/2}} - \frac{1+3u^2-10u^4}{(1-u^2)^2}$$

Then

$$\left|\frac{W^{IV}(u_c)}{W^{III}(u_c)}\right| = \left|\frac{1+\frac{i}{2\rho}\,\frac{1+u^2}{(1-u^2)^{1/2}}}{1-\frac{i}{3\rho}(1-u^2)^{5/2}(1+2u^2)}\;\frac{1-u^2}{3u}\right|_{u=u_c}$$

If $u_c < \frac{1}{2}$ this ratio behaves as $1/3u_c$ and if u_c approaches +1 this ratio approaches zero. In other words only as the saddle point approaches the origin can we use the first order method. As the saddle point approaches the branch point at +1 the second derivative no longer dominates and we expect the Q = 0 lines become sharply curved. In the same manner we find that $\left|W^{V}(u_c)/W^{IV}(u_c)\right| \sim \frac{1}{4}$ to $\frac{1}{3}$ for $u_c < \frac{1}{2}$ and vanishes as u_c approaches +1. This indicates that the second order saddle point method will not give much improvement. If it were desireable the actual errors in neglecting the higher order derivatives could be related to the error in the approximate integral, but this is somewhat tedious and the purpose here is merely to indicate the limitations of the saddlepoint method.

Analyzing the error by comparing terms of the Taylor series expansion can be cumbersome as the above sketch indicates. A qualitative insight as to the errors involved can be obtained by examining the

$Q = 0$ lines of the function $W(u)-W(u_c)$. If we include the logarithm term we must deal with transcendental equations. However if the saddle point is not close to $+1$, we can drop the logarithm term without affecting the analysis very much. Otherwise we could include the logarithm term in $F(s)$. In any case we shall take the exponent in the form $W(u)-W(u_c) = iau + \rho i(1-u^2)^{\frac{1}{2}} - i(\rho^2+a^2)^{\frac{1}{2}}$. The saddle point is easily found to be $u_c = a(\rho^2+a^2)^{-\frac{1}{2}} = (1 + k^2)^{-\frac{1}{2}}$ where we let $\rho/a = k$. We shall also use the abbreviation $(1 + k^2)^{\frac{1}{2}} = A$ in what follows.

Now to obtain the steepest descent lines, substitute $u = x + iy$ in

$$iau + i\rho(1-u^2)^{\frac{1}{2}} - iaA = P + iQ \tag{3-6}$$

and solve for Q. After some algebra we obtain the solution

$$y = \pm\left(x - A - \frac{Q}{a}\right)\left[\frac{1}{k^2} - \frac{1}{k^2x^2 + \left(x - A - \frac{Q}{a}\right)^2}\right]^{\frac{1}{2}} \tag{3-7}$$

The quantity in brackets will be real when $(1 + k^2) < (A + Q/a)^2$ as can be shown by differentiation of the second term in the brackets with respect to x. When $Q = 0$ equation 3-7 gives us the steepest descent and steepest ascent lines which pass through the saddle point. These are given in Figure 3-4 for $k = 4$. The dashed lines of that figure indicate the position of the lines if the branch cuts are moved. If for any reason we did not want to stay exactly on the $Q = 0$ line during integration, then equation 3-7 will tell us how far off the $Q = 0$ line we can stray and still keep Q within, say, ten degrees of its initial zero value.

If we let $(Ax-1)/k = \gamma$ then the steepest descent line can be put in the parametric form

$$(3\text{-}8) \qquad x = \frac{k\tau + 1}{A}, \qquad y = \frac{\tau(\tau - k)}{A(1+\tau^2)^{\frac{1}{2}}} .$$

Several steepest descent lines for different values of k are given in Figures 3-5 and 3-6. In the former the slopes and points used for the rapid construction of these curves are given for any k.

Whether we integrate along a Q = 0 line or some other Q = constant line, we shall want to know how far out the integration must be carried until there is negligible contribution to the integral. This information is given by the level lines for P = -D where D is a positive constant. As we follow a Q line away from the saddle point, we cross P lines with consecutively larger values of D. Beyond D = 4 or 5 the contribution to our integral is negligible since the factor e^P of the integrand will have a strong attenuating effect. By an analysis similar to that for the Q lines, we obtain from 3-6 the equation of the level lines for P:

$$(3\text{-}9) \qquad x = \pm\left(y + \frac{P}{a}\right)\left[\frac{1}{k^2} + \frac{1}{\left(y+\frac{P}{a}\right)^2 + y^2}\right]^{\frac{1}{2}}$$

Setting P = -D gives us the desired level lines. The level lines for D = 4 with a = 0 and $\rho = 8$ is given in Figure 3-7. The case D = 4 with a = 16 and $\rho = 64$ is given in Figure 3-8.

From the figures, the increase of the curvature of the steepest descent lines as the saddle point approaches the branch point at +1 is quite apparent. The segment of the Q = 0 line which can be approximated by the straight line of the first order method rapidly decreases in length. For instance when the saddle point is at the origin and

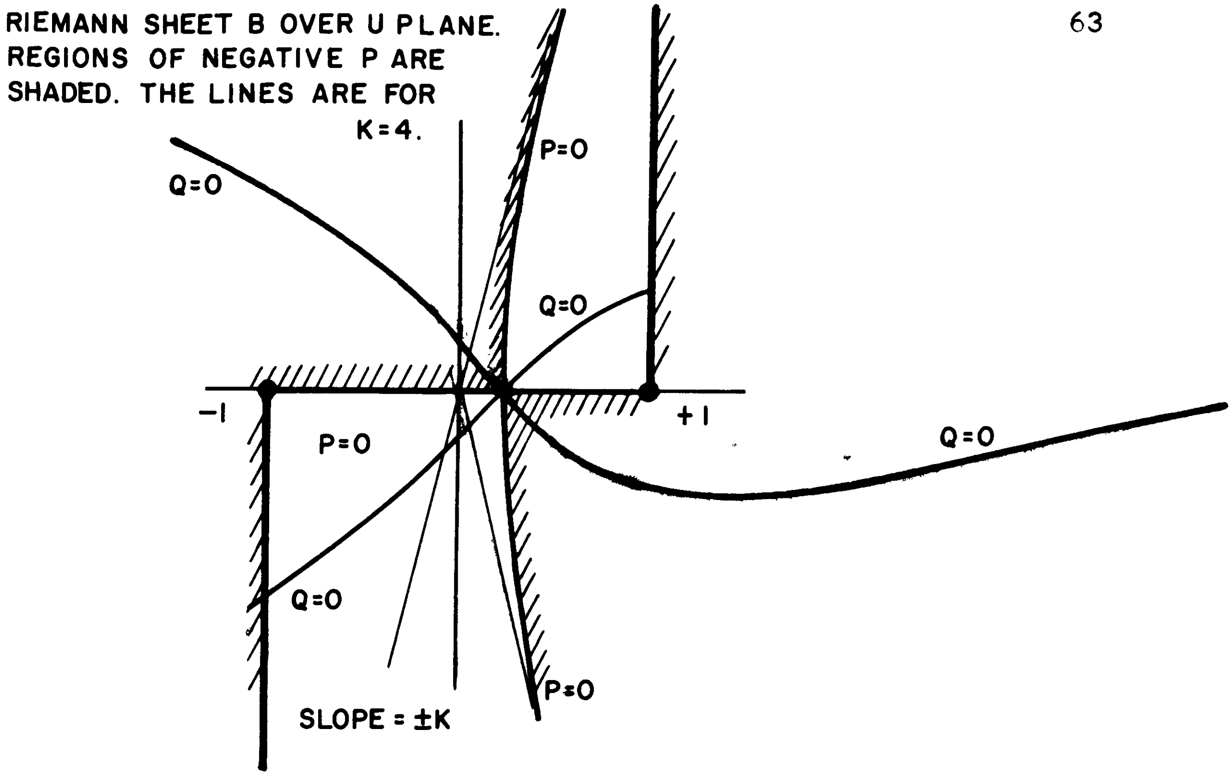


FIGURE 3-4

PLOT OF g(τ) ALONG Q=0 LINE FOR a=0

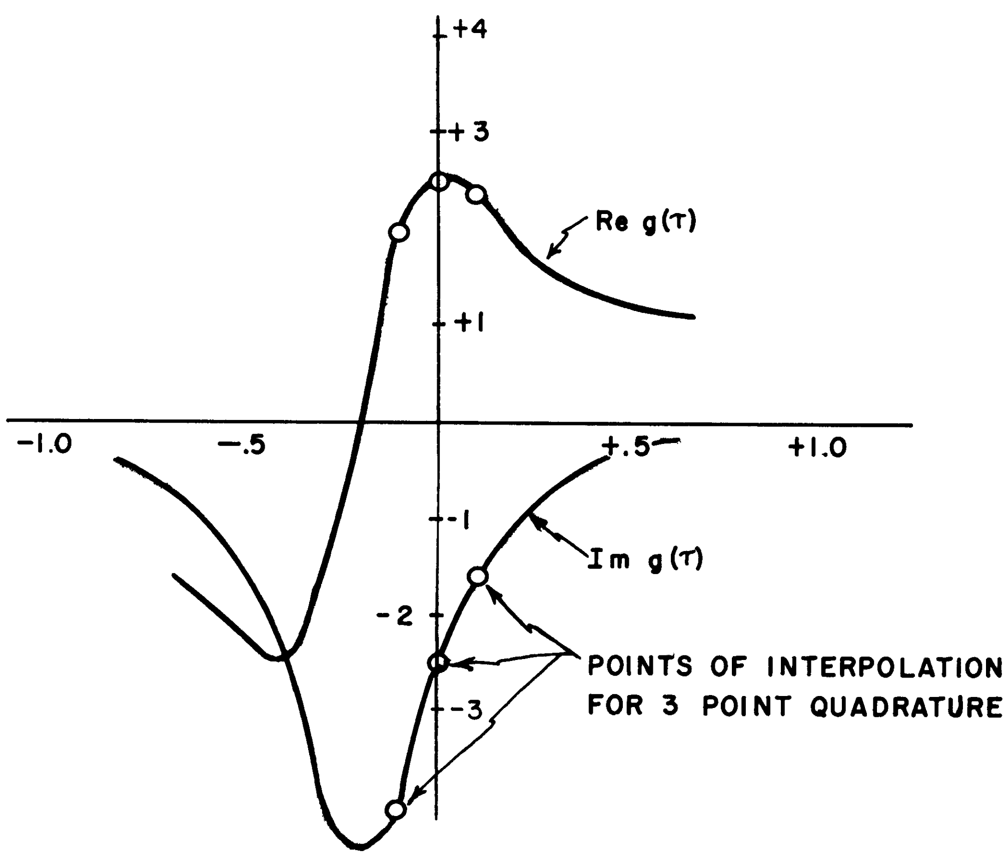


FIGURE 3-9

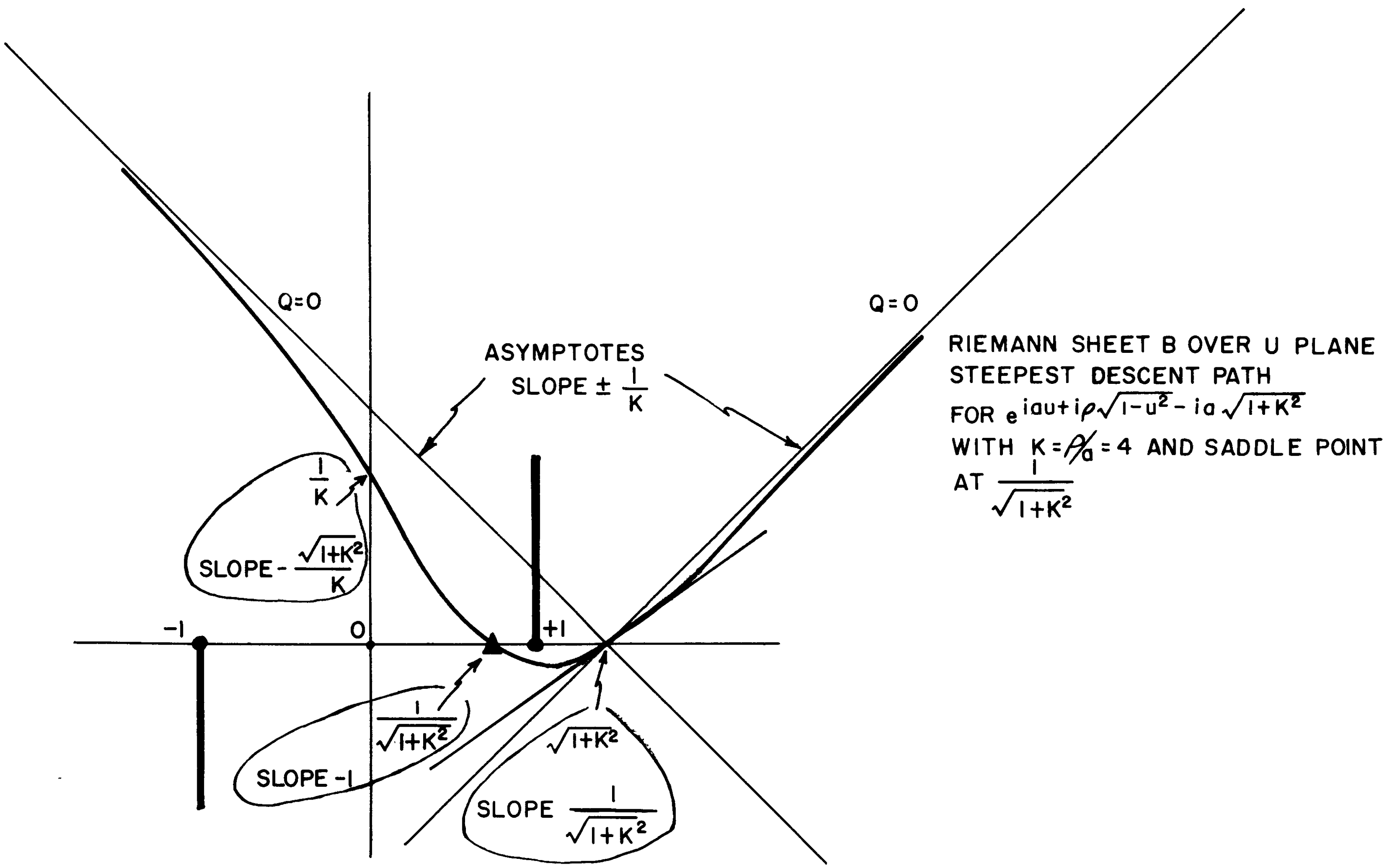

Q=0
Q=0
ASYMPTOTES
SLOPE ± $\frac{1}{K}$
RIEMANN SHEET B OVER U PLANE
STEEPEST DESCENT PATH
FOR $e^{iau+i\rho\sqrt{1-u^2}-ia\sqrt{1+K^2}}$
WITH $K=\rho/a=4$ AND SADDLE POINT
AT $\frac{1}{\sqrt{1+K^2}}$
$\frac{1}{K}$
SLOPE $-\frac{\sqrt{1+K^2}}{K}$
-1
0
+1
$\frac{1}{\sqrt{1+K^2}}$
SLOPE -1
$\sqrt{1+K^2}$
SLOPE $\frac{1}{\sqrt{1+K^2}}$

FIGURE 3-5

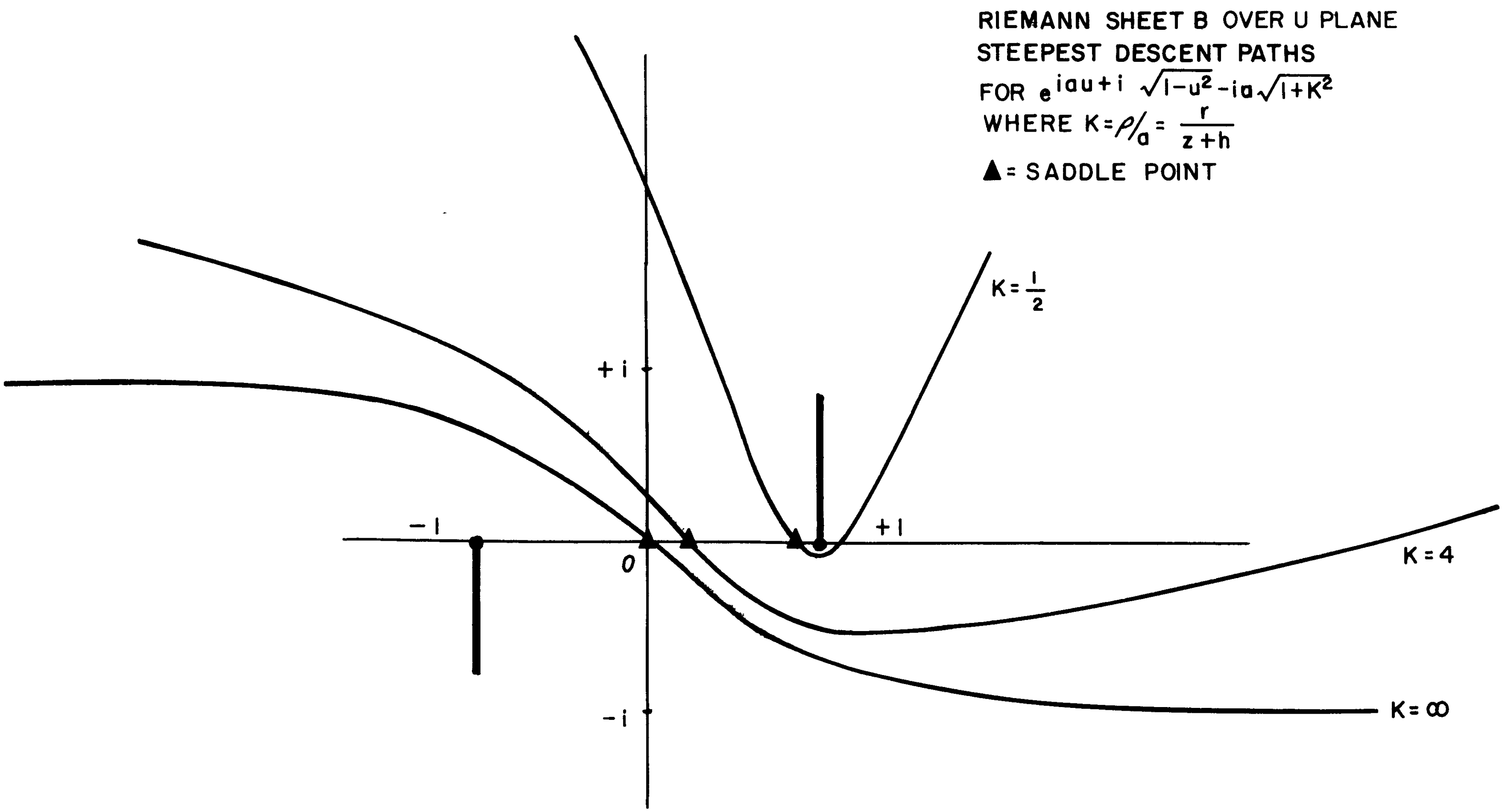

RIEMANN SHEET B OVER U PLANE
STEEPEST DESCENT PATHS
FOR $e^{iau+i\sqrt{1-u^2}-ia\sqrt{1+K^2}}$
WHERE $K = \rho/a = \frac{r}{z+h}$
▲= SADDLE POINT
$K=\frac{1}{2}$
$K=4$
$K=\infty$
+i
-i
-1
+1
0

FIGURE 3-6

$i\rho\sqrt{1-U^2} - i\rho = -D$ LINES FOR $D=4, \rho=8$

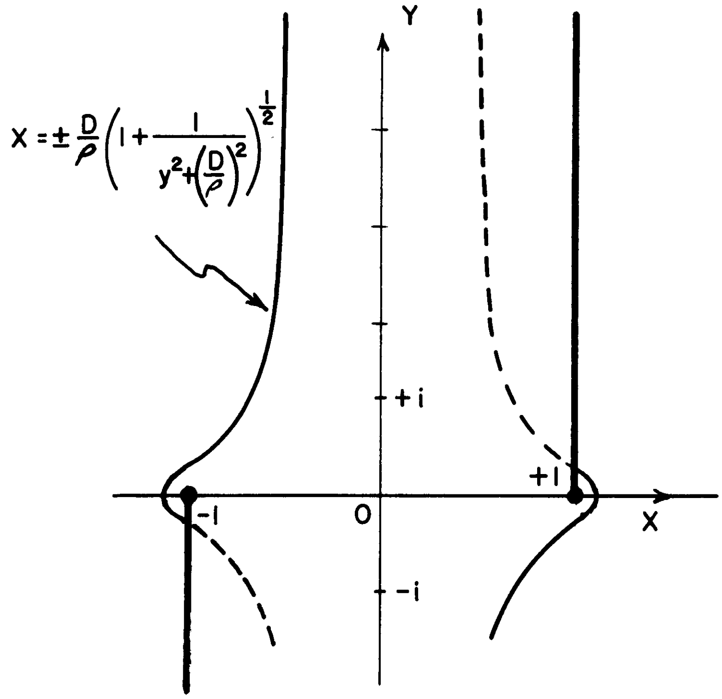


FIGURE 3-7

$iau + i\rho\sqrt{1-U^2} - i\sqrt{a^2+\rho^2} = -D$ LINES FOR

$K = \frac{\rho}{a} = 4$, $a = 16$, $D = 4$

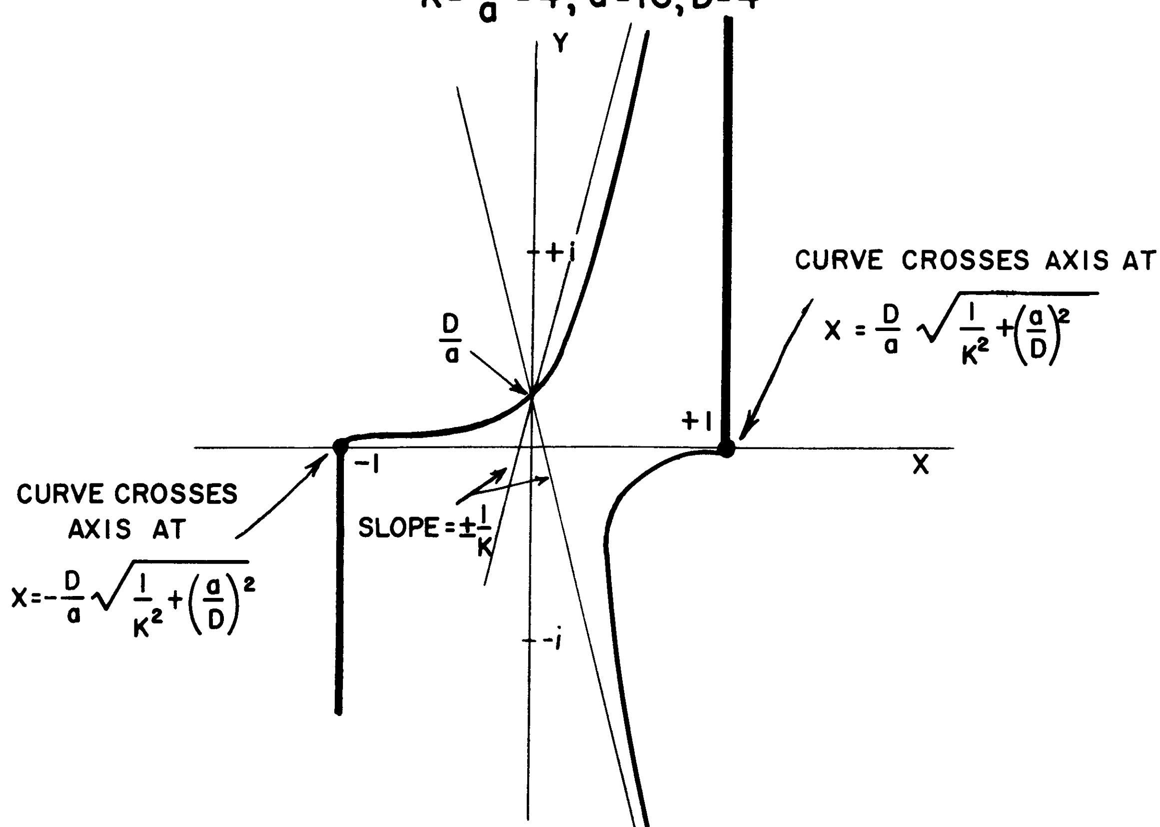


FIGURE 3-8

consequently a = 0, we see from Figure 3-6 that the approximation breaks down when $|u| = r_0 > \sim 1$. In the notation of the first section of this chapter, if we require $hr_0^2 > 4$ so that the exponential factor will dominate, then $h > 4$. Recalling that $h = \left|\frac{W^{II}}{2}(u_c)\right| \simeq \rho/2$ we find that $\rho > 8$ in order to apply the first order saddle point method. As expected, the analysis with the higher order derivatives of the Taylor expansion gives approximately the same answer.

QUADRATURE SADDLE POINT METHOD

The preceding methods have several disadvantages. For one thing the solutions are asymptotic which can cause difficulty if they must be differentiated. Also it may not be possible to represent W(s,t) by only a few terms of the Taylor expansion. Since along a Q = 0 line of W(s,t) the integrand does not oscillate, it may be possible to approximate the integral by some quadrature rule such as Simpson's rule. In this way the integration can be carried out on the actual Q = 0 line of W(s,t) no matter how sharp the curvature at the saddle point. Also the actual form of W(s,t) is used. If the quadrature rule is suitably chosen, it can be shown that the solution will always be convergent (see Lanczos, p. 402). Although this is a method of numerical analysis, the solution is analytic in that the parameters are left in completely general form. When F(s) has a polynomial representation,only a few terms are needed. Unfortunately if F(s) has a pole near the saddle point, this method converges slowly. However by suitably combining this method with the first order saddle point method this situation also can be handled.

For the type of integral appearing in the Sommerfeld problem the

Gaussian quadrature is probably the best suited and will be illustrated in the following example.

We shall evaluate the integral of equation 3-3 by the use of a modified Hermite-Gauss quadrature and then compare the answer to that obtained by the first order saddle point method. Since we want to integrate along the steepest descent or Q = 0 line, we shall use the relations of 3-8 to express the integral in terms of the variable τ. From 3-8 we can obtain the following relations valid on the Q = 0 line.

$$
(3\text{-}10)\quad \left.\begin{aligned}
v &= \frac{K\tau+1}{A} + i\,\frac{\tau(\tau-K)}{A(1+\tau^2)^{\frac{1}{2}}} \longrightarrow \tau\left(1-\frac{i}{(1+\tau^2)^{\frac{1}{2}}}\right)\\
\frac{dv}{d\tau} &= \frac{K}{A} + i\,\frac{\tau^3+2\tau-K}{A(1+\tau^2)^{\frac{3}{2}}} \longrightarrow 1-\frac{i}{(1+\tau^2)^{\frac{3}{2}}}\\
(1-v^2)^{\frac{1}{2}} &= \frac{K-\tau}{A} + i\,\frac{K\tau^2+\tau}{A(1+\tau^2)^{\frac{1}{2}}} \longrightarrow 1+i\frac{\tau^2}{(1+\tau^2)^{\frac{1}{2}}}\\
iav+i\rho(1-v^2)^{\frac{1}{2}}-iaA &= \frac{-aA\tau^2}{(1+\tau^2)^{\frac{1}{2}}} \longrightarrow -\rho\frac{\tau^2}{(1+\tau^2)^{\frac{1}{2}}}
\end{aligned}\right\}\ \text{if } a=0
$$

Upon substituting these relations into 3-3 and employing the asymptotic form of the Hankel function, we obtain the integral

$$
(3\text{-}11)\quad I \simeq \sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}+i\rho}\int_{-\infty}^{+\infty} \frac{e^{-\rho\frac{\tau^2}{(1+\tau^2)^{\frac{1}{2}}}}\left[1-\frac{i}{(1+\tau^2)^{\frac{3}{2}}}\right]}{\left[1+i\frac{\tau^2}{(1+\tau^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}\left[\tau-i\frac{\tau}{(1+\tau^2)^{\frac{1}{2}}}-v_p\right]}\,d\tau
$$

We shall take $\rho \gg 1$ and $u_p = -.40$ which choice will overlap the range in which the first order saddle point method is valid. A glance at Figure 3-6 for the steepest descent line with k = ∞ or a = 0 shows that we pass fairly near the pole u_p. Nevertheless a three term quadrature formula will give sufficient accuracy. If the path were much closer to the pole it would be better to subtract off from the integrand the effect of the pole, handling this by the first order method, and treat the balance by the quadrature method. In our present case, in terms of the quadrature formula our integral 3-11 becomes

$$I \simeq \sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}+i\rho} \sum_{j=-1}^{j=+1} \frac{(1+\tau_j^2)^{\frac{5}{4}}}{(1+\tau_j^2/2)}\, g(\tau_j)\, \mu_j \tag{3-13}$$

where $g(\tau) = \left[1 - \frac{i}{(1+\tau^2)^{\frac{3}{2}}}\right]\left[1 + i\frac{\tau^2}{(1+\tau^2)^{\frac{1}{2}}}\right]^{-\frac{1}{2}}\left[\tau - i\frac{\tau}{(1+\tau^2)^{\frac{1}{2}}} - v_\rho\right]^{-1}$

and μ_j are weight factors. Both τ_j and μ_j depend on ρ.

This formula is obtained from the results of Chapter 8 of Hildebrand. In that chapter the following Hermite-Gauss formula is given for an integral of the type:

$$\int_{-\infty}^{+\infty} e^{-s^2} f(s)\,ds \simeq \sum_{j=-n}^{j=+n} f(s_j)\mu_j' \tag{3-14}$$

For a three term formula the weight factors μ_j' and the abscissas s_j have the values

$\mu_0' = 1.1816\ldots \qquad s_0 = 0$

$\mu_{\pm 1}' = .29541\ldots \qquad s_{\pm 1} = \pm 1.2247\ldots$

The integral we have has the form

$$M = \int_{-\infty}^{+\infty} e^{-\rho\,\tau^2/(1+\tau^2)^{\frac{1}{2}}}\, g(\tau)\,d\tau$$

If we let $\rho\tau^2/(1+\tau^2)^{\frac{1}{2}} = s^2$ in this integral we obtain

$$M = \int_{-\infty}^{+\infty} e^{-s^2} g[\tau(s)] \frac{d\tau}{ds}\, ds \,.$$

Comparison with 3-14 shows that we must identify $g[\tau(s)]\frac{d\tau}{ds}$ with f(s) so that the quadrature formula for M is

$$M \simeq \sum_{j=-n}^{j=+n} g(\tau_j) \frac{(1+\tau_j^2)^{\frac{5}{4}}}{(1+\tau_j^2/2)} \mu_j$$

where $\mu_j = \mu_j'/\rho^{\frac{1}{2}}$ and $\tau = \frac{s^2}{\rho 2^{\frac{1}{2}}}\left(1+\left(1+\frac{4\rho^2}{s^4}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

The nature of these formulas is such that if there are m terms in the formula, then the formula is exact for any polynomial f(s) of degree less than 2m.

In some situations it may be preferable to use the Gaussian quadrature formula with equal weights. Then we would have in place of 3-13,

$$I \simeq \sqrt{\frac{2}{\pi\rho}}\, \frac{e^{-i\frac{\pi}{4}}}{2} \sum_j e^{-\rho\tau_j^2/(1+\tau_j^2)^{\frac{1}{2}}} g(\tau_j)\, \mu_j$$

where the weights μ_j and the τ_j can be obtained from tables given in part by Hildebrand, for instance. In this formula the complete form of the integrand including the exponential factor is preserved. However the convergence is somewhat slower than the Hermite-Gauss formula if the exponential varies rapidly.

Upon inserting the value $\rho = 500$ in the right side of 3-13 we obtain

$$(3\text{-}15) \qquad I \simeq -\sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}} e^{i\rho} \left(-.22 \quad +i.20 \quad \right)$$

In Figure 3-9 the variation of the real and imaginary parts of $g(\tau)$ along the path of integration is shown together with the points of interpolation used in obtaining 3-15. Now the first order saddle point method applied to 3-3 yields to two significant figures,

$$I \simeq -\sqrt{\frac{2}{\pi\rho}}\, e^{-i\frac{\pi}{4}} e^{i\rho} \left(-.22 \quad +i.19 \quad \right)$$

which agrees, as expected, with the quadrature method.

For smaller values of ρ we can see from Figure 3-9 that the effect of the pole is greater since our interpolation points are spread out more. For ρ = 20 the three term formula is not accurate enough, but calculations which are similar to the foregoing show that a five term formula gives satisfactory results. For still smaller values of ρ or for a pole very near the saddle point,it is better to subtract off the effect of the pole before applying the quadrature formula. For ρ = 20, say, as we increase a, the effect of the pole is diminished since the pole to saddle point distance increases. Therefore our five term formula for ρ = 20 and a = 0 will actually improve as a becomes larger. It turns out that a five or seven term quadrature formula, combined when necessary with the ordinary saddle point method, will at least handle the whole region for which we can use the asymptotic form of the Hankel function. In fact the method is not limited to this region. In the next chapter, for the Sommerfeld dipole radiation problem, the region of validity of the quadrature saddle point method is compared graphically with that of the ordinary saddle point method.

When a pole lies very near the saddle point, the quadrature saddle point method converges slowly, but this can remedied by subtracting off the effect of the pole and handling this term by the ordinary saddle point method. It is not always necessary to combine the first order saddle point method with the quadrature method. Sometimes it is sufficient to indent the contour of integration near the pole and still apply the quadrature method. The Q line analysis given earlier in this chapter will indicate when this is possible.

SUMMARY

In this chapter we have shown the relationship between the various modifications of the (asymptotic) saddle point methods. We have also shown how the saddle point methods can be extended to handle the case of broad saddles and curved steepest descent paths by employing quadrature methods. This extension was described as the quadrature saddle point method.

The advantage of the quadrature saddle point method is that it will work where the first order saddle point method breaks down. (In the example of the previous section this occured when the saddle point approached the branch point at +1). Furthermore if there is a pole near the saddle point, the first order method requires a table of the error function or error function complement for complex arguments. If the arguments are not multiples of 45^{o} then considerable labor is involved, as complete tables are not available (see Rosser). The quadrature method requires no special functions other than those given in the integrand.

The relation between the quadrature saddle point method and some other methods given in the literature will be brought out in the next chapter for the solutions to the Sommerfeld dipole radiation problem.

The relationship between the cliff method and the quadrature method applied to steepest descent paths was illustrated in the numerical example of the previous chapter. As shown in Appendix E, the cliff method compared very favorably with the quadrature methods. A quantitative appraisal of the situation can only be made by comparing error analyses. While such analyses are available for the quadrature methods, no equivalent analysis has been developed for the cliff methods. As explained in the previous chapter, one of the principal advantages of the cliff method is that it can often be applied to integrals for which it is difficult to develop a good quadrature.

Chapter IV

APPLICATION TO THE SOMMERFELD PROBLEM

Since the object in developing the methods of approximate integration is to apply them to physical problems, the present chapter is included to illustrate the various additional steps which are necessary. For instance, before we can apply the integration techniques, we must choose the proper Riemann sheet for integration. Also certain poles must be located if contour deformations are involved. These and other analytical steps will be illustrated in detail for the Sommerfeld dipole radiation problem. This particular problem was chosen as an illustration because it has features which appear in a number of geophysical problems concerned with wave propagation. Furthermore it is a problem that has been thoroughly analyzed in the literature.

INTRODUCTION

In 1909 Arnold Sommerfeld published a solution to the problem of finding the electromagnetic field due to an electric or magnetic dipole element located above a flat earth of finite conductivity. Certain parts of his solution, however, were controversial, particularly his expression for a surface wave. In 1919 H. Weyl obtained an independent solution which did not agree with Sommerfeld's solution. Since that time a host of papers, both theoretical and experimental, have appeared on this subject which is certainly one of the most thoroughly considered aspects of radio wave propagation. No attempt at a review of this work will be made here, as there exist several historical accounts (see, for instance, the final chapter of Electromagnetic Theory by J. A. Stratton).

A number of explanations have been published for the discrepancy between Sommerfeld's 1909 solution and the independent solutions obtained by H. Weyl in 1919, A. Sommerfeld in 1926, Balth. van der Pol and K. F. Niessen in 1930 and W. H. Wise in 1931. These latter solutions all agree and have been verified experimentally, whereas the 1909 solution does not agree with experiment. The correct reason for this discrepancy was given by K. F. Niessen in 1937. In order to understand the situation fully, I went carefully through the mathematics of the above papers. As the results are of interest, I will mention them briefly.

First, no one bothers to mention that in the 1909 paper by Sommerfeld there is a mistake in the form of the boundary conditions in equation 5 of his paper. This propagates through to his final answers which can be corrected by multiplying them by a factor of 2. It is probably one of those errors which enter when a manuscript is revamped for publication. Otherwise Sommerfeld's 1909 general solution given in equation 47 is <u>entirely</u> <u>correct</u> and agrees with his 1926 solution. The discrepancy which has caused so much comment is due to an error in sign which appears when the general solution is specialized by replacing a parameter α by $\rho^{\frac{1}{2}}$. From the definition given by Sommerfeld in equation 41, it is clear that $\alpha = -\rho^{\frac{1}{2}}$ whereas Sommerfeld used $\alpha = \rho^{\frac{1}{2}}$. Wherever $\rho^{\frac{1}{2}}$ appears in Sommerfeld's paper, the correct forms can be obtained by replacing it by $-\rho^{\frac{1}{2}}$.

As will be shown here, the contour chosen by Sommerfeld was such that he had to add the contribution of a pole to his branch cut integral. Superficially, we can correct Sommerfeld's solution (in terms of ρ) by dropping the pole contribution, as this gives the same result as changing the sign of ρ. This coincidence has led some authors to claim the pole does not exist. However since we can show the pole must be included, the proper way to correct Sommerfeld's solution (in terms of ρ) is to make the above mentioned change of sign.

FORMAL SOLUTION

First we shall set up the dipole radiation problem in the form in which it was originally solved. Cylindrical coordinates will be employed with the z axis perpendicular to the air-earth interface which is at z=0. For simplicity, the dipole element is assumed to be at the origin. Only the case of a vertical electric dipole will be considered as the other cases can be worked out in a similar manner. The current in the dipole element then flows in the z direction and is assumed to vary sinusoidally with time. The propagation constants of the air and earth are, respectively $k_0 = \omega\left[\mu_0 \epsilon_0\right]^{\frac{1}{2}}$ and $k_1 = \omega\left[\mu_1 \epsilon_1 \left(1 + i\frac{\sigma_1}{\omega \epsilon_1}\right)\right]^{\frac{1}{2}}$ where $\mu_1, \epsilon_1, \sigma_1$ are the permeability, permittivity and conductivity of the earth (in MKS units) while ω is the angular frequency of the source.

From Maxwell's equations we obtain the Helmholtz equation which the components of the field vectors must satisfy. Following Sommerfeld we shall use the Hertz vector $\vec{\Pi}$ which in this problem has only a z component and is related to the other field vectors by $\vec{E} = \text{grad div}\vec{\Pi} + k^2\vec{\Pi}$ and $\vec{H} = (-ik^2/\omega\mu)\text{curl}\vec{\Pi}$. The usual method of solution is to solve the Helmholtz equation div grad $\Pi + k^2\Pi = 0$ by separation of variables in cylindrical coordinates and then to apply the boundary conditions. These boundary conditions, obtained from the continuity of E_r and H_φ at $z = 0$, are $\Pi_0 = n^2 \Pi_1$ and $\partial \Pi_0/\partial z = \partial \Pi_1/\partial z$ where $n^2 = k_1^2/k_0^2$. $\vec{\Pi}_0$ is the Hertz vector for the air and $\vec{\Pi}_1$ is the Hertz vector for the earth. Finally the radiation condition is applied to the solution. When these steps are carried out we obtain the solutions

$$(4\text{-}1) \qquad \Pi_0 = 2Cn^2 \int_0^\infty \frac{e^{ik_0 z(1-w^2)^{\frac{1}{2}}}}{N} J_0(rk_0 w)\, w\, dw \qquad z > 0$$

$$(4\text{-}2) \qquad \Pi_1 = 2C \int_0^\infty \frac{e^{-ik_0 z(n^2-w^2)^{\frac{1}{2}}}}{N} J_0(rk_0 w)\, w\, dw \qquad z < 0$$

where $C = -I\omega\mu_0 dz/4\pi k_0$, I = current in dipole element dz, $N = (n^2-w^2)^{\frac{1}{2}} + n^2(1-w^2)^{\frac{1}{2}}$. The factor C will be suppressed in what follows. Actually Sommerfeld used the variable of integration $\lambda = k_0 w$, but the above form of the solution is more convenient for this discussion.

RIEMANN SURFACE

The above solution is not yet completely specified since the integrands contain the multivalued functions $(1-w^2)^{\frac{1}{2}}$ and $(n^2-w^2)^{\frac{1}{2}}$. At every point of the w plane (except singular points) each function has two values differing only in sign. The four combinations of these signs correspond to the four sheets of the Riemann surface which renders the integrand single valued. On each sheet the integrand has a different value, but as we shall see, only one sheet has physical meaning.

When the double valued functions are considered separately, they are defined on Riemann surfaces of two sheets. These sheets are defined in Figure D-1 . The four combinations of sheet A or B of $(1-w^2)^{\frac{1}{2}}$ and sheet A or B of $(n^2-w^2)^{\frac{1}{2}}$ correspond to the four sheets of the Riemann surface of the integrand.

It is apparent that the integral for Π_0 converges only for sheet B of $(1-w^2)^{\frac{1}{2}}$, while that for Π_1 converges only for sheet B of $(n^2-w^2)^{\frac{1}{2}}$. Since the two integrals are connected by the boundary conditions, they must converge together. Consequently the integral for Π_0 must be evaluated on a path which lies on the sheet of the Riemann surface of the integrand corresponding to the choice of sheet B for both multivalued functions. The integrals are now single valued on this chosen sheet which is the only sheet having physical meaning.

LOCATION OF THE POLE

The integrands contain pole singularities at $w = \pm n(1 + n^2)^{-\frac{1}{2}}$. The important point is that these poles do not lie on all four sheets of the Riemann surface. In fact we can show that for the branch cuts in the position shown in Figure 4-1, there is no pole on the sheet on which the integration must be carried out. From Appendix B we see that $|n| > 1$ so that the pole can only lie within the unit circle. Also the branch points at $\pm n$ must lie outside the unit circle in the angular sectors $0 \leq \arg w \leq \pi/4$ and $\pi \leq \arg w \leq 3\pi/4$.

Next let $(1-w^2)^{\frac{1}{2}} = A_r + iA_i$, $(n^2-w^2)^{\frac{1}{2}} = B_r + iB_i$ and $n = x + iy$. Then we have for the denominator in 4-1, $N = xA_r-yA_i+B_r+i(xA_i+yA_r+B_i)$. Now a glance at the signature diagrams in Figure D-1 will show that A_r and B_r are always positive within the unit circle. In the first and third quadrants A_i is negative within the unit circle. Since x and y are positive numbers, it follows that the real part of the denominator

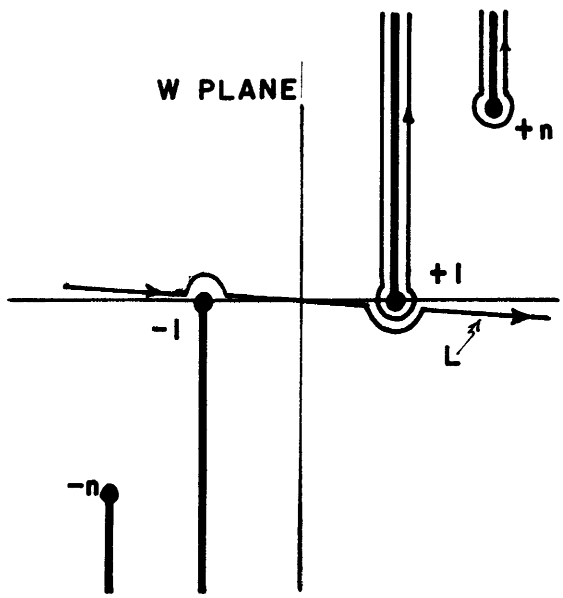


FIGURE 4-1

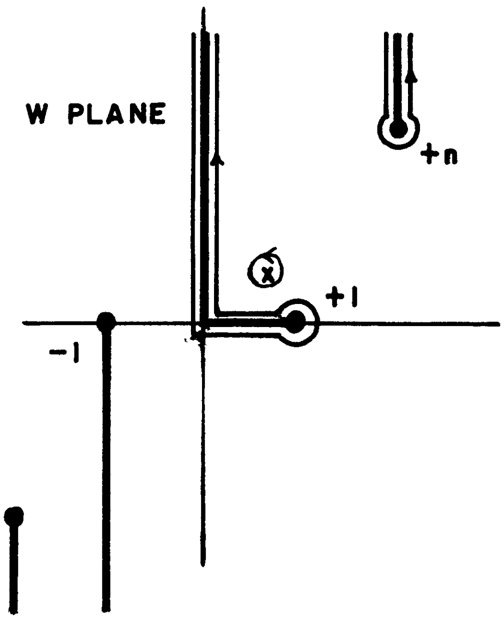


FIGURE 4-2

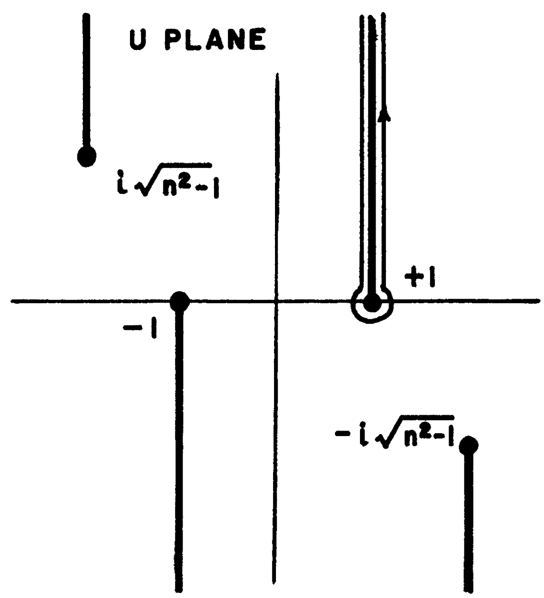


FIGURE 4-3

cannot vanish in the first and third quadrants within the unit circle. In the second and fourth quadrants A_i and B_i are positive so that the imaginary part of the denominator cannot vanish here, and the proof is completed.

We have now shown that if the branch cutting for the integrands of 4-1 and 4-2 is chosen as in Figure 4-1, then no poles appear on the sheet on which we must integrate. By the same type of analysis we can show that for the branch cutting given in Figure 4-2, a pole does appear on the sheet on which we integrate. This pole lies within the unit circle as indicated in Figure 4-2. Now as demonstrated by Sommerfeld, the real axis integrals of 4-1 and 4-2 can be replaced by the following integrals

(4-3)
$$\Pi_0 = Cn^2 \int_L \frac{e^{ik_0 z(1-w^2)^{\frac{1}{2}}} H_0^{(1)}(k_0 r w)}{N} w\,dw \qquad z > 0$$
$$\Pi_1 = C \int_L \frac{e^{-ik_0 z(n^2-w^2)^{\frac{1}{2}}} H_0^{(1)}(k_0 r w)}{N} w\,dw \qquad z < 0$$

where the contour L is indicated in Figure 4-1. The contour L can now be deformed onto the two branch cuts as shown in Figure 4-1. Alternatively, if we use the branch cutting of Figure 4-2, the contour L deforms onto the two branch cuts plus a loop around the pole. This latter choice is the one used by Sommerfeld (with h=z=0). Thus we see that he was correct in including a pole contribution. Sommerfeld could have taken the branch cuts in vertical position in which case he would have had no pole contribution, though of course the solution must be the same.

It is tempting to relate the pole residue to a surface wave since the residue has the proper form for such a wave. However a proper (physically significant) surface wave should appear in the far field solution. As several authors have shown (see Kay, for instance) this is not the case. This can be seen from the saddle point method solution. If we consider a layered media--in contrast to a simple half space--then surface waves, or guided waves, do appear in the far field solution and can be related to poles of the integrand (see Kay and Lo). The important point is that for the half space problem the contribution from the pole does not have special physical significance.

APPROXIMATE METHODS OF SOLUTION

Since we are interested in applications, it will be interesting to consider all of the integrals appearing in the Sommerfeld problem with respect to the various methods of approximate integration. We shall limit ourselves at first to the solutions for the field above the earth. The derivations of the integral solutions are given, for instance, by Sommerfeld and will not be repeated here. For compactness, let $k_0 r = \rho$ and $k_0(h+z) = a$. C was defined earlier in this chapter.

Vertical Dipole

The Hertz vector component is

$$\Pi_z = C\left\{\frac{e^{ik_0R_1}}{ik_0R_1} + \frac{e^{ik_0R_2}}{ik_0R_2} - 2A_v\right\} = C\left\{\frac{e^{ik_0R_1}}{ik_0R_1} - \frac{e^{ik_0R_2}}{ik_0R_2} + 2A_v'\right\}$$

where

$$A_v = \int_0^\infty e^{ia(1-w^2)^{\frac{1}{2}}} \frac{J_0(\rho w)(n^2-w^2)^{\frac{1}{2}} w\,dw}{N\,(1-w^2)^{\frac{1}{2}}} = \int_{L_w} e^{ia(1-w^2)^{\frac{1}{2}}} \frac{H_0^{(1)}(\rho w)(n^2-w^2)^{\frac{1}{2}} w\,dw}{2\,N\,(1-w^2)^{\frac{1}{2}}}$$

$$= \int_{L_v} e^{iav} \frac{H_0^{(1)}[\rho(1-v^2)^{\frac{1}{2}}]\,(n^2-1+v^2)^{\frac{1}{2}}\,dv}{n^2 v + (n^2-1+v^2)^{\frac{1}{2}}}$$

$$A_V' = n^2 \int_0^\infty e^{ia(1-w^2)^{\frac{1}{2}}} \frac{J_0(\rho w)}{N} w\,dw = n^2 \int_{L_w} e^{ia(1-w^2)^{\frac{1}{2}}} \frac{H_0^{(1)}(\rho w)}{2N} w\,dw$$

$$= \frac{n^2}{2} \int_{L_u} e^{iau} \frac{H_0^{(1)}[\rho(1-u^2)^{\frac{1}{2}}]\, u\,du}{n^2 u + (n^2-1+u^2)^{\frac{1}{2}}}$$

$$R_1^2 = r^2 + (z-h)^2 \;, \quad R_2^2 = r^2 + (z+h)^2 .$$

The contour L_w is given in Figure 4-1 while L_u is given in Figure 4-3. The integrals in the u plane were obtained by the transformation $u = (1-w^2)^{\frac{1}{2}}$ as explained in Appendix D. In the u plane the contour L_u surrounds only one branch cut while in the w plane L_w must surround two branch cuts. For some applications the representation in the u plane is more convenient because the algebraic factor of the integrand is simplified in that it contains only one double valued function. In the u plane there is no problem in locating the pole, as is described in Appendix B.

When a = 0, the above integrals can be solved in very compact form for the complete range of the parameters ρ and n by expanding the algebraic factors in ascending or descending powers of u and integrating term by term. When $\rho = 0$, the integrals are also not hard to handle. When $\rho \neq 0$ and $a \neq 0$, solutions can be obtained by the saddle point method of Chapter III. This is the asymptotic solution obtained by Sommerfeld and is valid only for a restricted range of the parameter values. Either the cliff method or quadrature saddle point method described in Chapter III for the integral A_V' enables us to extend the solutions to a much greater range of the parameter values.

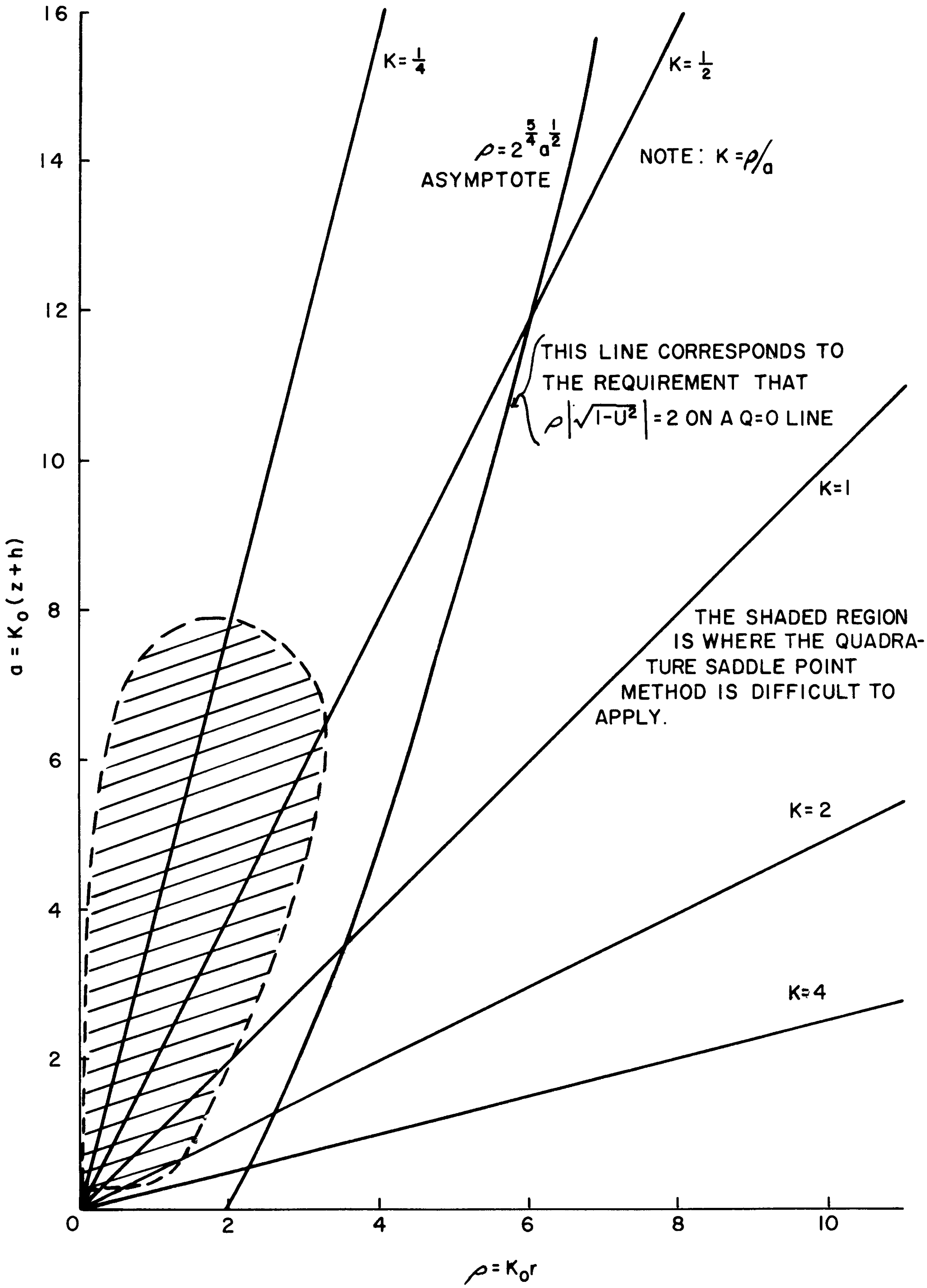
In what follows we shall only speak of the quadrature saddle point method, although the remarks apply to the cliff method as well. The difference between the quadrature saddle point method and the ordinary saddle point method of solution is illustrated in Figures 4-4 and 4-5. The region indicated in Figure 4-4 where both the saddle point method and the quadrature method are difficult to apply can be handled by expanding the integrand of the original real axis integral in polynomials or exponentials and integrating term by term with the Bessel function as a weight factor.

Horizontal Dipole

In this case the Hertz vector has both a vertical and a horizontal component. The solution will not be given in detail here as we are only interested in the type of integrals appearing. The integrals for the vertical component are similar to those for the vertical dipole if we replace $J_0(\rho w)$ by $J_1(\rho w)w$. No new problems arise,so that the solutions can be carried out in the same manner as those for the vertical dipole. The integrals for the horizontal component are simpler analytically in that the integrands do not possess poles in the finite plane. For $a = 0$ these integrals can be evaluated exactly. For $a \neq 0$ either the saddle point method or the quadrature method can be used. The latter works especially well because there is no pole near the saddle point. The difference in the range of parameter values these two methods can handle is the same as for the vertical dipole and is given in Figures 4-4 and 4-5.

The solution for the fields in the earth is more difficult than that for the field s in the air because the exponentially varying factors of the integrands are more complicated. The ordinary saddle point method can still be applied. However if we try to extend our solution by using

SOMMERFELD DIPOLE RADIATION PROBLEM



THIS DIAGRAM CORRESPONDS TO PHYSICAL SPACE WITH THE SURFACE OF THE EARTH AT Z=0. THE DIPOLE SOURCE IS AT Z=h. THE ORIGIN AT Z=-h IS THE POSITION OF AN IMAGE SOURCE FOR A PERFECT CONDUCTOR.

FIGURE 4-4

SOMMERFELD DIPOLE RADIATION PROBLEM

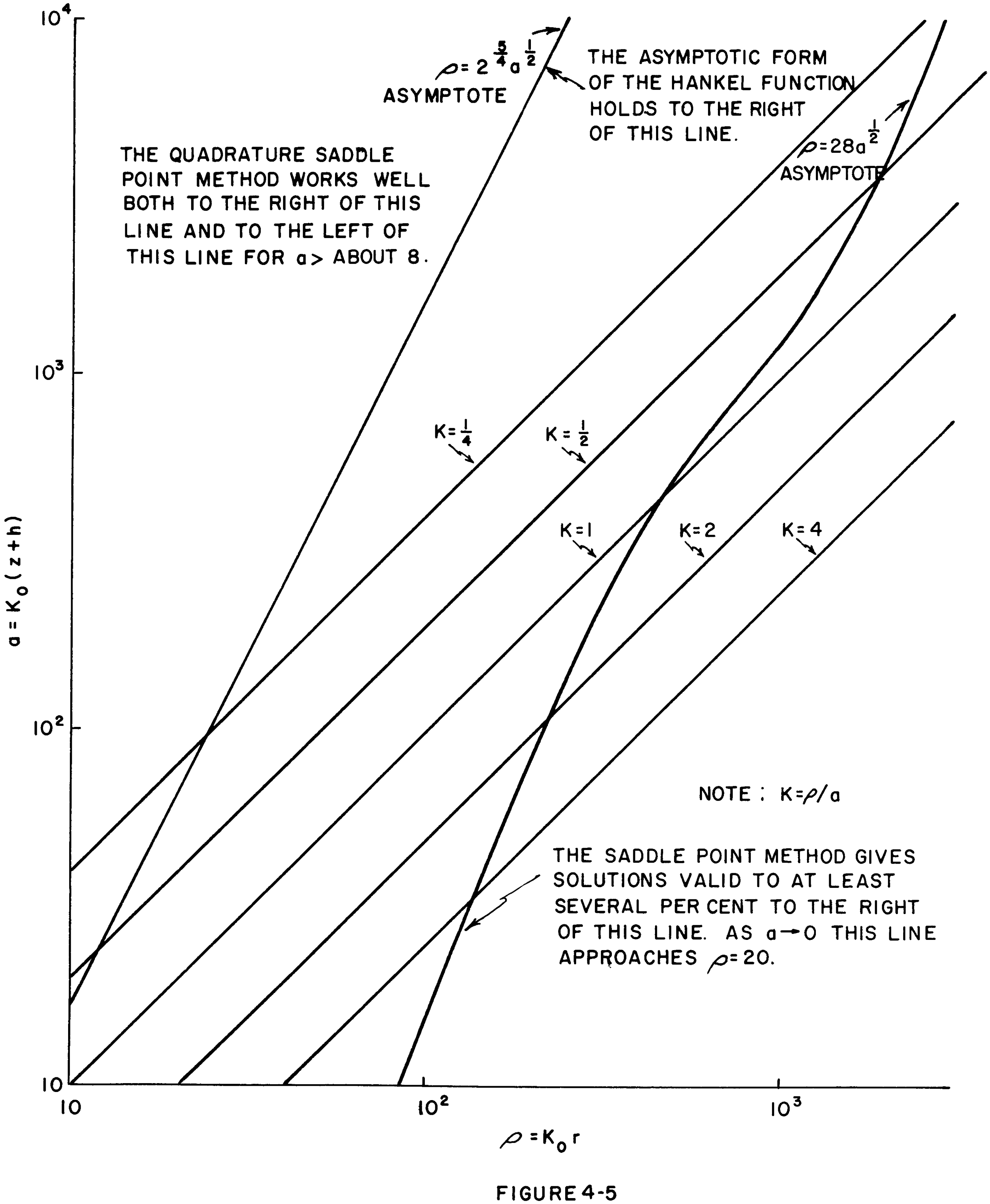


FIGURE 4-5

the quadrature saddle point method we run into trouble, because we have to solve a general quartic equation to find the analytic expressions for the steepest descent lines. Though this is not impossible, it is a laborious operation.

In other types of problems we may have to solve transcendental or very high order equations to obtain the steepest descent lines. This is a serious limitation on the quadrature saddle point method. In some cases we can solve for the asymptotes to the steepest descent lines, and this may be sufficient. In general we need to know the location of the steepest descent lines more accurately to apply the quadrature saddle point method. If the problem is being solved on a modern high speed computer, it is possible to overcome these limitations by programming the computer to find the steepest descent lines and then having it apply the quadrature saddle point method. These remarks apply also to the cliff method on steepest descent paths. In some cases the cliff method will be easier to apply than the quadrature method.

It will be enlightening to compare our (topological) methods with published solutions to the Sommerfeld problem which are intended to extend the range of validity beyond that of the saddle point method. None that I have seen employ quadrature or cliff methods along steepest descent paths. Instead the treatment is usually non-topological and consists in various series developments.

Two such treatments are those given by K. Norton of the Federal Communications Commission and C. Burrows of Bell Laboratories. Norton's solutions are valid only for $(a^2 + \rho^2)^{\frac{1}{2}} \gtrsim 20$ or for $a \ll \rho$ so that his results do not overlap the whole region covered by the quadrature or cliff methods. Furthermore his solution requires the evaluation of an untabulated function. As he admits, the method used by him does not permit a reliable error estimate. The quadrature methods, on the other hand, can yield an error estimate. Burrows solution, for $(h + z) \ll 1$, requires three different series developments for as many ranges of the distance r along the surface. A glance at his expression for the field intensity of a vertical dipole (page 63 of his paper) shows how involved the method becomes. In the quadrature or cliff method approach we require only one primary form for the solution--in fact we can write down the general form of the solution directly from the integrand.

In contrast to the methods of these papers, the quadrature saddle point method or cliff method on steepest descent paths is a general technique and applies to a large class of integrals.

CONCLUSIONS

The class of integrals which can be conveniently handled by approximate integration methods has been substantially increased by the cliff methods developed in this thesis. It has been shown that both the cliff and and quadrature methods of integration can be used to remove the limitations of ordinary saddle point methods. This has been accomplished by exploiting the inherent power of topological methods.

The cliff method and extended cliff method offer a convenient means of evaluating many branch cut integrals which do not readily yield to non-topological methods such as Gaussian quadrature. As shown by the examples of Appendix E, the accuracy of the cliff methods compares favorably with that of standard methods, although a tight error analysis is still lacking.

Both the cliff method and quadrature methods can be applied to integrands with broad saddles and curved steepest descent paths--cases for which the ordinary saddle point methods break down. The solutions are convergent and reasonably compact.

As shown in Appendix G, the cliff methods can be successfully applied to singular integral equations when these are impractical to solve by standard techniques.

Finally it has been shown that the general cliff method needs further development before it can be a useful integration tool.

SUGGESTIONS FOR FUTURE WORK

The practicality of approximate integration techniques depends on the labor involved and the errors committed. The methods presented in this thesis need to be studied more extensively along these lines--both from the theoretical standpoint and from working actual problems. Specifically, the following points should be investigated:

Cliff and Extended Cliff Methods. A tight error analysis is needed in order to compare these methods with quadrature methods. Also the manner in which the distribution function is approximated needs to be carefully studied in order to minimize the errors. The application of the cliff methods to singular integral equations needs further development. Here again a comparison with techniques such as Gaussian quadrature is needed to determine which techniques, for a given tolerance, require less labor for a given problem.

Saddle Point Methods. The improvements possible in the first and higher order saddle point methods by using expansions for $W(s,t)$ other than the Taylor series need more consideration. The extension of the quadrature saddle point method and cliff methods to problems where the steepest descent paths cannot be found explicitly needs development. In particular, a program for a high speed computer could be developed to determine the steepest descent paths. This would not only extend the usefulness of these methods but has important applications to higher order saddle point methods. Specific problems that could be handled are seismic and electromagnetic field solutions for dipole or multipole sources above or in multilayered media and wave guide problems.

General Cliff Method. Means of overcoming the limitations due to the summations required in the exponents should be investigated. Some work could be done on eliminating the sensitivity of the approximations in the exponents to the integration around the essential singularities. For instance an expansion which appropriately considers the oscillations around the essential singularities would be needed. Unpublished work by Dr. Cerrillo indicates this could be done. The possibility of handling the artificial branch cut integrals described in the section on General Cliff Method of Chapter I needs further pursuit along the lines suggested in that section.

Miscellaneous. To say that an approximation is uniformly convergent does not necessarily imply that its derivative will be even a reasonable approximation to the derivative of the function being approximated. This problem is well known for polynomial approximations. As the cliff methods are based on topological considerations, the possibility that they give better approximations than non-topological methods from the standpoint of differentiation should be worth investigating. This is important when we have an approximation to a potential and wish to obtain field solutions. From a broader viewpoint, a study of the underlying relationships between approximate methods of solving a problem in integral form and approximate methods for dealing with the problem in differential or other form should prove fruitful.

Appendix A

HANKEL FUNCTIONS

This is a brief summary of some of the important properties of the Hankel functions used in this thesis. We are primarily concerned with functions of integral order since the functions of half integral order can in terms of elementary functions.

The Bessel functions (of the first kind) can be expressed in terms of Hankel functions:

$$J_n(z) = \frac{H_n^{(1)}(z) + H_n^{(2)}(z)}{2} \qquad (n = 0,1 \ldots) \tag{A-1}$$

This can be reduced to Hankel functions of the first kind by the relation

$$H_n^{(2)}(z) = (-1)^{n+1} H_n^{(1)}(e^{i\pi} z) \tag{A-2}$$

When we apply the asymptotic formulas to $H_n^{(1)}(1-u^2)^{\frac{1}{2}}$, it will be necessary to relate the phase of u to that of $z = (1-u^2)^{\frac{1}{2}}$. In Appendix D we define the Riemann sheets A and B for $(1-u^2)^{\frac{1}{2}}$. We shall now relate these sheets and the branch cut positions to the phase of z. A constant factor in front of $(1-u^2)^{\frac{1}{2}}$ in the Hankel function argument will not essentially alter the analysis. To agree with Watson (1952) we shall take the principal value of z to be $-\pi < \arg z < +\pi$.

First consider how the Riemann surface of $(1-u^2)^{\frac{1}{2}}$ maps onto the z plane. Sheet B is shown in Figure A-1 for straight line branch cuts.

Figure A-2 shows how sheet B maps onto the shaded region of the z plane which is doubly covered. The unshaded portion of the z plane is the region into which sheet A maps. For our purposes we can consider the branch cut angle to be $\pi/2 \geq \varphi \geq 0$. Then points in the shaded region always lie within $\pi > \arg z > -\pi/2$.

From these results we have the convention,

(A-3) $\qquad$ (points on sheet A) = $e^{-i\pi}$(points on sheet B)

Also $(1-u^2)^{\frac{1}{2}}$(on a_1 bank of branch cut) = $e^{-i\pi}(1-u^2)^{\frac{1}{2}}$(on a_3 bank of cut) so that

$$\text{(A-4)} \qquad H_n^{(1)}(1-u^2)^{\frac{1}{2}} \text{ (on } a_1 \text{ bank)} = (-1)^{n+1} H_n^{(1)}(1-u^2)^{\frac{1}{2}} \text{ (on } a_3 \text{ bank)}$$

where $(1-u^2)^{\frac{1}{2}}$ is taken on sheet B.

For large value of z (and hence u) we have Schläfli's asymptotic formula with an upper bound on the error. The general form given by Watson becomes for order zero:

$$\text{(A-5)} \qquad H_0^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\pi/4 + iz} \left\{ \sum_{m=0}^{p-1} \frac{\Gamma^2(m+\frac{1}{2})}{\pi m! (2iz)^m} + \frac{\theta_1 \Gamma^2(p+\frac{1}{2})}{\pi p! (2iz)^p} \right\}$$

The equality sign holds if $3\pi/2 > \arg z > -\pi/2$ so that this result is valid on sheet B for $z = (1-u^2)^{\frac{1}{2}}$. Also $|\theta_1| < 1$ or $|\sec \arg z|$ according as Im z is positive or negative. For p = 1 we have for the expression in the brackets $\left\{ 1 + \theta_1/8iz \right\}$, so that if we neglect the second term, the error for $|z| > 1$, is less than about 17%. From this we see that the usual asymptotic expression obtained by setting p = 0 in A-5 is valid in a rough way for $|z| > 1$.

Appendix B

LOCATION OF BRANCH POINTS AND POLE

The complex index of refraction n is defined by

$$n^2 = \frac{k_1^2}{k_0^2} = \frac{\epsilon_1}{\epsilon_0}\left(1 + i\,\frac{\sigma_1}{\epsilon_1\omega}\right)$$

where k_0 and k_1 are the propagation constants in the air and earth, ϵ_0 and ϵ_1 are the permittivity in the air and in the earth, σ_1 is the conductivity of the earth and ω is the angular frequency, all in MKS units.

From the possible values of these physical parameters we can locate the positions of the branch points of $(n^2-w^2)^{\frac{1}{2}}$ and $(n^2-1+u^2)^{\frac{1}{2}}$ which are, respectively, $w = \pm\, n$ and $u = \pm\, i(n^2-1)^{\frac{1}{2}}$. Also we can find the permissible region for the pole of one of the integrals, which is at $-1/(n^2 + 1)^{\frac{1}{2}}$. After some algebra and using the fact that $\epsilon_1 \geq \epsilon_0$ and $\sigma_1/\epsilon_1\omega \geq 0$ we obtain the results in Figures B-1, B-2, and B-3.

Appendix C

BASIC THEOREMS

A basic theorem used by Aronszajn in his work on the decomposition of analytic functions will be repeated here, as it justifies some of the developments in this thesis. Essentially this theorem is a restatement of a theorem due to Runge.

RUNGE'S THEOREM "Let B be the union of a finite number of open connected domains which together contain the entire singular set of an analytic function f(z). If R(z) is a rational function, then the difference f(z) - R(z) can be made arbitrarily small in the complement of the closure of B. The poles of R(z) may be chosen completely arbitrarily within B as long as there is at least one in each of the domains composing B."

Another form of the theorem is:

"If G is the region on which an analytic function f(z) is defined and if F is the set disjoint with G, then there is a rational function R(z) which in G is arbitrarily less than f(z) and on F is arbitrarily less than zero. Moreover if $R_n(z)$ is a series of rational functions they converge uniformly to the limit." It is assumed that no essentially singular points are on the boundary of G in this latter form of the theorem.

Several theorems which form a rigorous basis for some of the developments on cliff methods of integration are presented here. They are given without proof since they are either proven by Behnke and Sommer or can be derived from theorems by the same authors. The theorems are not the most general but will serve for many purposes. The symbol $\{f_n(z)\}$ will mean "the sequence of functions $f_n(z)$". The path of integration L is assumed to be simple, closed and rectifiable. Uniform convergence implies that the limit is bounded.

THEOREM C-1 If $\{f_n(z)\}$ are continuous and uniformly convergent for $z \in L$ and if L is finite, then

$$\int_L \lim f_n(z)\, dz = \lim \int_L f_n(z)\, dz$$

THEOREM C-2 If L is the path from a to ∞ , and if on each bounded segment of L, $\{f_n(z)\}$ are continuous and uniformly convergent and if further

$$\{ p_n(z)\} \equiv \left\{ \int_a^z f_n(s)\, ds \right\}$$

is uniformly convergent on the closed path L, then

$$\int_L \lim f_n(z)\, dz = \lim \int_L f_n(z)\, dz$$

THEOREM C-3 The conditions of Theorem C-2 are satisfied if $f_n(z) = F(z)h_n(z)$ $(n = 1,2,\ldots)$ where F(z) on each bounded closed segment of L is continuous,

$$\int_L |F(z)|\, dz$$

exists and $\{h_n(z)\}$ are continuous and uniformly convergent for $z \in L$.

THEOREM C-4 Let $\{f_n(z)\}$ satisfy the first conditions of Theorem C-2 and let $\int_L f_n(z)dz$ exist. Then and only then $\{p_n(z)\}$ are uniformly convergent on the closed path L if for every ϵ an n_o and a z_o exist such that for all $n \geq n_o$ and all $z \in [z_0, \infty]$

$$\left| \int_{z_0}^{z} f_n(z)\, dz \right| < \epsilon$$

<u>THEOREM C-5</u> If $\{f_n(z)\}$ is continuous and uniformly convergent and $\{g[f_n(z)]\}$ and g(limf(z)) are continuous, then $\{g[f_n(z)]\}$ is uniformly convergent and $g(\lim f_n(z)) = \lim g(f_n(z))$, for finite z.

This last theorem enables us to apply the preceding theorems to the type of approximations used in the general cliff method.

Appendix D

RIEMANN SURFACES

The type of integral considered in this thesis contains multivalued functions. In order to employ certain analytic tools, these functions are rendered single valued by defining them on a Riemann surface. This surface in turn is cut up into separate sheets by appropriate branch cuts. It is important to know the analytic structure of the functions on these sheets for the following reasons: only certain sheets correspond to the physical problem at hand, integrals do not converge on every sheet, contour deformation requires the knowledge of which sheets contain poles and the behaviour of the function as a branch cut is crossed. It is assumed the reader is familiar with the concept and terminology of Riemann surfaces as described, for instance, by Courant (1925), Nehari (1952) or Ahlfors (1953).

DOUBLE VALUED FUNCTIONS

The Riemann surface of a double valued function will have two sheets which correspond to the two branches of the function. To be specific, consider the function $(1-w^2)^{\frac{1}{2}}$ which has branch points at $w = \pm 1$. The two sheets and consequently the two branches of this function can be uniquely characterized by specifying the branch cutting and the sign of the $\mathrm{Re}(1-w^2)^{\frac{1}{2}}$ at the origin. We shall define sheet A to be the sheet for which $\mathrm{Re}(1-w^2)^{\frac{1}{2}}$ is negative at the origin. Then sheet B will correspond to the branch having $\mathrm{Re}(1-w^2)^{\frac{1}{2}}$ positive at the origin. It follows from a property of analytic functions, that defining sheets A and B in the above manner automatically determines the complete sign distribution of the real and imaginary parts of $(1-w^2)^{\frac{1}{2}}$ on these sheets. These sign distributions or signatures are given in Figure D-1 for sheet B. The zero lines of the real and imaginary parts are indicated by circles while the solid lines are branch cuts. It is only necessary to reverse all the signs in these diagrams to obtain the signature for sheet A.

In order to see how these signature diagrams are developed, consider $P + iQ = (1-w^2)^{\frac{1}{2}}$ where P and Q are the real and imaginary parts of this function. If we let $w = he^{i\varphi}$ then we have $P + iQ = \left(1-h^2\cos 2\varphi - i h^2 \sin 2\varphi\right)^{\frac{1}{2}}$. In order to find the zeros of P and Q set $\sin 2\varphi = 0$. Then $P + iQ = (1-r^2)^{\frac{1}{2}}$ for $\varphi = 0$ or π and $P + iQ = (1+r^2)^{\frac{1}{2}}$ for $\varphi = \pi/2$ or $3\pi/2$. The zeros follow directly.

Having determined the zero lines (which always terminate in branch points in our case) and chosen the position of the branch cuts, we now can determine the signature by studying the signs of the real and imaginary parts of the function in the vicinity of a branch point while making a complete circuit around the branch point. The results are shown in in Figure D-1.

In the case of $(n^2-w^2)^{\frac{1}{2}}$ let $n = pe^{i\theta}$ and $w = he^{i\varphi}$. Then in $P + iQ = \left(\rho^2\cos 2\theta + i\rho^2\sin 2\theta - r^2\cos 2\varphi - ir^2\sin 2\varphi\right)^{\frac{1}{2}}$ set $\rho^2\sin 2\theta - r^2\sin 2\varphi = 0$ to obtain the zeros of P and Q. The solutions of this last equation are hyperbolas with the real and imaginary axes as asymptotes. Then on these hyperbolas

$P + iQ = (\rho^2\cos 2\theta - \rho^2\sin 2\theta\cot 2\varphi)^{\frac{1}{2}}$ so that P = 0 when $\sin 2(\varphi - \theta)/\sin 2\varphi < 0$ and Q = 0 when $\sin 2(\varphi - \theta)/\sin 2\varphi > 0$. If we choose $\mathrm{Re}(n^2-w^2)^{\frac{1}{2}}$ to be negative at the origin on sheet A of this function, the complete signature follows as shown in Figure D-2. The signatures for $(n^2-1+w^2)^{\frac{1}{2}}$ and $(1-w)^{\frac{1}{2}}$ given in Figures D-3 and D-4 were developed in the same manner. For the applications in this thesis the parameter n of the signature diagrams has the meaning of a complex index of refraction. Because n can take on only certain physically realizable values, the branch points in these diagrams are restricted to certain regions which are determined in Appendix B. For convenience the positions of the branch points given in the signature diagrams are compatible with these requirements.

INTEGRAL TRANSFORMATIONS

Some of the integrals encountered in wave propagation problems have the form

$$\text{(D-1)} \quad I = \int_0^\infty e^{ia(1-w^2)^{\frac{1}{2}}} g\left(w, (1-w^2)^{\frac{1}{2}}, (n^2-w^2)^{\frac{1}{2}}\right) J_n(\rho w)\, w\, dw$$

where g is itself a single valued analytic function of its arguments and is not of exponential order for large w. Since the integrand contains two double valued functions, its Riemann surface will have four sheets. In general the integral I will have four different values depending on which sheet is used for the integration. In practice the physical conditions of the wave propagation problem will dictate which of the four sheets over the w plane is the appropriate one.

For certain applications, the analytic form of D-2 will be simplified by applying the transformation $u = (1-w^2)^{\frac{1}{2}}$. Then we have

$$\text{(D-3)} \quad I = \int_L e^{iau} g\left[(1-u^2)^{\frac{1}{2}}, u, (n^2-1+u^2)^{\frac{1}{2}}\right] J_n\left(\rho(1-u^2)^{\frac{1}{2}}\right)(-u)\, du$$

The integrand now has a four sheeted Riemann surface over the u plane. The original contour on one of the sheets over the w plane is mapped by this conformal transformation onto each of the four sheets over the u plane. However the same physical considerations which singled out one of sheets over the w plane will also single out the appropriate sheet over the u plane.

If the original contour over the w plane is as shown in Figure D-5 then the conformal image of this contour will have the positions shown in Figure D-6. If we choose a to be a positive number in the integral of D-2, then only the upper contour designated by L will lead to convergent integrals.

One further transformation will prove useful. We shall assume that the integrand of D-2 has the same value on both sheets of $(1-u^2)^{\frac{1}{2}}$ and replace the Bessel function by the relation given in equation A-1 of Appendix A. The contour L can be deformed into the position shown by the dotted line in Figure D-6. The integral I can now be written as the sum of two integrals along the left bank the cut from the branch point +1 :

$$\text{(D-3)} \quad I = -\int_{+1} e^{iau} g\, \frac{H_n^{(1)}[\rho(1-u^2)^{\frac{1}{2}}]}{2}\, u\, du - \int_{+1} e^{iau} g\, \frac{H_n^{(2)}[\rho(1-u^2)^{\frac{1}{2}}]}{2}\, u\, du$$

From Appendix A we see that the second integral in D-3 can be replaced by an integral, on the right bank of the cut, with a Hankel function of the first kind in its integrand. Since the integrals on the left and right banks of this cut then have the same integrands, it is easy to show that the integral of D-3 and consequently that of D-1 is equivalent to the following intergral taken on the lancet contour indicated:

$$\text{(D-4)} \quad I = \int_{+1} e^{iav} g\left[(1-v^2)^{\frac{1}{2}}, v, (n^2-1+v^2)^{\frac{1}{2}}\right] \frac{H_n^{(1)}\left[\rho(1-v^2)^{\frac{1}{2}}\right]}{2} v\,dv$$

If we had performed the contour deformations in the w plane, we would have the less compact form:

$$I = \int_{+1} e^{ia(1-w^2)^{\frac{1}{2}}} g\left[w, (1-w^2)^{\frac{1}{2}}, (n^2-w^2)^{\frac{1}{2}}\right] \frac{H_n^{(1)}(\rho w)}{2} w\,dw$$

$$\text{(D-5)} \quad + \int_{+n} e^{ia(1-w^2)^{\frac{1}{2}}} g\left[w, (1-w^2)^{\frac{1}{2}}, (n^2-w^2)^{\frac{1}{2}}\right] \frac{H_n^{(1)}(\rho w)}{2} w\,dw$$

which is valid only if the integrand of D-1 is an odd function of w.

Appendix E

EXAMPLES

CLIFF METHOD

We shall evaluate the integral representation of the Bessel function $J_0(z)$,

$$'I = \frac{2}{\pi} \int_1^{\infty} e^{-z(t-t^{-1})} \frac{\cos z + t \sin z}{t(1-t^2)^{\frac{1}{2}}} dt$$

by the cliff method. As mentioned in Chapter I it is simplest to throw this into a Stieltjes form rather than expand the double valued function in a rational function with poles inside a lancet contour. We then have

$$\text{(E-1)} \qquad 'I = \frac{2}{\pi} \int_1^{\infty} e^{-z(t-t^{-1})} (\cos z + t \sin z)\, d\varphi(t)$$

where $\varphi(t) = \pi/2 - \sin^{-1} 1/t$ is given in Figure E-1. The integration only needs to be carried out to t_N due to the damping effect of the exponential. We determine t_N by $z(t_N - 1/t_N) = 4$ or $t_N = 2/z + (1 + (\frac{2}{z})^2)^{\frac{1}{2}}$.

The next step is to approximate $\varphi(t)$ by a staircase function $\varphi_n(t)$ as shown in Figure E-1 by the solid line approximation. For convenience we shall take the jumps J_q to be of equal size. (We could take the abscissas t_q to be equally spaced instead). If we use this form of $\varphi_n(t)$ the error will be quite large. We obtain a great improvement if we shift the graph of $\varphi_n(t)$ down one half jump to the position indicated by the dotted line in Figure E-1. Then the errors tend to cancel.

All the jumps are the same size except the two end jumps J_o and J_N which are half size.

We have for the above spacing, $J_q = \varphi(t_N)/N \quad (q \neq 0,N)$ and $J_o = J_N = \varphi(t_N)/2N$. From the figure we see that the equation for the abscissas t_q should be $\varphi(t_q) = q\varphi(t_N)/N$ so that $t_q = 1/\cos[q\frac{\varphi(t_N)}{N}]$.

One word of caution here is that if a Stieltjes integral has the range of integration $[a,b]$, then the staircase distribution function $\varphi_n(t)$ cannot have jumps at <u>both</u> t = a and b . This follows directly from the properties of the integral. In the present case, however, we can consider the range of integration to go beyond $t = t_N$ so that we can have jumps at both t_o and t_N . A similar artifice will hold for other integrals. The particular problem will determine what is the best procedure. This actually is a very important point if we wish to keep errors small.

On substituting $\varphi_n(t)$ into E-1 we obtain

$$^1I \simeq \frac{2}{\pi}\sum_{q=0}^{N} e^{-z(t_q - t_q^{-1})}(\cos z + t_q \sin z)J_q$$

where t_q and J_q are defined above. In Table E-1 we have plotted the cliff method approximation to ^1I obtained with N = 4. The accuracy is better than 1% beyond $z \simeq \pi/4$.

To show how conservative the error analysis of Chapter I is, we shall apply equation 1-7 and 1-8 to this problem. Here lub $|\varphi(t) - \varphi_n(t)| = \varphi(t_N)/2N$. We can take $|g(b)-g(a)| \simeq |\cos z + \sin z| \leq 2^{\frac{1}{2}}$. Since $\varphi(t)$ and $\varphi_n(t)$ have the same values at the ends of our range of integration, we have for the error E

$$|E| < \frac{\varphi(t_N)}{2^{\frac{1}{2}} N}$$

For large z, $|\varphi(t_N)| = \left|\left[\frac{\pi}{2} - \sin^{-1}(t_N^{-1})\right]\right| \simeq \frac{1}{z^{\frac{1}{2}}}$ so

$$|E| < \frac{1}{2^{\frac{1}{2}} N} \cdot \frac{1}{z^{\frac{1}{2}}}$$

Since the Bessel function behaves as $(2/\pi z)^{\frac{1}{2}}$ for large z, the relative error is $\sim 1/N$ which is much too conservative.

It is desireable to compare the cliff solution with the standard methods of handling ${}^1I$. Since ${}^1I$ has a singular integrand, Simpson's Rule will not converge very fast. Also ${}^1I$ is not in convenient form to apply a quadrature rule. However if we let $\tau^2 = t^2 - 1$ in ${}^1I$ we obtain

$${}^2I = \frac{2}{\pi} \int_0^{\infty} e^{-z\tau^2/(1+\tau^2)^{\frac{1}{2}}} \left[\frac{\cos z}{1+\tau^2} + \frac{\sin z}{(1+\tau^2)^{\frac{1}{2}}}\right] d\tau$$

which is well suited to Simpson's rule. With a five term Simpson's rule over the range $[0, \tau_N]$ where $\tau_N^2 = t_N^2 - 1$, we obtain the values listed in Table E-1. Evidently the cliff method gives about the same error as Simpson's rule (though of course we used a different integral to give Simpson's rule a good chance). The important point is that there are many cases--integral equations, for instance--when the cliff method will be the easiest technique to apply.

We shall next take an example in which there is no exponential damping term in the integrand. Again, for the Bessel function $J_0(z)$ we have the representation

$$(\text{E-2}) \qquad {}^3I = \frac{1}{\pi}\int_{-1}^{+1} \frac{\cos zt}{(1-t^2)^{\frac{1}{2}}}\,dt = \frac{2}{\pi}\int_0^1 \cos zt \; d(\sin^{-1} t)$$

By the cliff method we have

$${}^3I \simeq \frac{2}{\pi}\sum_{q=0}^{N} \cos(zt_q)\, J_q$$

where the jumps are taken equally spaced as shown in Figure E-2. Here $J_q = \pi/2N$ $(q \neq 0, N)$ and $J_o = J_N = \pi/4N$. We have $\sin^{-1} t_q = q\pi/2N$ or $t_q = \sin q\pi/2N$. The approximation with N = 2 and 4 is shown in Table E-1. The accuracy increases very rapidly as we increase N.

These results can be compared with a Chebyshev-Gauss quadrature for the first integral of E-2. For a four point formula we have

$${}^3I \simeq \frac{1}{2}\sum_{k=1}^{2} \cos\left[z \cos\left(\frac{2k-1}{8}\pi\right)\right]$$

where we only have two terms because cos zt is an even function. This formula is equivalent to fitting a seventh degree polynomial to cos zt over $[-1,+1]$. For a ten point formula, which is equivalent to fitting a nineteenth degree polynomial over $[-1,+1]$ we have

$${}^3I \simeq \frac{1}{5}\sum_{k=1}^{5} \cos\left[z \cos\left(\frac{2k-1}{20}\pi\right)\right]$$

The error E in this last formula is bounded by

$$|E| < \frac{1}{(10\pi)^{\frac{1}{2}}}\left(\frac{ze}{40}\right)^{20}$$

which is conservative. In Table E-1 we show the results for the four point formula. The five term (ten point) formula breaks down between 4π and 4.5π as the error analysis indicates. Only two check points are given. It is apparent that the cliff method is not as good as a quadrature rule of the same number of terms but still is quite accurate for many purposes. When it is difficult to apply a quadrature then the cliff method comes into its own.

EXTENDED CLIFF METHOD

In Figure E-3 the extended cliff method with two straight lines approximating the distribution function $2\sin^{-1}t$ is compared with a four point Chebyshev-Gauss quadrature for the integral of E-2. The reason the cliff method stays with the Bessel function longer becomes clear if we look at the basic difference between the methods.

Suppose we call $(1-t^2)^{\frac{1}{2}}$ the "weight" and cos zt the "kernel". Then the quadrature takes care of the weight exactly but approximates the "kernel" by a polynomial. The cliff method handles the "kernel" exactly but approximates the weight (actually the integral of the weight is approximated). Since polynomials do not follow an oscillatory function very long, the quadrature rule does not do as well as the cliff method when z is increased.

GENERAL CLIFF METHOD

Consider the integral representation of the Bessel function $J_1(z)$

$$\text{(E-3)} \qquad \frac{1}{2\pi i}\int_C e^{iz(1-u^2)^{\frac{1}{2}}}\,du$$

where C is a clockwise contour encircling the real axis cut from -1 to +1. We take the branch of $(1-u^2)^{\frac{1}{2}}$ for which the real part is positive in the upper half plane. We already have the rational approximant to $(1-u^2)^{-\frac{1}{2}}$

given in equation 1-3. This is

$$(1-u^2)^{-\frac{1}{2}} \simeq \frac{u}{\pi i} \sum_{q=1}^{N} \frac{J_q}{t_q^2 - u^2}$$

where the distribution function $\varphi(t) = 2\sin^{-1} t$. Thus the jumps are

$$J_q = 2\sin^{-1}(t_q) - 2\sin^{-1}(t_{q-1})$$

We shall redefine $\varphi(t)$ to be $\sin^{-1} t$ and employ the definition of jumps given in Figure E-2. Then

$$(1-u^2)^{\frac{1}{2}} = \frac{2(1-u^2)u}{\pi i} \sum_{q=0}^{N} \frac{J_q}{t_q^2 - u^2} \tag{E-4}$$

where $J_q = \pi/2N$ $(q \neq 0,N)$ and $J_o = J_N = \pi/4N$. Also $t_q = \sin(q\pi/2N)$. To apply the general cliff method we substitute E-4 into E-3 and collapse C_1 onto the essential singularities at $u = t_q$. We then have

$$J_1(z) \simeq \frac{1}{2\pi i} \sum_{j=0}^{N} \int_{\odot_{t_j} + \odot_{-t_j}} e^{\frac{2z}{\pi}(1-t^2)t\left[\sum_{\substack{q=0 \\ q\neq j}}^{N} \frac{J_q}{t_q^2 - t^2} + \frac{J_j}{t_j^2 - t^2}\right]} dt$$

Next we can use the method of Appendix F to evaluate these integrals around the essential singularities by setting $t = t_j + v$. If we drop all terms of o(v) and greater we obtain by the analysis in Appendix F,

$$J_1(z) \simeq \frac{2z}{\pi} \sum_{j=0}^{N} \cosh\left[\frac{2z}{\pi}(1-t^2)t \sum_{\substack{q=0 \\ q\neq j}}^{N} \frac{J_q}{t_q^2 - t_j^2}\right](1-t_j^2) J_j$$

where $t_j = \sin j\pi/2N$, $J_j = \pi/2N$ $(j \neq 0,N)$ and $J_o = J_N = \pi/4N$. For $N = 2$ we obtain $J_1(z) \simeq z/2$ which is the correct first term in the Bessel function series. For $N = 4$, we obtain

$$J_1(z) \simeq \frac{z}{2}\left(\frac{1}{3}+\frac{1}{2}\cosh\left(\frac{z}{12}\right)+\frac{1}{6}\cosh\left(\frac{3^{\frac{1}{2}}}{36}z\right)\right)$$

which is no improvement although it is good for small z since $\cosh x = 1 + x^2/2! + \ldots$

If we use the method of Appendix F with only the term in $1/v$ -- we drop the constant term together with terms of $o(v)$ and greater--we obtain

$$J_1(z) \simeq \frac{2}{\pi} z \sum_{q=0}^{N} (1-\tau_q^2) J_q = \frac{z}{2}(.998) \ .$$

It is evident from this example that we need a careful analysis of the effect of terms in the exponent on the integration around the essential singularities at t_j. Though this example does not give an optimistic picture of the practicality of the general cliff method, more research is needed before definite conclusions are reached.

CLIFF METHOD AND INTEGRAL EQUATIONS

The cliff method can easily be adapted to handle singular integral equations, particularly those which are "weakly" singular. In this respect the cliff method is more versatile than Gaussian quadrature methods since the form of the kernel is not limited to the standard weighting functions of known orthogonal polynomials. We shall illustrate the procedure for Abel's integral equation. This equation is

$$f(y) = \int_0^y \frac{g(x)}{(y-x)^{\alpha}}\,dx$$

where $0 < \alpha < 1$ and $f(y)$ is a known function. The solution is which will enable us to check our approximate solution. We can rewrite Abel's equation to be

$$f(y) = y^{1-\alpha} \int_0^1 \frac{g(yv)}{(1-v)^\alpha} dv .$$

To apply the cliff method, we put this integral in Stieltjes form so that we have

$$f(y) = \frac{y^{1-\alpha}}{1-\alpha} \int_0^1 g(yv) \; d\left[1 - (1-v)^{1-\alpha}\right]$$

If we take our jumps in the same manner as in Figure E-2, we have the cliff solution

$$f(y) \simeq \frac{y^{1-\alpha}}{1-\alpha} \sum_{q=0}^{N} g(yv_q) J_q$$

where $J_q = 1/N \quad (q \neq 0,N)$, $J_0 = 1/2N$, $J_N = 1 - 1/2N$ and the abscissas are found from $1-(1-v_q)^{1-\alpha} = q/N$ ($q = 0,1,\ldots N$). We obtain $v_q = 1-\left(1-\frac{q}{N}\right)^{\frac{1}{1-\alpha}}$. Taking $\alpha = \frac{1}{2}$ and $f(y) = y$ we obtain finally

$$\frac{y^{\frac{1}{2}}}{2} = \sum_{q=0}^{N} g^*(yv_q) J_q$$

where we replaced $g(yv_q)$ by its approximation $g^*(yv_q)$. We shall take $N = 2$ and 4 to see how accurate the method may be. With $N = 2$ we find

$$y^{\frac{1}{2}} = \frac{1}{2}\left[g^*(0) + g^*(y)\right] + g^*\left(\tfrac{3}{4}y\right)$$

while for $N = 4$ we find

$$y^{\frac{1}{2}} = \frac{1}{4}\left[g^*(0) + g^*(y)\right] + \frac{1}{2}\left[g^*\left(\tfrac{7}{16}y\right) + g^*\left(\tfrac{12}{16}y\right) + g^*\left(\tfrac{15}{16}y\right)\right] .$$

These equations must now be solved for $g^*(y)$. A general way to do this is to expand the left hand side--in general this will be $f(y)$--and $g^*(y)$ in power series and equate coefficients. In this particular case, if we expand in powers of $y^{\frac{1}{2}}$ we obtain for N = 2

$$g^*(y) = \frac{2}{1+3^{\frac{1}{2}}}\, y^{\frac{1}{2}}$$

For N = 4 we obtain

$$g^*(y) = \frac{2}{\frac{1}{2}+\frac{1}{4}\left(7^{\frac{1}{2}}+12^{\frac{1}{2}}+15^{-\frac{1}{2}}\right)}\, y^{\frac{1}{2}} .$$

The exact solution is $g(y) = 2y^{\frac{1}{2}}/\pi$ so the per cent errors are 15% and 4.7% for N = 2 and 4, respectively.

Appendix F

GENERAL CLIFF METHOD

CONTRIBUTION OF BRANCH CUTS

In the section of Chapter I on the general cliff method it was stated that in the method of approximation, certain integrals along banks of the branch cutting in Figure 1-6 could not be neglected. The following example will demonstrate this for a fairly simple case. Consider the integral

$$I = \int_L e^{-at} \ln\left[\rho\,(1-t^2)^{\frac{1}{2}}\right] dt$$

where L is the lancet contour taken in the counterclockwise sense about the cut from +1 to $+\infty$ on the real axis. If the contour L is collapsed onto the cut we obtain the finite value $I = i\pi \exp(-a)/a$ by straight forward means.

Suppose we use the general cliff method by employing the rational function expansion $(1-t^2)^{\frac{1}{2}} \simeq (1+t)^{\frac{1}{2}} R_n(t)$ where the poles and zeros of $R_n(t)$ lie in the position of the original cut from +1 to $+\infty$. The function $\ln\left[\rho(1+t)^{\frac{1}{2}}R_n(t)\right]$ will have a series of branch cuts issuing from the zeros and poles of $R_n(t)$ in a pattern similar to that of Figure 1-6. If we collapse the contour L onto the singularities of $\ln\left[\rho(1+t)^{\frac{1}{2}}R_n(t)\right]$, it is not hard to show that the contributions from the zeros and poles of $R_n(t)$ vanish and that we are left only with the contribution along the banks of the cuts for $\ln\left[\rho(1+t)^{\frac{1}{2}}R_n(t)\right]$. This contribution must be nonvanishing since we know the original integral is finite.

INTEGRATION AROUND ESSENTIAL SINGULARITIES WITH BRANCH POINTS

In the section on the general cliff method, we had to evaluate the contributions from certain branch points of the Hankel function as shown in equation 1-18. The general form of the terms to be evaluated is

$$\text{(F-1)} \qquad \oint_{z_j} \frac{e^{i\rho R_n(z)}}{\left(R_n(z)\right)^{\frac{1}{2}}}\, g(z)\, dz$$

where g(z) is slowly varying and $R_n(z)$ is the rational function approximant with poles $\overset{x}{z}_j$, or simply z_j. First we make the linear transformation $z = z_j + v$ so that from the general form for a rational function we have

$$\text{(F-2)} \qquad R_n(z_j+v) = \sum_{m\neq j} \frac{a_m}{z_j - z_m + v} + \frac{a_j}{v} + h(z_j+v)$$

$$\text{(F-3)} \qquad R_n(z_j+v) \simeq A_j + B_j v + a_j/v$$

where A_j and B_j are independent of v. As v becomes small, F-3 becomes a very good approximation to $R_n(z)$. If we substitute F-3 into F-1 we obtain

$$\text{(F-4)} \qquad \frac{e^{i\rho A_j}}{(a_j)^{\frac{1}{2}}} \oint_{v=0} v^{\frac{1}{2}} \frac{e^{i\rho B_j v + i\rho a_j/v}}{\left[1 + \frac{A_j}{a_j} v + \frac{B_j v^2}{a_j}\right]^{\frac{1}{2}}}\, g(z_j+v)\, dv$$

If we next assume we can remove the slowly varying factors from the integrand, we are left with the following integral to evaluate:

(F-5) $$\int_{\circlearrowleft\, v=0} v^{\frac{1}{2}} e^{i\rho B_j v + i\rho a_j/v}\, dv$$

where we have the restriction $2\pi > \arg R_n(z) > \pi$ or $2\pi > \arg \frac{a_j}{v} > \pi$ due to the fact that we have employed the asymptotic form of the Hankel function to obtain 1-18. The integral F-5 can be put in the form of a known integral by letting $i\rho a_j/v = t$. Then F-5 becomes

(F-6) $$-(i\rho a_j)^{\frac{3}{2}} \int_L t^{-\frac{5}{2}}\, e^{t - c_j^2/t}\, dt$$

where $c_j^{\,2} = \rho^2 a_j B_j$ and L is the contour in the t plane shown in Figure F-1.

The cut in Figure F-1 is chosen so that $2\pi + \frac{\pi}{2} > t > -\frac{\pi}{2}$ to satisfy the restriction on arg a_j. In order to identify this integral with the Bessel function we must take the cut along the negative imaginary axis. Then since our integral vanishes in the left half plane, we are left with the open contour in Figure F-2. With this contour, we can identify our integral F-5 with the Sonine integral representation:

(F-7) $$\frac{2\pi i\, J_\nu(K)}{\left(\frac{K}{2}\right)^\nu} = \int_{c-i\infty}^{c+i\infty} t^{-\nu-1}\, e^{\left(t - K^2/4t\right)}\, dt$$

Thus we have

(F-8) $$\int_{\substack{\circlearrowleft \\ v=0,\ \theta}} v^{\frac{1}{2}}\, e^{i\rho B_j v + i\rho a_j/v}\, dv = (i\rho a_j)^{\frac{3}{2}}\, 2\pi i\, \frac{J_{\frac{3}{2}}\left[2\rho (a_j B_j)^{\frac{1}{2}}\right]}{\left[\rho\sqrt{a_j B_j}\right]^{\frac{3}{2}}}$$

where the angle θ is determined from the relation arg v = arg $t - \frac{\pi}{2}$ - arg a_j and the fact that t = $-\pi/2$ on the branch cut in Figure F-2. Thus θ = - arg a_j.

Our solution and the form F-7 show the sensitivity of integrations around an essential singularity. In F-5 we might have argued that we could neglect the term $i\rho B_j v$ in the exponent because it is dominated by $i\rho a_j/v$ for small v. Then from F-8 we see our solution would have been $(i\rho a_j)^{\frac{3}{2}}\, 2\pi i\ 4/3\pi^{\frac{1}{2}}$ since $J_{\frac{3}{2}}(2x)/x^{\frac{3}{2}} \to 4/3\pi^{\frac{1}{2}}$ for $x \to 0$. Since $\rho(a_j B_j)^{\frac{1}{2}}$ is not necessarily a small quantity, we see that indiscriminately dropping terms of o(v) in F-5 can lead to incorrect solutions. The proper criterion for dropping this term is independent of v and is that $\left|\rho(a_j B_j)^{\frac{1}{2}}\right| \ll 1$. Similar statements can be made for higher order terms in v in the exponent. The method of expansion of $R_n(z)$ must justify in some way the dropping of the higher order terms.

The reason for this sensitivity around an essential singularity can be seen if we express the exponential factor in the integrand as the product of an exponential damping term and a rapidly oscillating term.

As we integrate around the essential singularity we are subtracting large terms of about the same size--due to the oscillations. Very small changes in the exponent of the damping factor are then magnified, since it is the differences of large terms which count.

Appendix G

INTEGRATION ON BANKS OF BRANCH CUTS

Let $I = \int_C g(z)f(z)dz$ where C is the contour of Figure G-1 and the cut belongs to f(z). We shall evaluate this integral by the cliff method. First expand f(z) in a rational function $R_n(z)$ with poles along the cut position. Now when we deform C as shown in Figure G-2 we are left with semi-circles about the poles and straight line segments between the poles. It might appear that we are still stuck with line integrals. However the approximation to the integral I is given by adding up the residues at these poles and multiplying by a weight factor. In other words we can still obtain our approximation by the Residue Theorem.

To see this, expand f(z) by the Cauchy Integral Formula so that it is approximated by a rational function $R_n'(z)$ in the unshaded portion of Figure G-3, with poles located as shown. In the wedge shaped section, the approximation $R_n'(z)$ becomes vanishingly small by Cauchy's Integral Theorem. Now if we deform C as indicated in Figure G-3, we are left with loops about the poles t_j' plus a contour C'. Now in the limit as the poles become dense (on the lines) the integral $\int_{C'} g(z)R_n'(z)dz$ must vanish by Cauchy's Integral Theorem (assuming appropriate convergence of the original integral). Therefore the approximation to I only needs to include the residues a_j' at the poles t_j'. For j sufficiently large, the term $\int_{C'} g(z)R_n'(z)dz$ can be made as small as we like. Thus we have our solution in terms of the residues a_j'.

The final step is to relate the residues a_j' to the residues a_j at the poles t_j in Figure G-2. This relation is obtained by comparing the rational function expansions of f(z). By Cauchy's Integral Formula we have

$$\text{(G-1)} \qquad f(z) = \frac{1}{2\pi i}\int_L \frac{f(t)}{t-z}\,dt = \frac{1}{2\pi i}\left\{ \int_{L_{ab}} \frac{d\varphi(t)}{t-z} + \int_{L-L_{ab}-L_{ca}} \frac{d\varphi(t)}{t-z} + \int_{L_{ca}} \frac{d\varphi(t)}{t-z} \right\}$$

$$\text{(G-2)} \qquad f(z) = \frac{1}{2\pi i}\int_M \frac{f(t)\,dt}{t-z} = \frac{1}{2\pi i}\left\{ \int_{L_{ab}} \frac{d\varphi(t)}{t-z} + \int_{M-L_{ab}} \frac{d\varphi(t)}{t-z} \right\}$$

where L and M are given in Figures G-4 and G-5 and $\varphi(t)$ is the distribution function for f(z). Now our method of producing poles is to approximate the distribution function $\varphi(t)$ by a stair-case function. Suppose on L_{ab} and M_{ab} we use the same stair-case approximation to $\varphi(t)$. Then from G-1 and G-2 we see that the integrals which generate the poles t_j and t_j' are, respectively,

$$\frac{1}{2\pi i}\int_{L_{ab}} \frac{d\left[\varphi_{ab}(t) - \varphi_{ca}(t)\right]}{t-z} \quad \text{and} \quad \frac{1}{2\pi i}\int_{M_{ab}} \frac{d\left[\varphi_{ab}(t)\right]}{t-z}$$

where we let L_{ab} and L_{ca} approach the branch cut so that $L_{ab} = -L_{ca}$. We can also take $M_{ab} = L_{ab}$. The subscripts on the distribution function refer to the value of $\varphi(t)$ on the opposite banks of the cut. Then the residues at t_j and t_j' are

$$a_j = \frac{1}{2\pi i} \left\{ \left[\varphi_{ab}(t_j) - \varphi_{ab}(t_{j-1}) \right] - \left[\varphi_{ca}(t_j) - \varphi_{ca}(t_{j-1}) \right] \right\} \tag{G-3}$$

$$a_j' = \frac{1}{2\pi i} \left[\varphi_{ab}(t_j) - \varphi_{ab}(t_{j-1}) \right] \tag{G-4}$$

If For a double valued function, $\varphi_{ca} = -\varphi_{ab}$, so that $a_j' = \frac{1}{2}a_j$, then our weight factor is $\frac{1}{2}$. In terms of Figure G-2, our approximate solution to the integral I is then given by taking one-half the residues at the poles t_j. The determination of the weight factor for any multivalued function is easily obtained from G-3 and G-4.

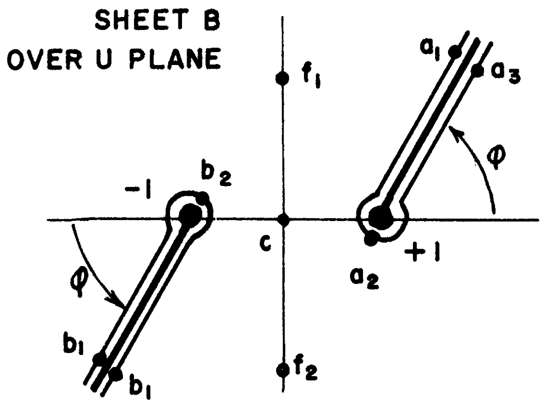


FIGURE A-1
BRANCH POINTS OF $\sqrt{n^2-1+U^2}$

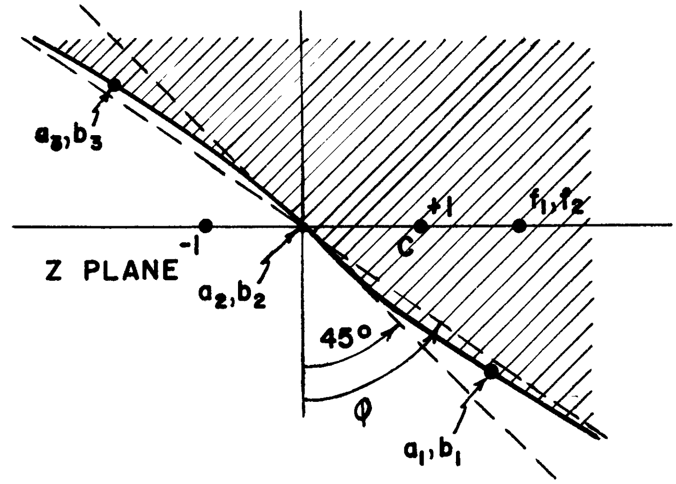


FIGURE A-2
BRANCH POINTS OF $\sqrt{n^2-W^2}$

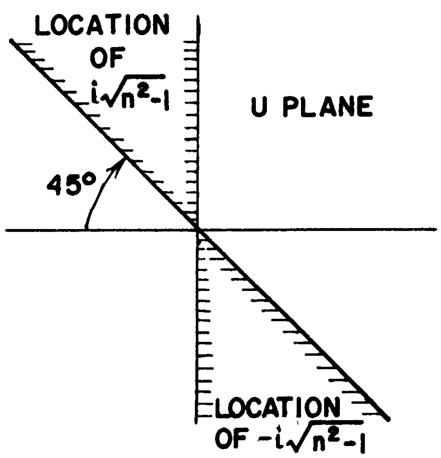


FIGURE B-1

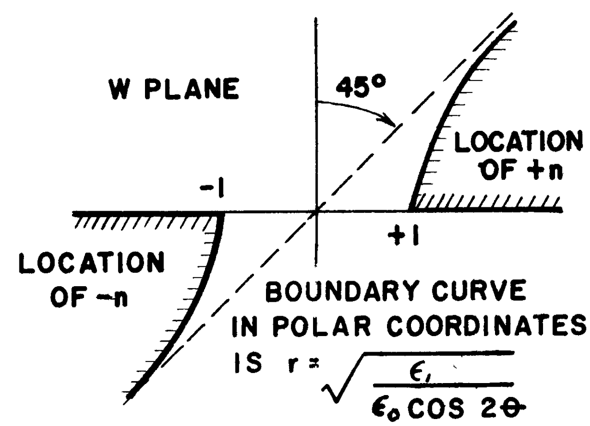


FIGURE B-2

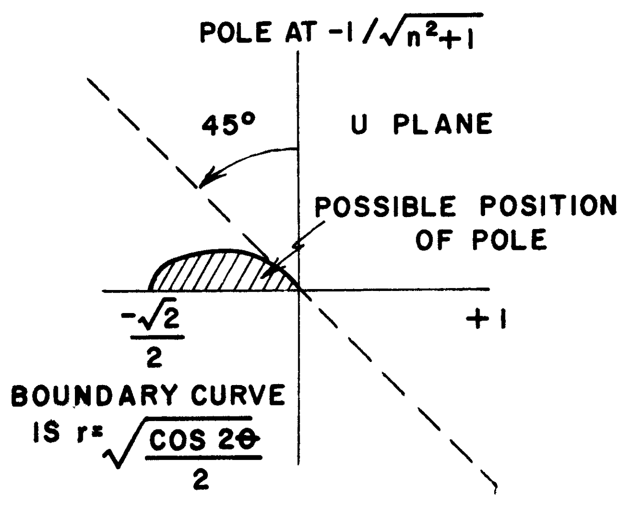


FIGURE B-3

SIGNATURE OF $\sqrt{1-W^2}$ ON SHEET B

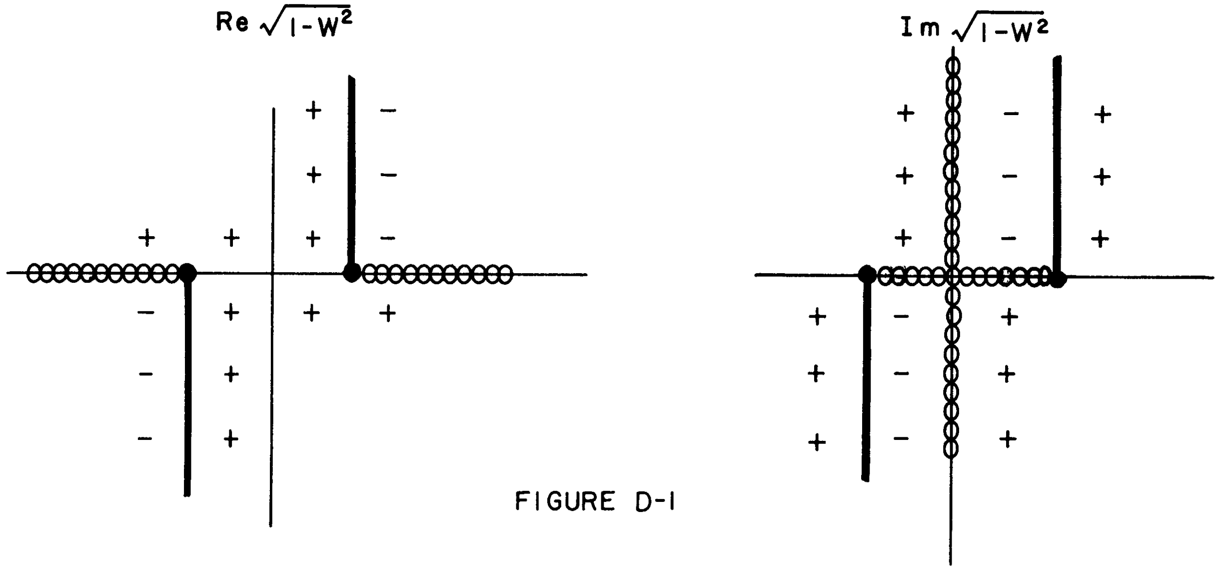


FIGURE D-1

SIGNATURE OF $\sqrt{n^2-W^2}$ ON SHEET B

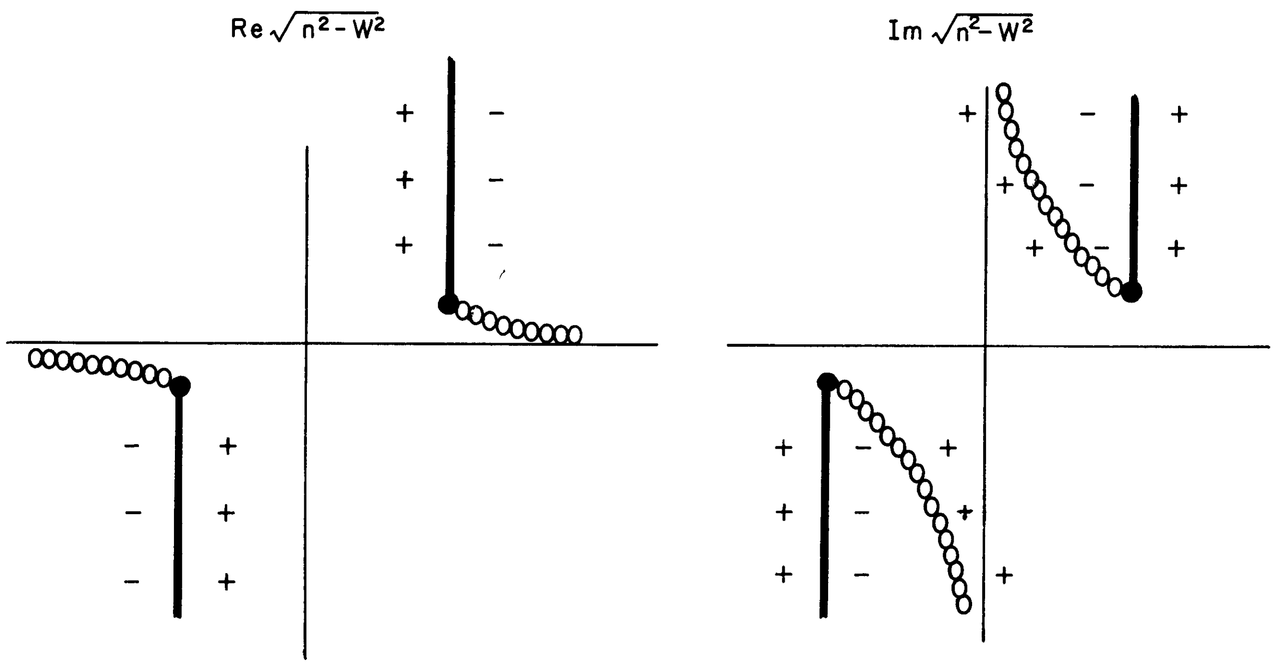


FIGURE D-2

SIGNATURE OF $\sqrt{n^2-1+U^2}$ FOR SHEET B

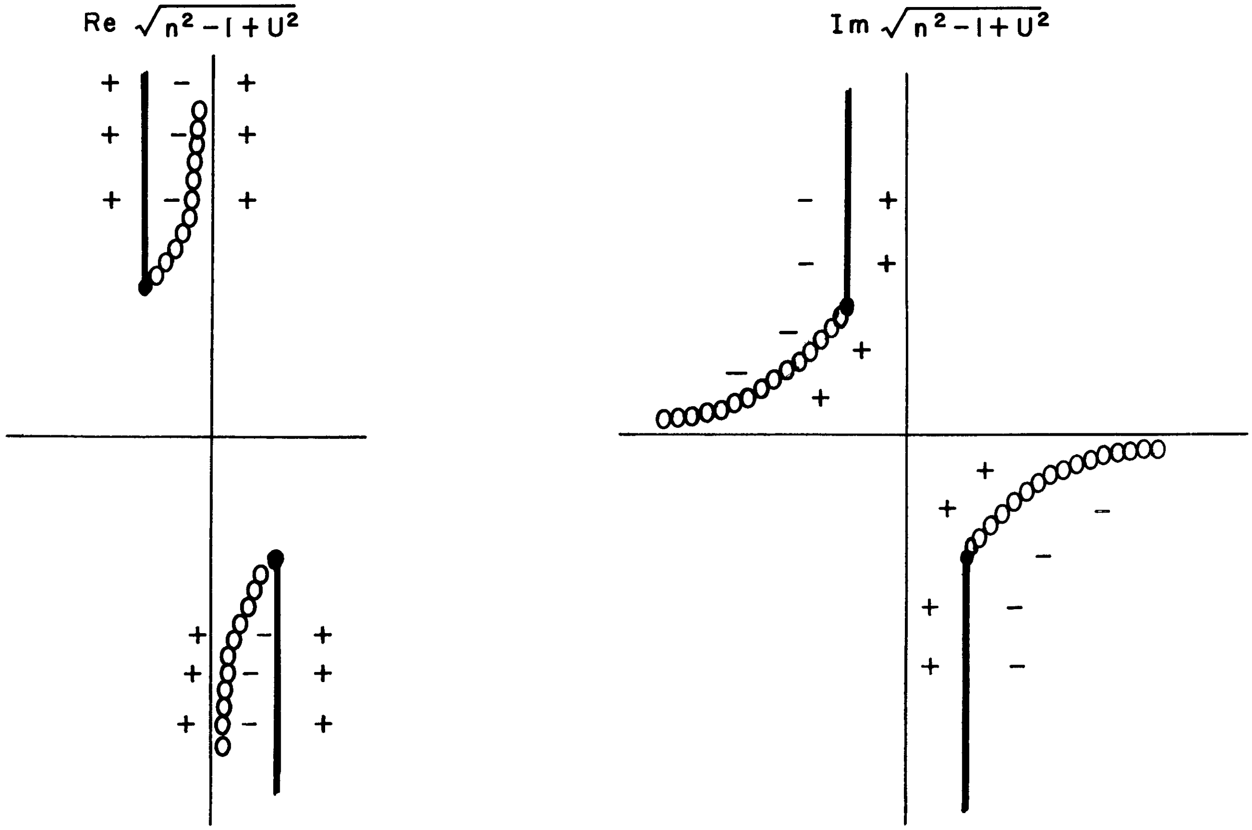


FIGURE D-3

SIGNATURE OF $\sqrt{1-U}$ ON SHEET B

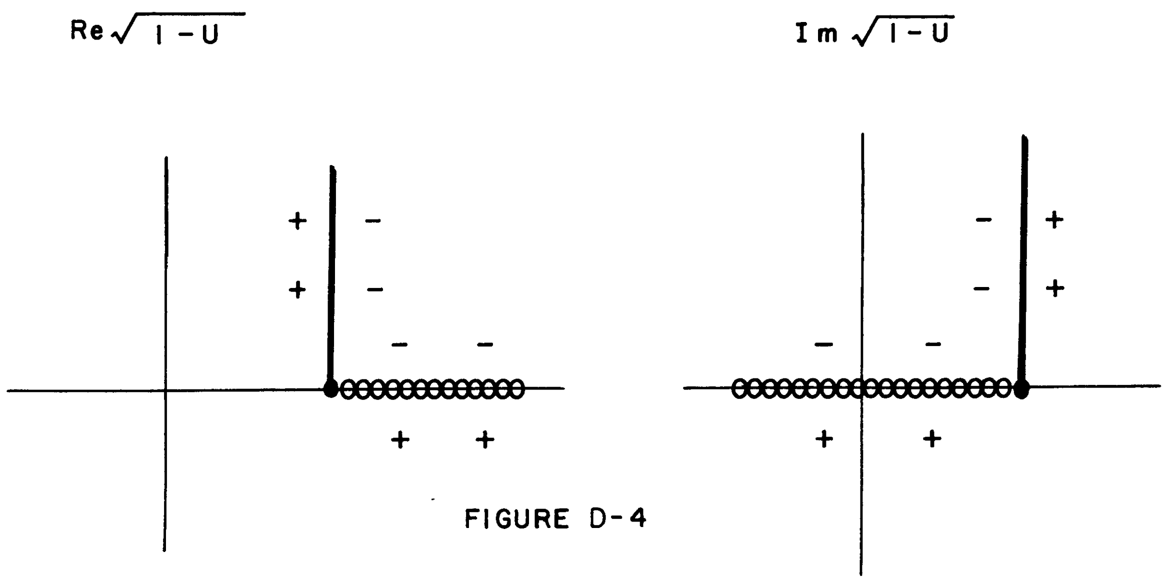


FIGURE D-4

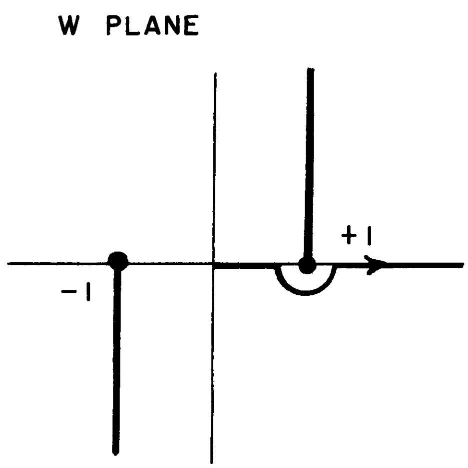


FIGURE D-5

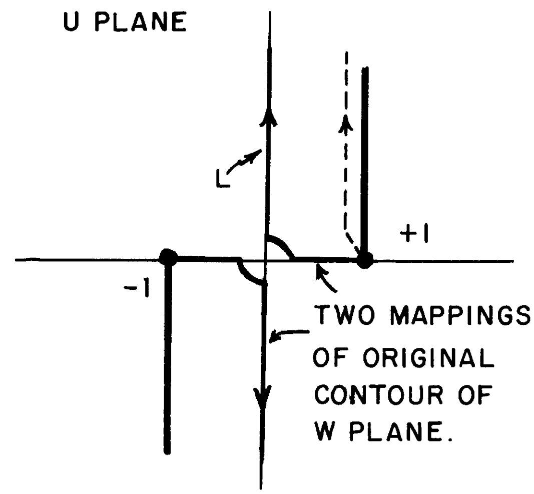


FIGURE D-6

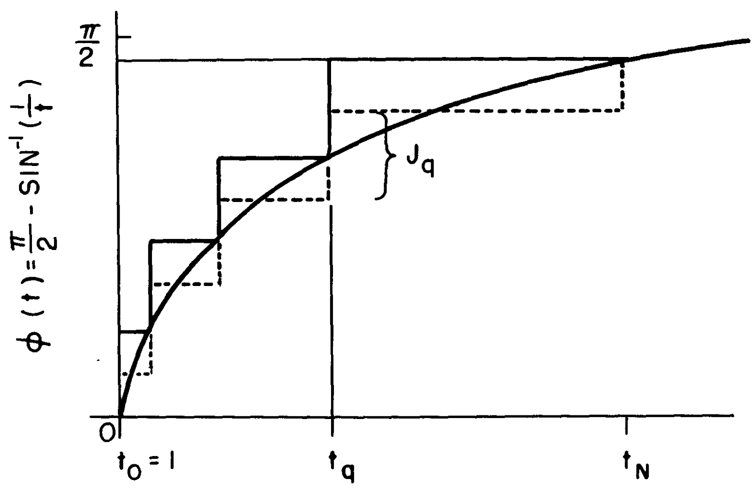


FIGURE E-1

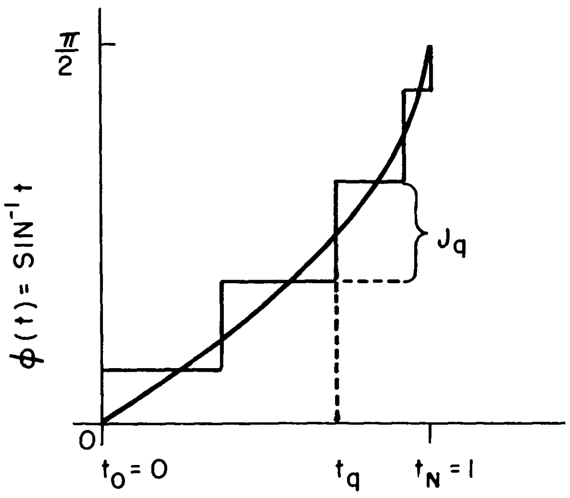


FIGURE E-2

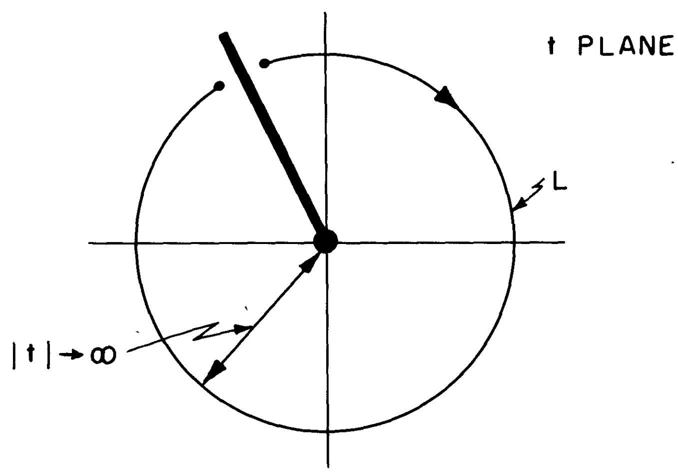


FIGURE F-1

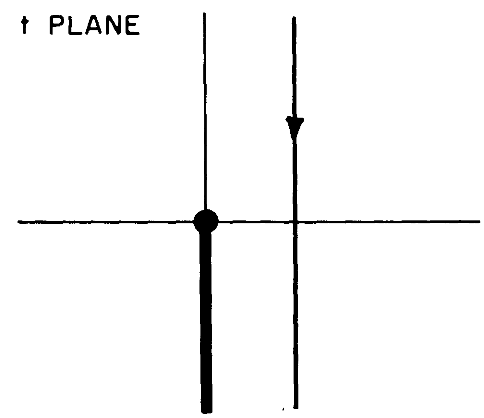


FIGURE F-2

APPROXIMATION TO $J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{(1-t^2)^{\frac{1}{2}}}\,dt$

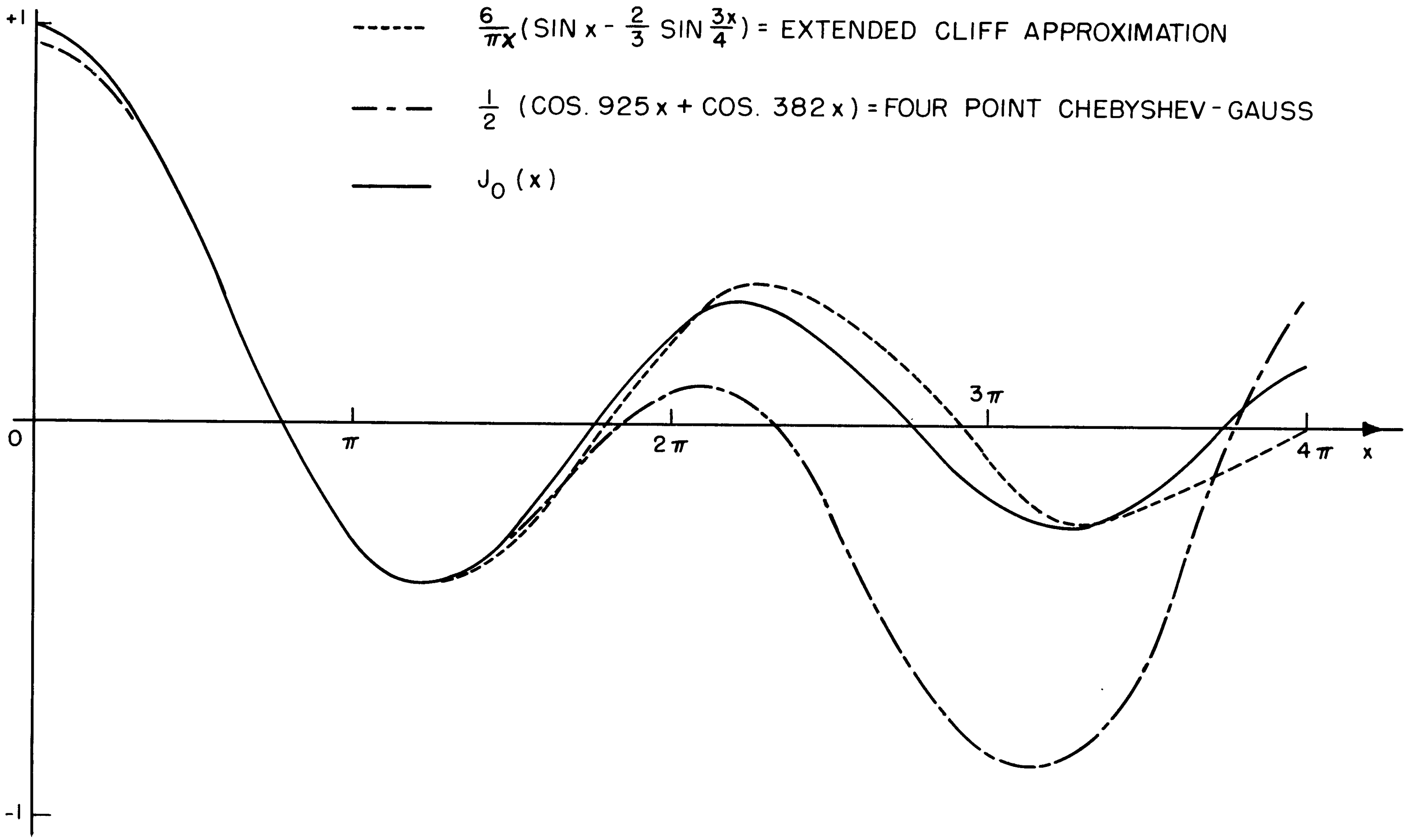


FIGURE E-3

TABLE E-1

z	$J_o(z)$	saddle point method	5 term Simpson's rule for ^2I	5 term cliff method for ^1I	2 term Chebyshev Gauss for ^3I	2 term cliff method for ^3I	5 term Chebyshev Gauss for ^3I	5 term cliff method for ^3I
0	1.0000		0.0000	0.8664	1.0000	1.0000	1.0000	1.0000
π/4	8516	.900	7727	8389	853	8536		8516
π/2	4720	450	4520	4669	472	5000		4720
3π/4	0255	000	0264	0244	024	1465		0255
π	<u>3042</u>	<u>318</u>	<u>2991</u>	<u>3028</u>	<u>305</u>	0000		<u>3042</u>
5π/4	<u>4009</u>	<u>403</u>	<u>3964</u>	<u>3976</u>	<u>406</u>			<u>4009</u>
6π/4	<u>2659</u>	<u>260</u>	<u>2634</u>	<u>2638</u>	<u>286</u>			<u>2659</u>
7π/4	<u>0076</u>	000	<u>0074</u>	<u>0072</u>	<u>070</u>			<u>0076</u>
2π	2203	225	2188	2191	082			2203
9π/4	2997	300	2977	2978				2997
10π/4	2043	202	2029	2028	<u>218</u>			2044
11π/4	0039	000	0038	0037				0044
3π	<u>1812</u>	<u>184</u>	<u>1802</u>	<u>1802</u>	<u>829</u>			<u>1798</u>
13π/4	<u>2495</u>	<u>250</u>	<u>2481</u>	<u>2479</u>				<u>2455</u>
14π/4	<u>1720</u>	<u>170</u>	<u>1710</u>	<u>1708</u>	<u>612</u>			<u>1618</u>
15π/4	<u>0025</u>	000	<u>0024</u>	<u>0023</u>				<u>0203</u>
4π	1575	159	1567	1566	338		158	2036
18π/4	1493						<u>319</u>	

negative numbers are underlined

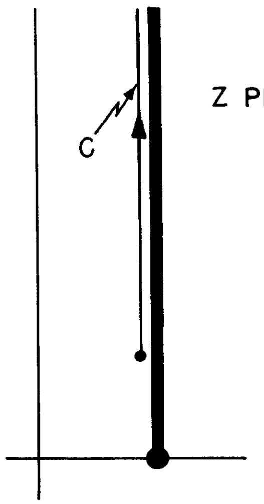


FIGURE G-1

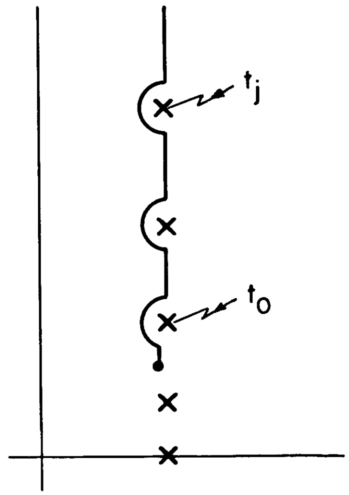


FIGURE G-2

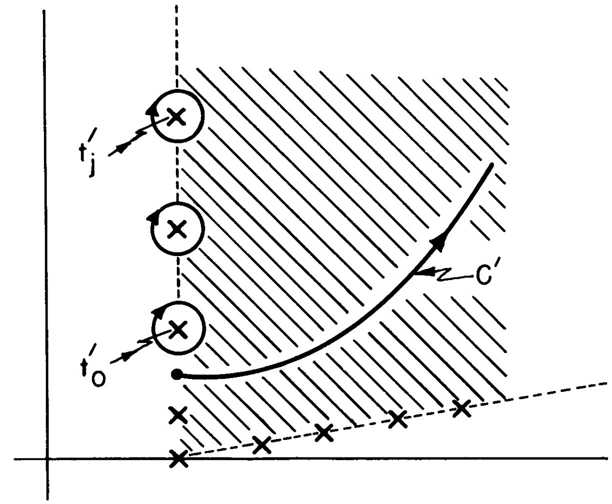


FIGURE G-3

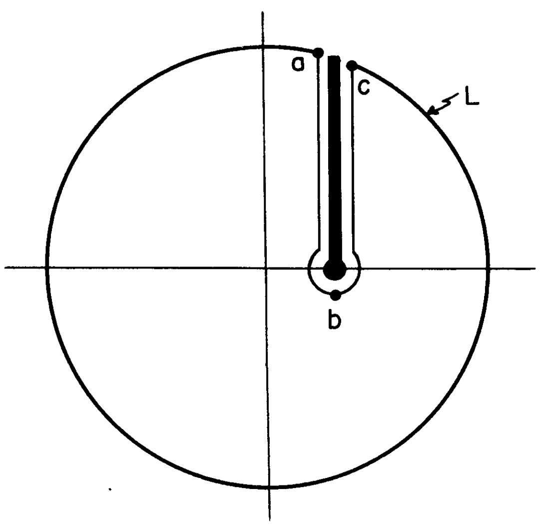


FIGURE G-4

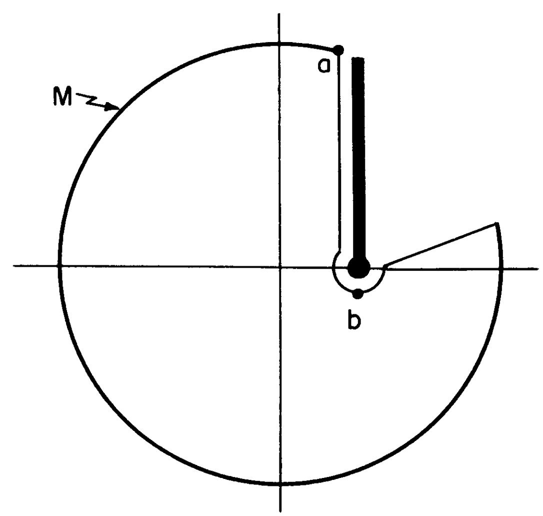


FIGURE G-5

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