EFFECT OF STRONG DENSITY GRADIENTS ON DENSITY WAVES

IN SPIRAL GALAXIES

by

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ABSTRACT

The purpose of this study is to determine more accurately the marginal stability condition for axisymmetric waves since it is believed that an unstable axisymmetric wave could affect the amplification of spiral waves. An axisymmetric wave with a gradient term of velocity dispersion (surface density) is derived. Analysis of this axisymmetric wave with a gradient term for our own galaxy shows that the wave is essentially stable in a medium which is slightly unstable according to the local criterion. Thus, the axisymmetric wave will probably have no effect (or at best only very slight effects) on the amplification of spiral waves.

Gereralization of the axisymmetric wave to the spiral wave is also discussed. It is conjectured that the spiral wave with a gradient term (for those galaxies with a large bending structure) could cause spiral wave emission and amplification.

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I. INTRODUCTION

(i) Density Wave Theory of Spiral Galaxy

The density wave theory is developed especially to resolve the problems of (a) the persistence of the spiral structure in the face of differential rotation, and (b) the large-scale nature of the phenomena, i.e. the existence of a grand design over the whole galactic disk. Lin and Shu (1964) therefore begins with the QSSS hypothesis (hypothesis of the existence of density waves with a quasi-stationary spiral structure), rather than attempt to study the problem of the origin of the spiral structure. The theory based on the calculation of the responses of an infinitesimally thin disk of stars and gas to a resultant graxitational field of a spiral form. The density wave of a spiral structure is assumed to be self-sustained at a small but finite amplitude. The wave involves instantaneous disturbances in mass density and in gravitational field. The propagation of the density wave is the combined effects of the gravitational field and stellar motions. An analogy to this is given by waves in collisionless plasma, which propagate across a strong magnetic field. An outline of the gravitational density wave theory is given in Appendix A. Lin and Shu (Lin and Shu 1964, 1966; Shu 1970) obtained a dispersion relation that connects the spacing between the spiral arms with the pattern frequency of the wave together with other known parameters in the lowest order of WKBJ (short wavelength) approximation. To the next order, relation giving amplitude as a function of galacto-centric distance was obtained. In particular, the Lin-Shu dispersion relation (c.f. Appendix A) gives for each permitted value of intrinsic frequency ν two solutions for wavelength λ: short and long waves. Lin and Shu identify the typical spiral structure in early-type spirals with the short wave solutions.

Toomre (1969) found that the density waves are not stationary, but that the wave action (energy density divided by a frequency factor) propagates radially with a kinematic group velocity (v_g). For trailing waves, he showed that the action propagates away from the corotation circle (the circle at which the particle and wave circular frequencies are equal) for short wave solutions, and propagates toward the corotation circle for long wave solutions (see Figure 2). For the radial group velocity of Lin's density wave at $\tilde{\omega} = 10$ kpc from the galactic center of our Galaxy, Toomre found $v_g \sim 10 \frac{\text{km}}{\text{sec}}$. If we identify the spiral structure with the short wave solutions, then we find that the energy of the wave will propagate through the galaxy in a time typically of the order of 10^9 years and is then absorbed (at the so-called "Lindblad resonances", c.f. Mark 1971). This indicates that density waves cannot be "Quasi-Stationary" waves unless some energy replenishment exists. Thus, a complete theory of density wave in galaxies requires a mechanism for the long-term maintenance of the wave.

(ii) Earlier Proposals for Possible Mechanisms of Wave Maintenance

There have been several suggestions on the maintenance mechanism for density waves. Toomre (1969) briefly discussed the possibilities of external forcing by a companion galaxy, of forcing from the central regions by barlike or oval distortions and of local amplification. Lin (1969) has suggested that the galaxy may be Jeans unstable near corotation and that this mechanism could fuel the waves. In the same paper Lin also proposed a feedback loop to reinforce the spiral density wave system in view of Toomre's result on group velocity. He suggests that the inward propagating short trailing wave would cause a distortion of the central regions of the galaxy. Furthermore this distortion would then excite the short wave at corotation either via direct bar-like gravitational field associated with the distortion or via a long trailing wave pattern (open spiral) propagating toward corotation. (The central distortion itself can be viewed as part of a long wave.) Inspired by Lin's suggestion, She et al. (1971), in their study of galaxy M51, found indications of the existence of the long waves in the form of a weak, wide-open spiral being superimposed on the main arms which are the short waves. This is shown schematically in Figure 2.

The feedback mechanism near corotation has been worked out by Feldman and Lin (1973) for direct forcing due to a bar potential and by Mark (1973, 1975a) for driving by the long wave.

Improving the work by Feldman, Lin and Mark, Lau and Lin (c.f. Lin 1974) also obtained results for the gas dynamical case on long wave driving. In this particular work, they used gas-dynamics to simulate the stellar dynamics.

(iii) The Need for Wave Amplification

It has been shown that the density wave is absorbed at the Lindblad resonance regions (Mark 1971, 1974a; Lynden-Bell and Kalnajs 1972) where the spiral waves terminate according to the density wave theory. There might be other dissipative processes in the galaxies. The generation mechanisms based on the driving influence of bar-like and oval distortions of galaxies or on the influence of external companions could be quite weak. Ordinary spiral galaxies by definition would have only rather small bar distrotions (if any). Moreover, companion galaxies are not usually so near one another. Observational evidence suggest that spiral wave frequencies are not random but correlated with galaxy type (Roberts, Roberts and Shu 1975). Thus, Mark (1973 and later papers) proposed that "amplifiers" of galactic density waves are needed for long term maintenance and for specific determinations of wave frequencies. A wave amplification process advanced by him (Mark 1973, 1975b) involves wave-wave interactions at the corotation region. This can yield large amplification of waves. We state here without proof (c.f. e.g. Mark 1974b) that the wave angular momentum (energy) density inside corotation region is negative while wave angular momentum (energy) density outside corotation is positive. Thus waves inside corotation are amplified when angular momentum is removed from that region, and waves outside corotation are amplified on receiving angular momentum. This is intuitively plausible since the waves inside corotation rotate slower than the disk matter and vice versa for the waves outside corotation. In the wave-wave interaction picture, an outward propagating long trailing wave approaching corotation excites two short trailing waves departing corotation in opposite directions (see Figure 5). In particular, the short wave inside corotation region propagating toward the galactic center carries more angular momentum than the incident long wave. The short wave is then amplified relative to the incident wave. Such amplification is possible because an exchange of angular momentum has taken place in the corotation region where the long and short waves have similar wavelengths. Mark has also shown that there is strong amplification if the dispersive velocities of the stars are sufficiently low. The characterization of this latter parameter is usually denoted through the dimensionless local stability parameter Q_*. For a stellar disk this is the ratio between the actual and critical velocity dispersion needed to surpress local axisymmetric Jeans instability. Axisymmetric instability occurs if Q_* is less than unity for a pure stellar disk. If there is gas present, it cause a similar net effect. We must nevertheless emphasize that such instability parameter obtained from local analysis need not be very accurate determinant of the global axisymmetric instability. This is because the so-called local instability wavelength around the corotation region is comparable to galactic dimension. As we will see in § II d and § III, the purpose of this note is to determine more accurately the marginal stability condition for axisymmetric waves.

Moreover, resonant stars in the corotation regions absorb wave angular momentum (Lynden-Bell and Kalnaj's 1972) and help amplify the wave (Mark 1973, 1975b). The wave interacts with the stars at all the resonant regions. It turns out that the resonant stars emit angular momentum at the inner Lindblad resonance regions and absorb angular momentum at corotation and outer Lindblad resonance regions. Another amplification process worked out by Mark (1973, 1975c) involves the interaction of waves with the stars of the bulge-halo subsystems (c.f. Ostriker and Peebles 1973). In the propagation region inside the corotation circle, the stars of the slowly rotating bulge-halo subsystem take away angular momentum from the waves during their passage to the disk. As a result, waves are amplified in that region. The disk-halo effect is important for galaxies where there are no inner Lindblad resonances (an inner region of the galaxy where spiral waves terminate as an evanescent structure). Application of this mechanism to galaxy M33 (possessing no inner Lindblad resonance) has been carried by Mark (1975c). The result of his study shows that the interaction indeed greatly enhances the amplification. (Further detail on this section is given in Appendix C.)

II. AMPLIFICATION PROBLEM FOR GALAXIES SIMILAR TO THE MILKY WAY SYSTEM

(i) What Is the Dominant Amplification Mechanism in Such Galaxies?

Due to the shielding effects of inner resonance in more centrally concentrated galaxies, the disk-halo amplification effect is probably important only for galaxies similar to M33. These galaxies have low central concentration and thus process no inner resonances. The Milky Way System, M31 and M81 are good examples of ones with inner resonance. For these galaxies, there are growing semi-empirical evidences (c.f. Shu et al. 1971; Roberts et al. 1975) that the waves in such galaxies corotate with the material near the edge of the bright optical disk. Presumably, such waves are determined by processes involving the outer regions of these galaxies. In addition, dissipative processes such as resonance absorption are especially a prominent feature in such galaxies (c.f. Mark 1971, 1974 for evidence of absorption in our Galaxy). Moreover, there is the need for specific processes which choose the above mentioned class of wave frequencies. Both of these reasons suggest wave amplification as a possible central issue. Also, we are naturally led to study such processes which are important in the outer regions. Earlier, we have already ruled out on pure dynamical grounds the disk-halo effects which prefer waves corotating with the inner or middle regions of galaxies. With possible exceptions such as M81 (perhaps), we might also rule out resonant particle effects because the gradients of the surface density are small. Thus we seem to be limited to the following amplification process which we now describe.

In the wave-wave interactions picture, Mark (1975a, b) considered a composite disk of stars and gas. He assumed a small value for the ratio $\frac{\sigma_g}{\sigma_*}$ of gaseous to stellar surface density near corotation. He found that wave amplification is quite sensitive to changes in the ratio $\frac{\sigma_g}{\sigma_*}$ and the local stability parameter Q_* as defined in the last section (c.f. Appendeces A and B). Indeed, large amplification is possible if $Q_* < 1$ or if $Q_* \sim 1$ but $\frac{\sigma_g}{\sigma_*}$ is comparable to each other. This situation corresponds to an axisymmetric Jean instability by approximate local analysis, but the unstable wavelengths turn out to be long (c.f. § IId). Moreover, he incorporated these effects into a single amplification parameter defined by

$$\mu = D\left\{(1-Q_*) + \frac{1.822}{1+0.9481\frac{a^2}{c^2}}\frac{\sigma_g}{\sigma_*}\right\} \tag{1}$$

where $D \simeq 3$ for our Galaxy and $\frac{a}{c}$ is the ratio between the gaseous and stellar dispersive speeds. Thus, when $\mu > 0$ (< 0) wave amplification is large (small). Near $\mu = 0$, there is a relatively sharp transition between these two possible cases (c.f. Figure 6). From (1) we see that the first situation requires that $Q_* \lesssim 0.95$ or that $\frac{\sigma_g}{\sigma_*} \gtrsim 0.1$. In the outer regions of many spiral galaxies, gas density is comparable to that of the stellar density and gas dispersion speed (random turbulent motion) is low because of dissipation. Indeed, $\frac{\sigma_g}{\sigma_*} \simeq 20\%$ at the distance 14 kpc from the galactic center in our Galaxy. Moreover, the local parameter Q_* might possibly be less than unity. In such a situation we may expect large amplification of spiral waves.

Note that the axisymmetric waves might now be unstable in that local region when $Q_* < 1$ or $\frac{\sigma_g}{\sigma_*} = 20\%$. We must examine this effect because axisymmetric instabilities could cause Q_* to increase and thus decrease amplification of spiral waves (c.f. §IIb). We define a local "stability" parameter for the combined gaseous and stellar disk, Q_o, which includes the effects of both dispersion velocity and gas density. It is given by (c.f. Appendix B)

$$\frac{1}{Q_o} = \frac{1}{Q_*}\left[1 + \frac{1.822}{1+0.9481\frac{a^2}{c^2}}\frac{\sigma_g}{\sigma_*}\right] \qquad (2)$$

Hence, the contribution to the case $Q_o < 1$ (corresponding to the local estimate to the region of Jean's axisymmetric instability) may come from Q_* or $\frac{\sigma_g}{\sigma_*}$, or both. Again, we emphasize that a more careful determination of the global criterion is part of our work. Note that Q_o is related to μ by the following relation $\frac{\mu}{D} = (\frac{1}{Q_0} - 1)$. Obviously, when $Q_o < 1$, $\mu > 0$. This corresponds to a large wave amplification. To demonstrate this possibility, we study the outer part of our Galaxy at around

14 kpc from the galactic center. At this distance, gas density is appreciable. For a fixed total surface density σ and a given $\frac{a}{c}$, we find from the local criterion that the wave is certainly unstable if $\frac{\sigma_g}{\sigma_*}$ exceeds an upper bound $\frac{aK}{2\pi G\sigma}$ where K is the epicydic frequency of the star and G is the gravitational constant. We take $\frac{a}{c}$ and $\frac{\sigma_g}{\sigma_*}$ to be $\frac{7}{30}$ (velocity in $\frac{km}{sec}$) and $\frac{7}{34.5}$ (surface density in $\frac{M_\odot}{pc^2}$) respectively, and the rest of the parameters can be obtained from the Schmidt mass model (1965). Note that the value of $\frac{\sigma_g}{\sigma_*}$ corresponds to a 20% of gas density over stellar density. In the inner part of our Galaxy this ratio may drop to as low as 1%. We substitute these parameters into the Lin-Shu dispersion relations for the local marginal condition (wave frequency $\nu = 0$) for neutral oscillations to determine whether the wave is stable. In this case the wave is slightly unstable. Indeed, $Q_o \simeq 0.80$ from (2) at that distance. In our calculation we have neglected other components of the gas (such as molecular hydrogen) because our purpose is just to give a rough idea on the importance of gas density to the stability of the wave. The detail calculation is shown in Appendix B. In the absence of amplification effects of disk-halo and resonant particles, $Q_o < 1$ is an interesting case because large wave amplification is possible.

(ii) The Possible Influence of Axisymmetric Wave on Amplification of Spiral Wave

For small amplitude density waves, the simultaneous existence of axisymmetric waves and non-axisymmetric waves (spiral waves) is allowed. Generally speaking, one might be fearful about the possibility that if the axisymmetric wave grows, it will heat up the medium. Therefore, this generates more random motions and may drive the values of Q_o to a higher one through the stellar dispersion velocity in Q_*. This "fear" follows from the experience of the classical Jeans instability mechanism. When the axisymmetric Jean instability starts, there will be a conversion of energy from gravitational collapse into "thermal" motion. The most direct wa y of investigating this matter is a detail numerical simulation of the dynamical evolution of galaxies. In his 10^5 particles numberical experiment allowing enough velocity dispersion to suppress the local axisymmetric instabilities according to Toomre's stability criterion ($c = c_{min}$ where C_{min} is the critical velocity to suppress the Jean's instability). Hohl (1971) found that the disk was unstable to a large scale bar-like mode, and finally a stable axisymmetric disk with a velocity dispersion much larger than that given by Toomre's criterion (4 C_{min}) was generated. Several other authors have also arrived at this result, among them Ostriker and Peebles (1973) recently. Note that the disturbance considered in this numerical experiment is of a rather large scale. If the axisymmetric wave does drive Q_o up, large wave amplification for the spiral waves in a long period basis is impossible.

(iii) Three Possible Cases Where Axisymmetric Waves Might Influence Amplification of Spiral Waves

There are several possibilities on the relation between the axisymmetric disturbance and the spiral wave. We list them in the following table.

Axisymmetric Waves (Stability Condition)	Spiral Wave (Amplification Lasts for A)
Stable	Long Time
Slightly Unstable	Long Time
Highly Unstable	Short Time

If the axisymmetric wave is highly unstable, this will likely drive Q_o to a value substantially higher than its previous one (unless $\frac{\sigma_g}{\sigma_*} \gtrsim 1$). If this is the case, we cannot have large wave amplification for the spiral wave lasting over long periods. In our Galaxy, we may rule out this possibility. In the solar neighborhood, the observed velocity dispersion in the radial direction is about 40 km/sec which is slightly larger than 37 km/sec obtained from the marginal condition of Q_* (i.e. $Q_* = 1$). This suggests that our Galaxy is locally stable. Also, we should expect some large scale chaotic motions of matter generated from this supposed highly unstable axisymmetric wave. Observations do not seem to support this. If this is the case, we have only two possibilities left for our Galaxy. In the first case, a slightly unstable axisymmetric wave will probably not change Q_o much. This implies that large wave amplification in a long basis is still possible. Finally, if the axisymmetric wave is stable, it will not have any effect on Q_o. Thus, wave amplification is the same as before. In view of the result of previous examples, we will examine the effect of a slightly unstable axisymmetric wave on Q_o more carefully in § III.

(iv) Axisymmetric Wave With a Gradient Term of Dispersion Velocity

In the outer region of our Galaxy, we found (§IIa) from a local dispersion relation that the axisymmetric wave is slightly unstable because of the presence of a substantial amount of gas. At 14 kpc from the galactic center, the length scale $\lambda_* \; (\simeq \frac{1}{2} \frac{4\pi^2 G\sigma_*}{\kappa^2})$ of such an axisymmetric Jeans instability is about 10 kpc. Toomre (1964) found that this is the wavelength which is hardest to stabilize by dispersion velocities. Note that this unstable length scale is comparable to the size of the galaxy. For a marginally unstable case, the implication of this length scale is questionable for the following reason. In his analysis of local stability, Toomre (1964) assumed that the surface density $\sigma_*(\varpi)$ and angular velocity $\Omega(\varpi)$ vary slowly with distance. In real situation, thes two parameters vary appreciably over the scale of $\lambda_* \simeq$ 10 kpc (see Schmidt model in Figure 4). Rather, the marginal stability case for axisymmetric waves might well depend on the non-local conditions and in particular on gradient terms. Indeed, the presence of the rapid variation of gas density adds to the degree of inhomogenity (c.f. FIG. B1 in Appendix B). One will be surprised if μ (or Q_o) does not vary at all over this distance (of a couple of kpc). Thus, it is natural to include the gradient term of Q_o in the study of axisymmetric waves over this region. Note that Q_o can be expressed in terms of the basic parameters of the disk such as dispersion velocity, epicydic frequency and surface density (c.f. Equation (2)).

III. AXISYMMETRIC WAVE WITH A GRADIENT TERM

(i) One-Component Model

In our study of the axisymmetric wave with the inclusion of a gradient term, we adopt an equivalent one-component model with only a stellar disk. The reason is simple because it is much easier to handle. From the fact that the local stability criterion is determined by a combined parameter Q_o, we feel that an effective one component disk is at least a good starting point. We mention that we also obtained such a wave for a pure gaseous disk. The composite star-gas disk might just be an extra formality. Thus we only consider variations of the basic parameters of the stellar disk for the gradient term.

(ii) Turning Point Equation

To derive an equation for the axisymmetric wave we make use of a relation which corrects certain ambiguities in the earlier Lin-Shu (c.f. Appendix A) analysis obtained by Mark (1975a). For example, Shu (1970) in the next order of this earlier WKBJ analysis, found that the wave amplitude is infinite at the corotation region of spiral waves. This is not surprised because he considered a single wave in the corotation region, where several waves are actually coupled together (Mark 1973 and later papers). A similar situation occurs near the classical WKBJ turning points when one insists on having only one wave passing through. Such ambiguity is removed if one considers a system of several interacting waves. Mark (1975a) gives the derivation of the improved relation. He also takes into account of the effect of resonant stars. He reduced it to a turning point equation around the corotation region (c.f. Appendix C). This equation then determines the interaction of waves at the region of corotation with the basic equilibrium disk of the galaxy. The factor μ in his turning point equation is treated as a constant because the corotation region is only a small distance (2 to 4 kpc depending on the galaxy). In our case we take into account the gradient term of the basic parameters because our scale of interest is of rather high inhomogeneity. Note that the magnitude of such term varies from disk to disk depending on the degree of inhomogeniety of the disk, which partially reflects the conditions in the rotation curve.

In axisymmetric wave there does not exist a corotation region as the dimensionless frequency $\nu(\varpi)\ (= \frac{\omega - m\Omega}{\kappa})$ is now of the form $\frac{\omega}{\kappa(\varpi)}$ because the number of arms m = 0. We follow Mark's procedure by expanding the improved relation in the outer part of the galaxy (corresponding to the region around the corotation in the case of nonaxisymmetric wave). The difference between his and our approach is that we include the gradient term of Q_* and neglect the effect of the resonant stars (which do not occur in the same region for the axisymmetric waves.) Note that our final result will be equally valid for both the axisymmetric and non-axisymmetric waves. This arises from the fact that the factor m appears only in the dimensionless frequency $\nu\left(=\frac{\omega-m\Omega}{\kappa}\right)$ in the resultant differential equation. When we deal with the axisymmetric wave, we merely set m = 0. The differential equation for the axisymmetric wave (c.f. Appendix D for derivation) is recorded as follows:

$$w''(\varpi)+\frac{1}{(\varpi\varepsilon)_c D_y}\left\{D_\nu\nu^2+2\left(\frac{y_T}{Q}\right)(1-Q)+i\frac{y_T(\varpi)}{Q}\frac{(\varepsilon\varpi)_c}{y_c Q_c}\frac{dQ}{d\varpi}\right\}w=0 \quad (3)$$

where $w(\varpi)$ is the wave function as a function of galacto-centric distance ϖ, the subscript "c" denotes parameters evaluated at some arbitrary distance r_c, and the rest is defined in Appendix D. Equation (3) can be written in another form.

$$v''(x)+\left\{\nu^2+\frac{2y_T}{D_\nu Q}\left[(1-Q)+i\frac{(D_\nu/D_y)^{\frac{1}{2}}}{2y_c Q_c}\frac{dQ}{dx}\right]\right\}v(x)=0 \quad (4)$$

by the transformation variables $x=\frac{(\varpi-r_c)}{l}$ where $l\left(=\left(\frac{D_y}{D_\nu}\right)^{\frac{1}{2}}(r_c\varepsilon_c)\right)$ is a dimensionless length scale. We are interested in the stability properties of the axisymmetric density waves as derived from (3). Note that we have deleted the subscript "*" in Q_* for convenience.

(iii) Results for Special Cases

In order to gain some idea on the stability properties of the axisymmetric wave, we adopt a rather crude model: that of a parabolic "potential" (in Q) which is of the form $Q(x) = ax^2 + bx + Q_c$. We shall examine the eigen value ν. We substitute the function into (4) and convert it to a parabolic cylinder equation of the form

$$u''(z) + [(n+\tfrac{1}{2}) - \frac{z^2}{4}]\, u(z) = 0 \tag{5}$$

where z is the new variable for distance and n is an integer. In particular, the parameter n is defined by

$$n+\frac{1}{2} = \frac{1}{2\sqrt{am}}\, [\nu^2 + (1 - Q_c - a l^2 + \frac{b^2}{4a})\, m] \tag{6}$$

where ν is the wave frequency and m = 1.2679. There is no loss of generality to center the potential at the x = 0 axis so that b = 0. We solve for

$$\nu^2 = 0.5305\sqrt{a} + 0.3851\frac{a}{Q_c^2} - 1.2679\,(1 - Q_c) \tag{7}$$

where we take the solution corresponding to the most unstable n = 0 case. The parameters a and Q_C correspond to the width and depth of the potential well respectively. Results show that the wave is unstable as the well gets deeper. We also performed a similar analysis on a square potential well and obtained similar conclusion. We must emphasize that both models are rather crude. The transition between the flat region and the bottom of the square well is sharp while the sides of the potential well do not flatten out.

(iv) Numerical Results

We present here a more detailed analysis of a truncated parabolic potential well. This is more physically realistic for our purposes. We let $\nu = \frac{\varpi - r_c}{L} + \Delta\nu$ where L^{-1} is some constant (zero for axisymmetric wave) and $\Delta\nu$ is complex. We also adopt $x = \varpi - r_c$ as the new variable for the galacto-centric distance. Suppose Q has the form

$$1 - Q = \begin{cases} (1-Q_c)(1-\beta^2 x^2) & , \ |x| \leq \beta^{-1} \\ 0 & , \ |x| > \beta^{-1} \end{cases} \tag{8}$$

where β^{-1} and Q_c are the width and depth of the potential well respectively. We define $u(x) = w(\varpi)$ so that (3) becomes

$$u''(x) + [(\gamma + \Delta\nu_c)^2 + h(1-\beta^2 x^2) + i\delta_c x]\, u(x) = 0, \quad |x| \leq \beta^{-1} \tag{9}$$

$$u''(x) + [(\gamma x + \Delta\nu_c)^2]\, u(x) = 0, \quad |x| > \beta^{-1} \tag{10}$$

Here, $\gamma = 0\ (m=0)$, $\gamma = \frac{0.09432}{Q_c}\ (m=2)$, $\Delta\nu_c = \frac{\Delta\nu}{1.7749 Q_c}$, $h = 0.3793 \frac{1-Q_c}{Q_c^2}$ and $\delta_c = 0.7645 \frac{(1-Q_c)}{Q_c} \beta^2 S_k$ where $S_k = \pm 1$ for leading and trailing waves respectively. By matching solutions between regions

$|x| > \beta^{-1}$ and $|x| \leq \beta^{-1}$, we obtained results by numerical analysis supporting the earlier claim. A typical set of values are listed in the following table.

	β^{-1} (kpc)	Q_c	$\Delta\nu_{Im}$	e - folding time (yr)
(A)	1.0	0.9	-0.53×10^{-1}	1.29×10^{9}
(B)	1.0	0.8	-1.03×13^{-1}	0.66×10^{9}
(C)	2.0	0.9	-1.24×10^{-1}	0.55×10^{9}
(D)	2.0	0.8	-2.00×10^{-1}	0.34×10^{9}

For a shallow potential well as in (A), the e-folding time of the axisymmetric wave is 1.29×10^{9} years. Such a time scale is much longer than the rotation period of our Galaxy (2×10^{8} years). On the other hand, the growth time for the spiral waves is probably less than 10^{9} years. We, therefore, conclude that this axisymmetric wave is essentially stable. For the same width but deeper potential well (c.f. (B)), the e-folding time is shorter. Similarly, this growth time gets shorter for a wider potential well with the same depth (c.f. (A) and (C)). Thus, this shows that the axisymmetric wave becomes more unstable as the potential well gets wider and deeper.

The difference between Q and $\frac{dQ}{dx}$ contributing to the instability of the waves must be distinguished. We discuss the meaning of Q in a more detailed manner first. In general terms, it can be said that rotation (i.e., the conservation of angular momentum) stabilizes the tendency of the disk to contract on a large scale. On the other hand, velocity dispersions stabilized small-scale collapses (Jeans instability). If the dispersion velocity is sufficiently large, the combination of dispersion velocity and rotation will be able to stabilize the axisymmetric gravitational collapse on all scales. In his earlier local analysis of axisymmetric disturbances, Toomre (1964) found that the minimum root-mean-square radial velocity dispersion C_{min} ($\equiv \langle \dot{c}_{\varpi} \rangle_{min}$) required in any vicinity of the disk for the complete suppression of all axisymmetric instabilities is $3.36 \frac{G\sigma_*}{K}$. Julian and Toomre (1967) defined the stability parameter Q_* ($\equiv \frac{c}{c_{min}}$) as the ratio of the actual value of the radial velocity dispersion compared to the minimum value required for stability. Lin and Shu (1966) then incorporated this parameter in their study of galactic density wave. Generally speaking, the Q term alone with value equal to or greater than unity suppressing the local gravitational collapse leads to wave stability if there are no other dissipative processes. When Q is marginally stable (perhaps slightly less than unity), the presence of $\frac{dQ}{dx}$ term will stabilize the Jeans instability. On the other hand, instability persists if Q deviates appreciably from unity (i.e. $Q < 1$).

(v) Conclusion

In § IIa we have shown that there is a possible weak axisymmetric Jeans instability as tested by the Lin-Shu dispersion relation in the presence of substantial amounts of gas $\left(\frac{\sigma_g}{\sigma_*} \simeq 20\% \right)$. Under this situation, large wave amplification is possible. We were afraid that the axisymmetric wave (if it is also unstable) might drive Q_o up resulting in only a short period of amplification for spiral waves. Since the region under consideration is very inhomogeneous, we include the gradient term in the equation of the axisymmetric wave. Our analysis of the axisymmetric wave with a gradient term shows that the wave is essentially stable in a medium which according to the local criterion is slightly unstable. Thus, the axisymmetric wave will probably have no effect (or very small effect) on the dispersion velocities. If this is the case, strong amplification can last for a rather long period. This is what we wish to see.

IV. NONAXISYMMETRIC WAVE WITH A GRADIENT TERM

(i) Two Motivations for Studying Spiral Waves With a Gradient Term

There have been indications (Hunter and Toomre, 1969; Rots, 1974; Rogstad et al, 1976) that some of the spiral galaxies, such as the Milky Way System, M81 and M33, have disk components whose outer edges appear to be distorted from planar form. Indeed, one promising explanation of the warp of our galaxy's disks appears to be that it is the remnant of a tidal distortion due to a close encounter with with our nearest neighboring galaxy, the large Magellanic Cloud. The bending structure of this class of galaxies raises interesting possibilities for the generation of spiral structures. As a preliminary conjecture we suggest that it is the strong gradients of equilibrium quantities near the bending which results in spiral wave emission and amplification. If this were true, we might first consider the effect of strong gradients in a plane-parallel disk galaxy. In the following paragraphs we will consider the possible effects of such gradients.

First, Lau et al (1975) showed recently that spiral waves could be directly emitted from a mildly Jeans unstable region. Our preliminary study suggests that a large gradient term could cause similar emissions. The strengths of these self-emitted spiral waves depend on the type of rotation curve of a given galaxy. In general, we expect that the gradient effect will be important for a galaxy with a large gradient of surface density.

Second, we have indications that if the gradient term is large enough, an incident spiral wave will be amplified as it passes through the corotation region (analogous to the 'stimulated effect' effect discussed by Mark 1976). In §I and §II, we have discussed various wave amplification mechanisms. The gradient effect on wave amplification might well compete with these under the appropriate circumstances. The question of which are the dominant mechanisms depends on the type of galaxies we consider.

In the next section, we will outline the mathematical procedure for the study of the gradient effect.

(ii) Equations Governing Emission and Amplification of Spiral Waves in the Presence of Strong Gradients

(a) Self-Emission

Assuming that Q is of a parabolic form as in equation(8), the governing equations for the spiral waves with a gradient term are given by equations (9) and (10) with $\gamma \neq 0$. (Note that Q has been expressed in a specific form for both the axisymmetric and spiral cases only for the sake of convenience. It can equally well be expressed in more general forms.)

The solutions to equation (10) are given by

$$u(x) = A\sqrt{\frac{\gamma}{2}}\left(x+\frac{\Delta\nu}{\gamma}\right) H^{(2)}_{\frac{1}{4}}\left[\frac{\gamma}{2}\left(x+\frac{\Delta\nu}{\gamma}\right)^2\right] \quad , \qquad x > \beta^{-1} \tag{11}$$

$$u(x) = B\sqrt{\frac{\gamma}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right) H^{(2)}_{\frac{1}{4}}\left[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^2\right] + C\sqrt{\frac{\gamma}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right) H^{(1)}_{\frac{1}{4}}\left[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^2\right] , \quad x < -\beta^{-1} \tag{12}$$

where A, B and C are constants and H is a Hankel function. For the self-emission case we have set C = 0 in equation (12) since it represents a driven solution. Instead of making both u(x) and u'(x) continuous at $x = \pm\beta^{-1}$, it is enough to make the ratio u'(x)/u (x) continuous at $x = \pm\beta^{-1}$. The ratios for both regions $x > \beta^{-1}$ and $x < \beta^{-1}$ (see figure 7) are given by

$$\frac{u'(x)}{u(x)} = \frac{3}{2}\frac{1}{x+\frac{\Delta\nu}{\gamma}} - \gamma\left(x+\frac{\Delta\nu}{\gamma}\right)\frac{H^{(2)}_{5/4}\left[\frac{1}{2}\gamma\left(x+\frac{\Delta\nu}{\gamma}\right)^2\right]}{H^{(2)}_{\frac{1}{4}}\left[\frac{\gamma}{2}\left(x+\frac{\Delta\nu}{\gamma}\right)^2\right]} \quad , \quad x > \beta^{-1} \tag{13}$$

$$\frac{u'(x)}{u(x)} = \frac{3}{2}\frac{-1}{-x-\frac{\Delta\nu}{\gamma}} + \gamma\left(-x-\frac{\Delta\nu}{\gamma}\right)\frac{H^{(2)}_{5/4}\left[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^2\right]}{H^{(2)}_{\frac{1}{4}}\left[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^2\right]} \quad , \quad x < -\beta^{-1} \tag{14}$$

The solutions for $\Delta\nu$, which can only assume certain eigenvalues in the region $-\beta^{-1} < x < \beta^{-1}$, may be obtained by numerically integrating the differential equaiton (9) and matching with the boundary conditions given by equaitons (13) and (14).

Mark (1975, private communications) applied this technique for our own galaxy and showed, however, that the gradient term is small. This is not surprising since he used the Schmidt model whose rotation curve is relatively smooth even in the outer galactic regions. On the other hand, the northern and southern rotation curves in the outer regions of the galaxy M81 are very different. This might be due to bending of M81 in the center parts. We intended to simulate one aspect of this bending in terms of strong density gradients in a two-dimensional disk. However, the calculation outlined above has not been completed.

(b) Amplification of Externally Driven Signals Incident on a Corotation Region

For an externally driven wave problem, the governing equations are still (11) and (12). Equation (11) represents a solution for an outgoing wave while equation (12) represents a solution for both an incident and an outgoing wave. In the regions $x > \beta^{-1}$ and $x < -\beta^{-1}$, the ratios $u'(x)/u(x)$ become

$$\frac{u'(x)}{u(x)} = \frac{3}{2}\,\frac{1}{x+\frac{\Delta\nu}{\gamma}} - \gamma\left(x+\frac{\Delta\nu}{\gamma}\right)\frac{H^{(2)}_{5/4}\left[\frac{\gamma}{2}\left(x+\frac{\Delta\nu}{\gamma}\right)^2\right]}{H^{(2)}_{1/4}\left[\frac{\gamma}{2}\left(x+\frac{\Delta\nu}{\gamma}\right)^2\right]}\,, \quad x > \beta^{-1} \qquad (15)$$

$$
\begin{aligned}
\frac{u'(x)}{u(x)} = -2\left(\frac{\gamma}{2}\right)^{\frac{1}{2}} \Big\{ & B\Big[\frac{3}{4}\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{-1} H^{(2)}_{\frac{1}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big] - \left(\frac{\gamma}{2}\right)^{\frac{1}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right) H^{(2)}_{\frac{5}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big] + \\
& + C\Big[\frac{3}{4}\left(\frac{\gamma}{2}\right)^{-\frac{1}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{-1} H^{(1)}_{\frac{1}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big] - \left(\frac{\gamma}{2}\right)^{\frac{1}{2}}\left(-x-\frac{\Delta\nu}{\gamma}\right) H^{(1)}_{\frac{5}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big]\Big\} \times \\
& \times \Big\{ B H^{(2)}_{\frac{1}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big] + C H^{(1)}_{\frac{1}{4}}\Big[\frac{\gamma}{2}\left(-x-\frac{\Delta\nu}{\gamma}\right)^{2}\Big] \Big\}^{-1}
\end{aligned}
\tag{16}
$$

Clearly, equation(15) differs from equation(14) in the self-emission case by an extra term $H^{(1)}_{1/4}$ which corresponds to the solution of the incident wave. The solution for this driven wave problem ($C \neq 0$) is again obtained by numerically integrating the differential equaiton(9) in the region $-\beta^{-1} < x < \beta^{-1}$ and matching with the boundary conditions given by equaitons (15) and (16). As is evident from equaiton(16), there is always a solution for any $\Delta\nu$ by picking the constants C/B properly. Thus the driven wave problem is not an eigenvalue problem. The ratio C/B determines the ratio amplitudes and angular momentum for the two waves in the region $x < \beta^{-1}$. (Mark, 1975b)

Again the numerical solution outlined above has not been completed.

(iii) Discussion

The above suggests that the gradient effect for spiral galaxies might allow the direct emission of spiral density waves of certain specific discrete frequencies. It might also contribute to the amplification of a continuous spectrum of incident signals whose pattern frequencies result in corotation circles within the region of strong gradients. Either of these effects might be important for the determination of wave frequencies in selected galaxies. Observations suggest that spiral wave frequencies are not random but correlated with galaxy type (Roberts et al, 1975). Thus Mark (1973 and later papers) proposed that 'amplifiers' of density waves are needed for long term maintenance of spiral waves and for specific determinations of wave frequencies. In his theory of amplification process with an incident long wave, the gradient term is ignored (assumed small) and the wave pattern is determined at the corotation radius where the short and long waves have similar wavelengths. For a non-vanishing gradient term located around the corotation region, we should expect more amplification of the spiral waves and hence the possibility of a new effect which results in even more sensitive dependence of the amount of amplification on wave frequency. This selective amplification could affect the determination of the pattern frequency. The above consideration applies similarly to those generation mechanisms based either on the driving influence of bar-like and oval distortions of galaxies or on the influence of external companions. For the self-emission case which does not involve internal or external driving agents, the pattern frequency is completely determined by the gradient term.

In our discussion of the gradient effect for spiral galaxies, we have neglected the effect of their finite thicknesses. If we allow for such an effect by a corresponding adjustment in the critical dispersive velocities, the gradient term will be different. Indeed, the thickness correction is important for galaxies with large bending. Further work is required in this direction.

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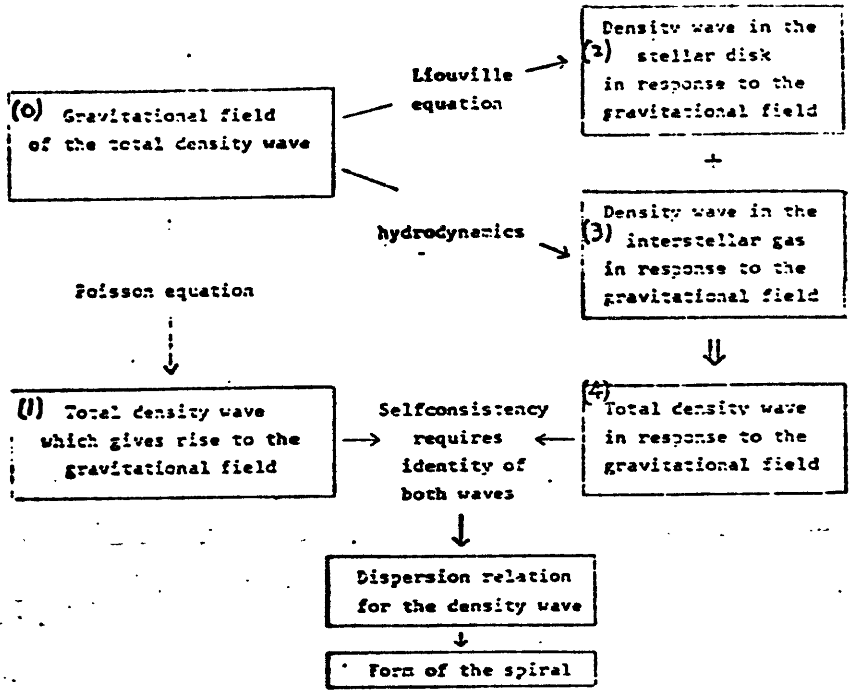


FIG. 1 — Lin's solution scheme.

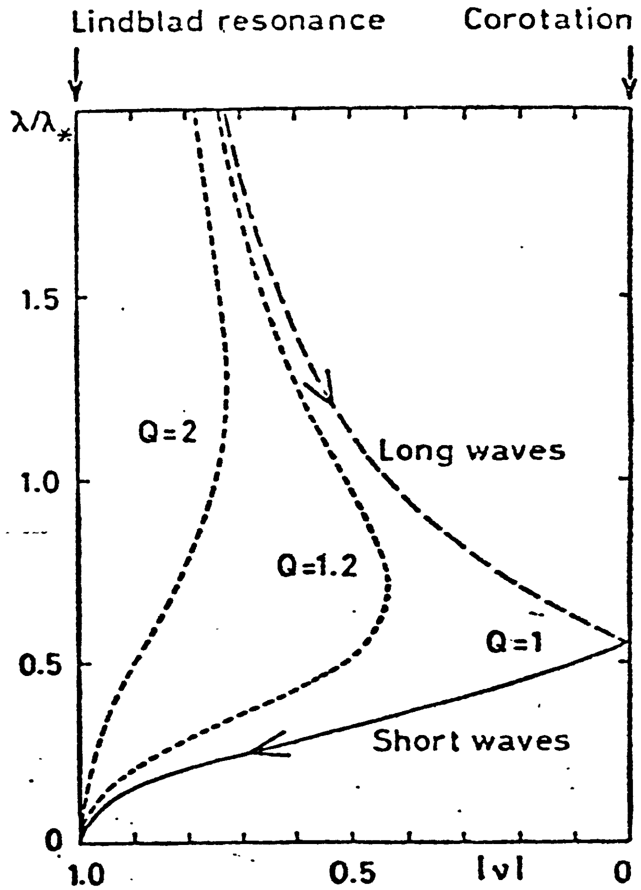


FIG. 2 — Dispersion relation λ(ν) for density waves (Lin and Shu). The arrows show the direction of propagation of the waves. Q measures the velocity dispersion.

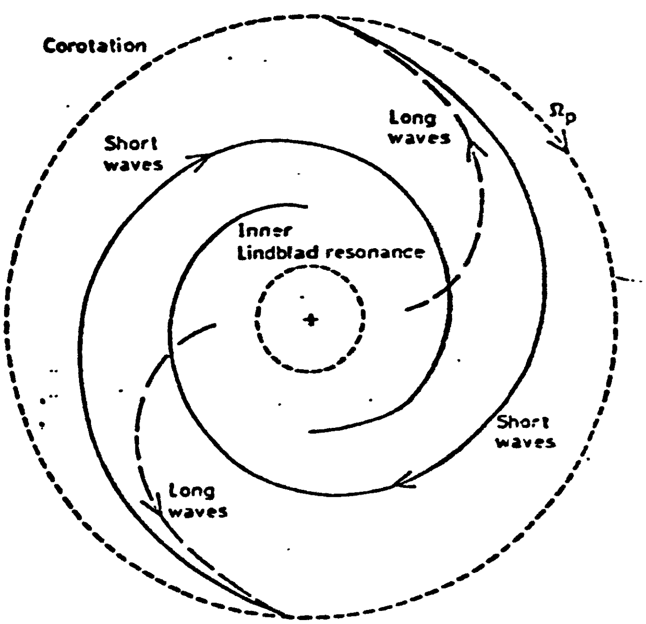


FIG.3 — Schematic spiral structure of a galaxy with superposition of short and long waves (Shu et al.).

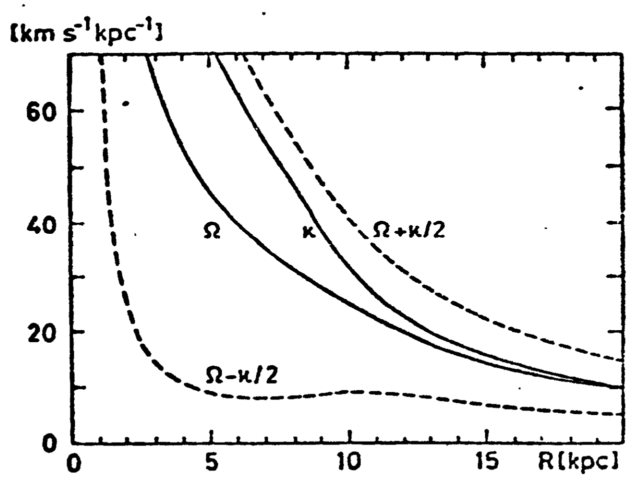


FIG. 4 — Rotational frequency Ω and epicyclic frequency κ in our Galaxy.

SCHMIDT MASS MODEL (1965)

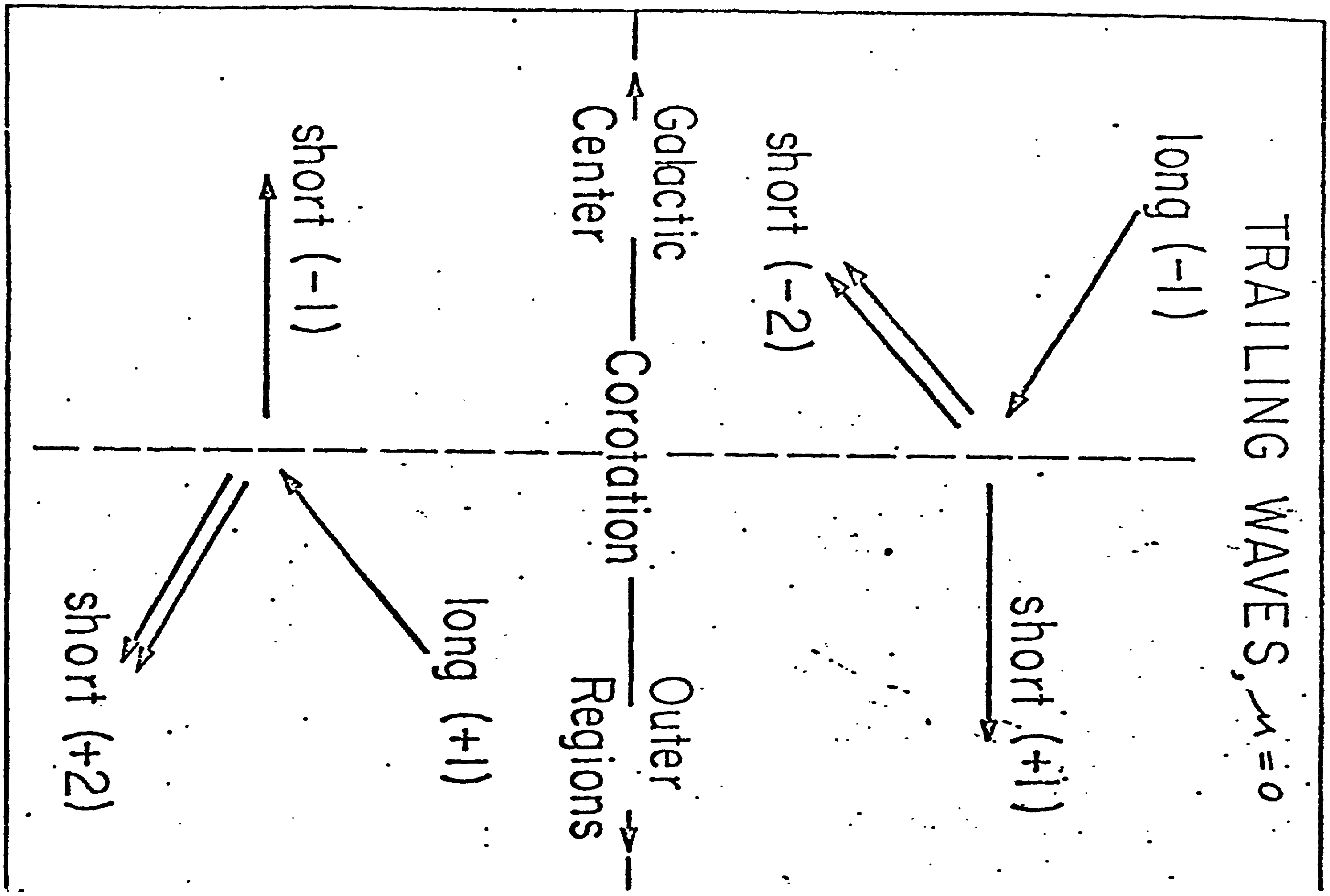
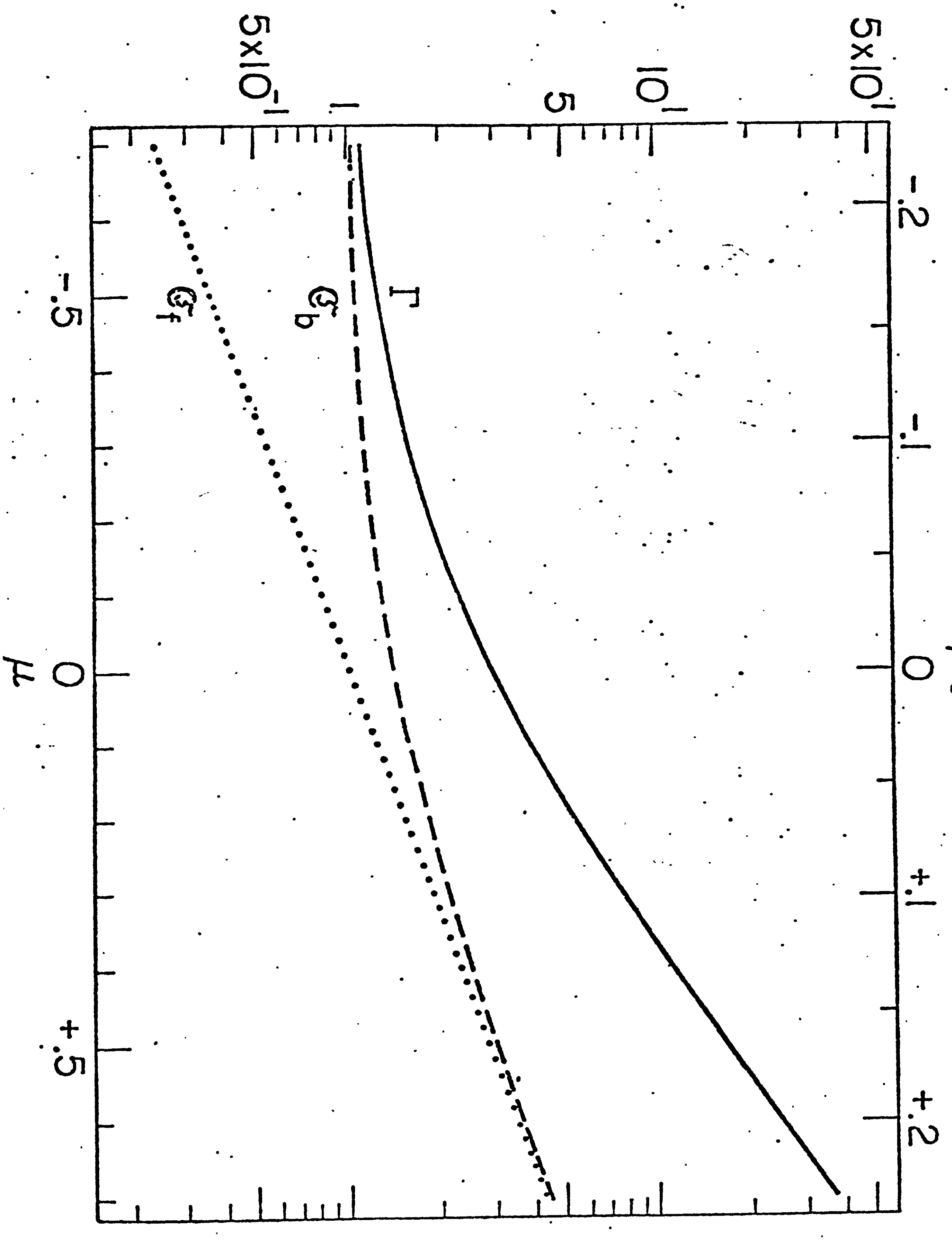


Figure 5 - Schematic diagrams illustrating the basic processes involving three interacting waves at corotation. Arrows indicate the direction (group velocity) and the intensity (number of arrows) of the waves. Intensities are measured by the amount of angular momentum per unit time carried into or out of corotation by the waves in the direction of their respective group propagation. They are also indicated in arbitrary units by the signed numbers that follow the designations "long" or "short" which identify the waves as members of the long or short wavelength branches. Wave angular momentum is conserved in this interaction. This case has $\mu_j = \mu_k = \rho = 0$.

Figure 6 - For processes involving three waves, this figure gives the total amplification factor Γ of the wave amplifier versus the amplification parameter μ (eq. 1). Also given are $|\ell_f|$ and $|\ell_b|$ which are respectively the amplitudes of the forward and backward emitted waves relative to the incident one. For galaxy models with a common rotation curve and a fixed total surface density of disk matter, the local surface density ratio of gas to stars can still vary together with the dispersive velocities. Under this circumstance, the parameter μ_s is more useful and it is given here as the top scale using the 1965 Schmidt model of our galaxy and for wave pattern frequency $\Omega_p = 13.3$ km s^{-1} kpc^{-1}.

$$\mu_s = \left\{ \left(1 + \frac{\sigma_g}{\sigma_*}\right) \frac{1}{Q_*} \left[(1 - Q_*) + \frac{1.822\ \sigma_g/\sigma_*}{1 + 0.944 \left(\frac{a}{c}\right)^2} \right] \right\}$$

Fig.6 AMPLIFICATION PROCESSES



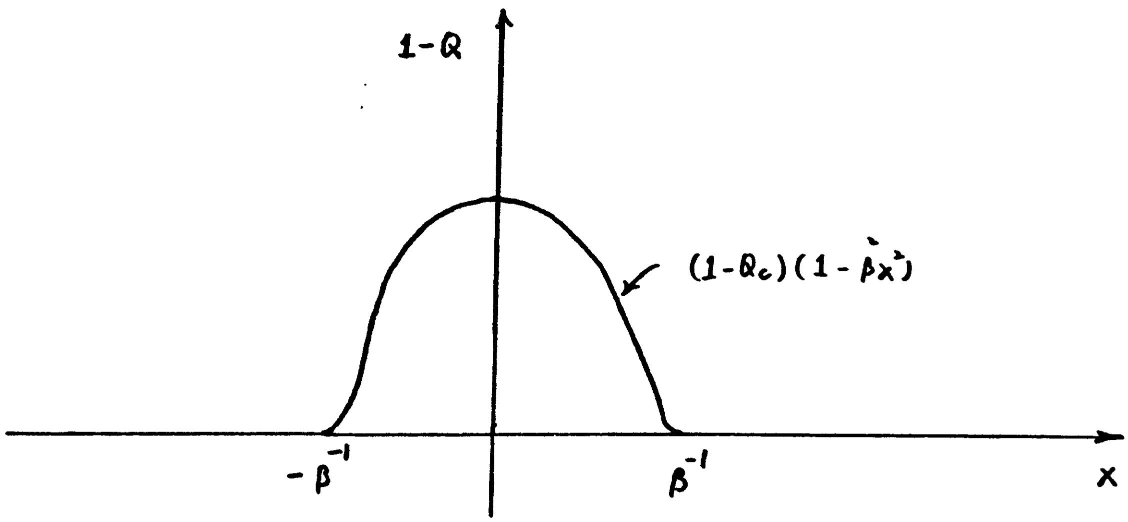


Figure 7. Q vs x

APPENDIX A

Summary of Density Wave

The winding dilemma associated with the spiral galaxies is well known. If the spiral arm contains the same material, it will wind up over a time scale of several revolutions. This is because the inner part of the galaxy rotates faster than the outer part. Moreover, the galactic magnetic field must steadily increase if the material arm winds up more tightly. The difficulties disappear if we identify the spiral structure with the crest of a wave pattern.

Lin and his co-workers considered a special mode of density wave in the form of a rigidly rotating, neutral tightly wound spiral. Such a density wave may be considered as a perturbation in the spatial distribution and in the motion of the material of an otherwise rotationally symmetric and stationary galaxy. The perturbation V_1 that this density wave causes in the gravitational potential V of a galaxy is given by

$$V_1 = A(\varpi)\, e^{i(\omega t - m\theta + \Phi(\varpi))} \tag{A1}$$

where (ϖ, θ) are the plane polar coordinators and $\Phi(\varpi)$ is the radial phase of the wave. Then the spiral pattern at each instant t has m arms. The lines of constant ReV are approximately given by the equation

$$m(\theta - \theta_0) = \Phi(\varpi) - \Phi(\varpi_0) \tag{A2}$$

which represents a spiral pattern with m arms. We assume that $\Phi(\varpi)$ varies rapidly and monotonically with ϖ in order to make a tightly wound spiral. The pattern given by (A1) rotates with an angular velocity $\Omega_p = \frac{\omega_r}{m}$ where ω_r is the real part of ω. The spacing between the arms is given by the "wave length" $\lambda = \frac{2\pi}{|k|}$ where $k = \Phi'(\varpi)$. If the motions of the stars is in the direction of increasing θ, trailing waves correspond to $k(\varpi) < 0$, and leading waves to $k(\varpi) > 0$.

As in any wave theory, a dispersion relation relating the wavelength λ and frequency ν is important. We outline Lin's method to obtain the dispersion relation. Lin and Sha consider a purely gravitational theory including both the gravitational field of the stars and of the gas. The density waves of a spiral structure is assumed to be self-sustained at a small but finite amplitude. If a wave were maintained, we would expect that there is a spiral gravitational field. In Fig. 1 the scheme for calculating the properties of the density wave is indicated. The resultant gravitational field (o) must be associated, according to Poisson's equation, with a certain distribution of matter (1), which consists of gas and stars. The distribution of stars (2) is calculated through the Lionville equation while the distribution of gas (3) is obtained by hydrodynamics. Since galaxies are almost isolated systems, we must require self-consistency: the sum of these two distributions yields a total distribution of matter (4) that must be identical with the density distribution (1) which is needed to give rise to the field. The dispersion relation obtained by the above scheme gives a connection between local wavelength $\lambda(\varpi)$ and the local relative frequency $\nu(\varpi)$ of the wave. Here $\nu = \frac{m(\Omega_p - \Omega)}{\kappa}$ discribes the frequency of passage of spiral arms past an observer who corotates with the material, measured in units of the epicyclic frequency. Throughout the calculation WKBJ approximation (tight spiral) is used. The dispersion relation for the stellar disk (assuming a Schwarzschield distribution of velocities) is

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G \mu |k| \tag{A3}$$

where G is the gravitational constant, and $\mu = \sigma_* \mathcal{F}_\nu(x)$.
Here $\sigma_*(\varpi)$ is the stellar surface density of the infinitesimally thin disk and $\mathcal{F}_\nu(x)$ is the reduction factor defined by

$$\mathcal{F}_\nu(x) = \frac{1-\nu^2}{x} \left\{ 1 - \frac{\nu\pi}{\sin\nu\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x(1+\cos s)} \cos\nu s \, ds \right\} \tag{A4}$$

where $x = \frac{k^2 c^2}{\kappa^2}$ and c^2 ($\equiv \langle c_\varpi^2 \rangle$) is the stellar dispersion velocity. The dispersion velocity is important in the spiral theory in that a density wave will be destroyed if it is not able to suppress the local gravitational collapse. Toomre (1964) has shown that a galaxy is locally stable against gravitational collapse if the local dispersion of the peculiar stellar velocities in the radial direction is greater than the critical value $c_{min} = \frac{3.36 G\sigma_*}{\kappa}$. Thus the ratio Q_* between the actual and critical velocity dispersion will insue a local stable region when it is equal to or greater than unity. Lin incorporated this factor into his dispersion relation which is shown in Fig. 2 for several values of Q_* . If we include the gaseous component of the disk, we replace (A3) by

$$\mathcal{M} = \sigma_* \mathcal{F}_\nu(x) + \sigma_g \mathcal{F}_\nu^{(g)}(x_g) \quad . \tag{A5}$$

In this equation, σ_g is the surface density of the gas, and

$$\mathcal{F}_\nu^{g)}(x_g) = [\, 1 + \frac{x_g}{1-\nu^2}\,]^{-1} \tag{A6}$$

where $x_g = \frac{k^2 a^2}{\kappa^2}$ and a is the acoustic speed of the gas. An equivalent version of (A3) for both the stellar and gaseous disk is

$$\frac{k_*}{|k|}(1-\nu^2) = \mathcal{F}_\nu(x) + \frac{\sigma_g}{\sigma_*}\mathcal{F}_\nu^{(g)}(x_g) \tag{A7}$$

where $k_* = \frac{\kappa^2}{2\pi G \sigma_*}$ is a dimensionless wavenumber.

We list below a number of properties of the dispersion relation without proof. For detail interested readers are referred to the original papers.

1. For the stellar disk solutions for density waves exist for $\nu^2 < 1$ i.e., $\Omega - \frac{\kappa}{m} < \Omega_p < \Omega + \frac{\kappa}{m}$
Note that the inner Lindblad and outer Lindblad resonance corresponds to $\nu = \mp 1$ or $\Omega_p = \Omega \mp \frac{\kappa}{m}$ respectively. The density waves are contained within the circles for which ϖ fulfills the condition $\Omega_p = \Omega \pm \frac{\kappa}{m}$. Moreover, their interactions between waves and (resonant) stars (c.f. item (4)) in these particular regions. For $\nu = 0$ or $\Omega = \Omega_p$, we have the corotation resonance where stars and waves corotate and interact. The graph of $\Omega(\varpi)$, $\kappa(\varpi)$, and $\Omega \pm \frac{\kappa}{2}$ of our Galaxy on the basis of Schmidt Model (1965) is shown in Fig. 4 . For any given value of the wave pattern speed up, the limiting radii of the Lindblad regions

can now be read off.

2. At Lindblad resonance $|\nu| = 1$, $\lambda = 0$ corresponds to ring structure at the center of the galaxy.

3. In the next order approximation, Shu (1970) obtained the amplitude distribution of the density waves as a function of galacto-centric distance. The amplitude distribution in the propagation region (between resonance regions) observes a principle of conservation of density of wave action (Toomre, 1969; Shu, 1970) and breaks down at the resonance regions.

4. Mark (1971; 1974b) removed the infinite amplitude difficulty in the Lindblad resonance by including the resonant star effects and showed that the waves are actually absorbed there. We remind the readers that there are interactions between the waves and stars in all the resonant regions (Lynden -Bell and KALNAJS 1972). In particular, the resonant stars emit angular momentum in the inner Lindblad regions while absorb angular momentum in the corotation regions.

APPENDIX B

The " Stability " Parameter Q_0

The Lin-Shu dispersion relation for a combined gaseous and stellar disk is given by

$$\frac{k_*}{|k|}(1-\nu^2) = \mathcal{F}_\nu(x) + \frac{\sigma_g}{\sigma_*}\mathcal{F}_\nu^{(g)}(x_g) \qquad (B1)$$

which connects $|k|$ and ν for a given galactic model (c.f. Appendix A). In the above equation, ν is the intrinsic frequency of the wave, $k_* = \frac{\kappa^2}{2\pi G\sigma_*}$ and $\mathcal{F}^{(g)}(x_g) = \frac{1}{1+x_g/1-\nu^2}$. The rest of the symbols may be found in Appendix A. In (B1) we set $\nu=0$ for the determination of the marginal condition for neutral oscillations at a given location. The result can be expressed as

$$\frac{a\kappa}{2\pi G\sigma}\left(1+\frac{\sigma_g}{\sigma_*}\right) = \frac{a}{c}x^{\frac{1}{2}}\mathcal{F}_0(x) + \frac{\sigma_g}{\sigma_*}\frac{\frac{a}{c}x^{\frac{1}{2}}}{1+\left(\frac{a}{c}\right)^2 x} \qquad (B2)$$

where a and c are the gaseous and stellar dispersion velocities; $\frac{\sigma_g}{\sigma_*}$ is the ratio between gaseous and stellar surface density; σ is the total surface density; κ is the epicylic frequency; and $x = \frac{k^2c^2}{\kappa^2}$. Thus, if the right hand side (r.h.s.) is less (greater) than the left hand side (l.h.s.), the wave is stable (unstable). For a fixed total surface density σ and a given dispersion velocity $\frac{a}{c}$, the ratio $\frac{\sigma_g}{\sigma_*}$ is an important factor to the stability of the wave. Generally speaking, $\frac{\sigma_g}{\sigma_*}$ is comparable to each other in the outer part of the galaxy. We want to determine the local stability of the wave at distance 14 kpc from the galactic center in our Galaxy. The surface density of neutral hydrogen at this distance is about 7 $\frac{M_\odot}{pc^2}$ (c.f. Mezger 1970) while the stellar density from the Schmidt model (1965) is about 35 $\frac{M_\odot}{pc^2}$ (see Fig. 4). This amounts to a 20% of gas density over stellar density locally. We choose $\frac{a}{c}$ to be $\frac{7}{30}$. The other parameters can be read off from the Schmidt model. The remaining undetermined variable in (B2) is x. We choose x such that $x^{\frac{1}{2}}\mathcal{J}_0(x)$ is a maximum. This occurs when $x = x_c\ (\simeq 0.9481)$. For this value of x, $x_c^{\frac{1}{2}}\mathcal{J}_0(x_c) \simeq 0.5345$. We substitute these values in (B2). The l.h.s. of (B2) gives 0.1329 while the r.h.s. yields 0.1685. The wave is therefore slightly unstable. We define a local "stability" parameter for the combined gaseous and stellar disk, Q_0, which includes the effects of both dispersion velocity and gas density. Q_0 is given by

$$Q_0 = \frac{aK}{2\pi G\sigma}\left(1+\frac{\sigma_g}{\sigma_*}\right) \Big/ \frac{a}{c}x^{\frac{1}{2}}\mathcal{J}_0(x) + \frac{\sigma_g}{\sigma_*}\frac{\frac{a}{c}x^{\frac{1}{2}}}{1+(\frac{a}{c})^2 x}\Bigg|_{x=x_c} \qquad (B3)$$

obtained from (B2), or explicitly

$$\frac{1}{Q_0} = \frac{1}{Q_*}\left[1+\frac{1.822\,\frac{\sigma_g}{\sigma_*}}{1+0.9481(\frac{a}{c})^2}\right] \qquad (B4)$$

We also find that is related to Mark's amplification parameter μ by

$$\mu = \frac{D}{Q_*}\left\{(1-Q_*) + \frac{1.822}{1+0.9481(\frac{a}{c})^2}\frac{\sigma_g}{\sigma_*}\right\} \qquad (B5)$$

where D is some constant. The significance of μ is discussed in the main text. Briefly, wave amplification is quite sensitive to the changes in the ratio $\frac{\sigma_g}{\sigma_*}$ and the local stability parameter Q_*. Indeed, large amplification is possible if $Q_* < 1$ or if $Q_* \simeq 1$ but $\frac{\sigma_g}{\sigma_*}$ is appreciable. This situation corresponds to $\mu > 0$.

If we assume Q_* constant, it is possible that μ (or $\frac{1}{Q_0} - 1$) has some peak value somewhere in the galaxy provided that $\frac{\sigma_g}{\sigma_*}$ is appreciable there. We demonstrate this possibility for our Galaxy. In (B2) we set $Q_* \simeq 1$ and $\frac{a}{c} \simeq \frac{7}{30}$ everywhere. The first assumption implies that the stellar disk is marginal stable against axisymmetric Jeans instability. The second approximation does not change μ much because $\frac{a}{c}$ appears in the denominator of (B5). Strictly speaking, we should set $C(\varpi) = C_{min}(\varpi)$ according to the first assumption. The values of σ_g and σ_* may be obtained from Van Woerden (,1965; c.f. Mezger,1970) and the Schmidt model respectively. Note that for the gas density we consider only the neutral hydrogen and neglect other components of the gas (c.f. Fig. B1). We plot ($\frac{1}{Q_0} - 1$) as a function of galacto-centric distance in Fig.B2. Clearly, ($\frac{1}{Q_0} - 1$) or μ has a maximum value centered at around 14 kpc. Let us estimate the total amplification factor Γ (the ratio of the sum of magnitudes of wave angular momentum fluxes in emergent waves to the sum of magnitudes of wave angular momentum fluxes in incident waves) for some value of $\frac{\sigma_g}{\sigma_*}$. We assume that $Q_* \sim 1$ and take $\frac{\sigma_g}{\sigma_*} \sim 5\%$. For our Galaxy, D=3. Thus, μ =0.26. We obtain $\Gamma \sim 7$ from Fig. 6 . This is a large amplification. Under the same assumption, $\mu \sim 1$ if $\frac{\sigma_g}{\sigma_*} \sim 20\%$. In this case, the total amplification factor Γ is even bigger. According to the local analysis of stability, the presence of appreciable amount of gas could lead to Jeans axisymmetric instability. Thus, the axisymmetric wave might be unstable. In a more detail analysis, we will show (main text) that this is not true for a marginal unstable axisymmetric wave.

Note added in proof.--We have also obtained a similar plot of ($\frac{1}{Q_0} - 1$) as a function of galacto-centric distance in the galaxy M31. The rotation curve is that of Rubin and Ford (1970) while the neutral hydrogen density is taken from Burke et al. (1964).

APPENDIX C

Wave Amplification at Corotation

Wave angular monentum (energy) inside corotation circle (where the wave pattern angular frequency equal to that of the material) is negative while that outside the corotation is positive. Wave amplification depends on the fact that the waves inside corotation are amplified when angular monentum is removed from that region while those waves outside of corotation have the opposite behavior in that they are amplified on receiving angular monentum. In the corotation region where several waves coupled together, amplification of spiral waves in both regions then occur because of an exchange of angular momentum. This coupling is achieved by an incident wave (long wave) which may be due to bar-like and oval distortions of the central regions of the galaxy or due to external companions. The amplification is further enhanced by the interaction of waves with resonant stars at the corotation region. Briefly, the resonant stars in the corotation region absorb angular momentum from the waves and thus amplify the wave. In the propagation region inside the corotation circle, the stars of the bulge-halo subsytem during their passage to the disk may take away angular momentum from the waves. Mark found that this amplification mechanism is important.

(The following is adapted from Mark's article)

In the neighborhood of corotation, the picture of a single propagating wave is no longer valid. Rather, at this region , several density waves of equal frequencies and nearly equal "wavenumbers " interact strongly even within the context of a linearized wave analysis (for details cf. Mark 1975a). The physical situation is more clearly seen by a discussion of the case where the disk is neutrally stable to local axisymmetric Jeans instability(Toomre 1964). Also resonant particle and disk-halo interactions are temporarily omitted for simplicity. Figure 5 shows the most simple configurations which involve three interacting waves, the incident signal and the two emitted waves. If the sense of winding is trailing, the upper left frame is for the case where the incident signal is a long wave due to bar-like or oval distortions in the central regions of galaxies. The lower left frame is the "mirror image" situation where the long wave is incident from the outer regions (perhaps due to external companions). In each case, the incident signal stimulates the emission of two short waves. The arrows indicate directions of group propagation while the number of arrows is proportional to the magnitude of angular momentum carried per unit time by each wave. The waves inside corotation are amplified by removal of angular momentum so that their angular momentum densities are negative. For the waves outside of corotation this density is positive. Thus even though there is an overall conservation of this quantity, the backward emitted (same side as incident one) wave is amplified relative to the incident signal.

As we can also see from this Figure 5, there is a systematic net outward flow of angular momentum of (+ 1) units per unit time (arbitrary units used here). Thus amplification proceeds at the expense of an outflow of angular momentum in the galaxy. We can also say that "redistribution" is a means whereby galaxies convert the angular momentum of circular rotation into that of the density waves. This is in rough correspondence with the expectations of Lynden-Bell et. al. (1972). However, our three-wave interaction process is very different from the resonant particle emission processes they described (they did briefly mention the "tunnelling" through corotation of a single long wave., but for dispersive velocities so high that it can hardly be related to our present effect). We also find that not all trailing waves carry angular momentum outwards in the galaxy. The long trailing waves do not.

In general, these wave-wave interactions at corotation together with the interactions with resonant stars and halo stars are all described by the turning point equation (Mark 1975c)

$$\frac{d^2 w(\varsigma)}{d\varsigma^2} + \left[\varsigma^2 + 1{\cdot}323\,(\mu_d + \mu_h) + \rho(\varsigma)\right] w(\varsigma) = 0, \tag{5}$$

where ς is a scaled variable measuring distance from corotation, $w(\varsigma)$ is the non-dimensional form of the "reduced wave function" of the waves , and $\rho(\varsigma)$ is a term giving the effects of interaction of density waves with resonant stars at corotation (for detailed formulae, cf.

Mark 1975b). The disk amplification parameter μ_d is the number

$$\mu_d = \left[\frac{\kappa^2/m}{c_* \,|d\Omega/d\varpi|}\left(1 - \frac{c_* \kappa}{3.358\, G \Sigma_*} + \frac{1.822\, \Sigma_g/\Sigma_*}{1 + 0.8481\, a_g^2/c_*^2}\right)\right]_{\varpi = r_c}, \quad (6)$$

where in addition to the symbols earlier defined, $\Sigma_*(\varpi)$ and $c_*(\varpi)$ are the surface mass density and radial dispersive velocity of the stars in the disk; G is the gravitational constant; $\Sigma_g(\varpi)$ and $a_g(\varpi)$ are the surface mass density and acoustical velocity of an equivalent one-phase gaseous medium (cf. Lin and Shu 1964). The amplification parameter μ_h describes the effects of interactions with the halo stars,

$$\mu_h = \left[\frac{3\,\kappa^3 D_h}{m\,\Sigma_*\, k^2\, v_h^2}\left|\frac{d\Omega}{d\varpi}\right|^{-1}\right]_{\varpi = r_c} M(\nu_{hc}), \quad (7)$$

where $k(\varpi)$ is the wavenumber of the Lin-Shu dispersion relation;

$$\nu_{hc} = \left(\frac{3}{2}\right)^{\frac{1}{2}} \frac{m\,\Omega_p}{v_h(r_c)\,|k(r_c)|}; \quad (8)$$

and $M(\eta)$ is a complex-valued function (Mark 1975c) which for $\eta \ll 1$ is approximately $M(\eta) \cong 1 - i\, 8\,\eta/(3\pi^{\frac{1}{2}})$.

The special case discussed earlier in Figure 5 corresponds to the case $\mu_d = \mu_h = \rho = 0$. The omission of the resonant particle term $\rho(s)$ is justified in those situations where non-linear saturation effects (Contopoulos 1972, 1973) have set in. Let us also briefly discuss the separate effects which arise if each of these terms are in turn taken to be non-zero. Corresponding to the forward (or backward) emitted waves which convects in the same (or opposite) direction as the incident signal, the forward (or backward) amplification factor Γ_f (or Γ_b) is the magnitude of angular momentum flux in this emitted wave relative to unit flux in the incident wave. Since these emitted waves propagate into different regions of the galaxy in question, they contribute to the wave angular momentum of these separate regions. Therefore we may also define the total amplification factor $\Gamma = \Gamma_f + \Gamma_b$. These amplification factors are evaluated by matching the solution of equation (5) for large $|\xi|$ with the WKBJ solutions outside of this corotation region. The angular momentum fluxes are evaluated using these WKBJ solutions and the formula for this flux given by Mark (1974).

If $\mu_d \neq 0$ but the effects of μ_h and ρ are still omitted, these amplification factors are given in Table 1. Clearly, higher amplifications are possible if $\mu_d > 0$. This amplification is still due to exchange of angular momentum between the emitted waves so that this quantity is conserved among the waves (as shown by $\Gamma_b - \Gamma_f - 1 = 0$). The presence of resonant stars (of the disk) introduces a non-conservative effect which increases the amplification of the waves inside corotation. For the $\mu_d = \mu_h = 0$ case, modifications of wave amplification factors are on the average of about 30% (using the Schmidt 1965 model of our galaxy and for $\Omega_p = 13.3$ km s^{-1} kpc^{-1}).

If we keep $\mu_d = \rho = 0$ and consider only the effects of the disk-halo interactions given by μ_h, then Table 2 gives the amplification factors as a function of corotation radius r_c or pattern frequency Ω_p for the simple disk-halo model discussed in Section 3. (Mark's article, 1974) The much larger amplifications at smaller corotation radii (higher pattern frequencies) is due partly to the central concentration of the halo and partly due to the near uniform rotation in the inner regions of this galaxy. In this Table 2, the quantity ($\Gamma_h - \Gamma_f - 1$) indicates the amount of wave angular momentum contributed by the halo stars. Thus for $r_c \lesssim 3$ kpc, this can be the major contribution to the wave angular momentum in the backward emitted wave. Also the quantity $(k_c r_c)^2 = [k(r_c)\, r_c]^2$ must be larger than $m^2 = 4$ in order for the approximations at corotation to be valid.

APPENDIX D

DERIVATION OF SPIRAL WAVE WITH A GRADIENT TERM

The short and long wavebranches of the Lin-Shu dispersion relation have almost the same wavelengths at the corotation region (Fig.3). We expect that strong interactions between these wavebranches occur between the wavelengths of these waves are almost equal to each other in the surroundings of the corotation region. The wave amplitude obtained by Shu (1970) is infinite at the corotation region as he assumed a single wave picture. Mark (1975a) removed the ambiguity by considering a serveral a several interacting waves picture. He also took account of the resonant stars effect in the corotation region. He obtained a relation for the surroundings of the corotation region

$$\mathcal{L}(y,\nu,\tilde{\varpi}) = \mathcal{M}(y,\nu,\tilde{\varpi}) + \eta(y, q+i\gamma,\tilde{\varpi}) \tag{D1}$$

where

$$\mathcal{L}(y,\nu,\tilde{\varpi}) = 2y_T(\tilde{\varpi}) - \frac{2S_x\, y(\tilde{\varpi})\, \mathcal{F}_\nu(y^2)}{1-\nu^2(\tilde{\varpi})} \tag{D2}$$

$$\mathcal{M}(y,\nu,\tilde{\varpi}) = i\,\varepsilon(\tilde{\varpi})\frac{y_T(\tilde{\varpi})}{y(\tilde{\varpi})}\frac{d}{d\tilde{\varpi}}\left[\tilde{\varpi} - 2\tilde{\varpi}S_x\frac{y(\tilde{\varpi})}{y_T(\tilde{\varpi})}\frac{\mathcal{F}_\nu(y^2)}{1-\nu^2(\tilde{\varpi})}\mathcal{D}_\nu(y^2)\right] + \frac{2S_x y(\tilde{\varpi})}{1-\nu^2(\tilde{\varpi})+x_g(\tilde{\varpi})}\frac{\sigma_g(\tilde{\varpi})}{\sigma_*(\tilde{\varpi})} \tag{D3}$$

$$\eta(y, q+i\gamma,\tilde{\varpi}) = \frac{i\,\varepsilon(\tilde{\varpi})\, m\, 2^{\frac{3}{2}}\,\Omega(\tilde{\varpi})\,L}{S_x\, y(\tilde{\varpi})\, K(\tilde{\varpi})\,\tilde{\varpi}}\int_0^\infty \exp\left[-\lambda^2 - i(q+i\gamma)\lambda - u\right] \times$$

$$\times\left\{I_0(u)\frac{d\ln}{d\ln\tilde{\varpi}}\left(\frac{\Omega\sigma_*}{Kc^2}\right) + \left[(1-\lambda^2-u)\,I_0(u) + u\,I_1(u)\right]\frac{d\ln}{d\ln\tilde{\varpi}}\left(\frac{c^2}{K}\right)\right\}d\lambda \tag{D4}$$

with $y_T(\tilde{\varpi}) = \dfrac{c(\tilde{\varpi})K(\tilde{\varpi})}{2\pi G\sigma_*(\tilde{\varpi})}$; $y(\tilde{\varpi}) = X(\tilde{\varpi})\,\tilde{\varpi}\,\varepsilon(\tilde{\varpi})$; $D_y = \left|\dfrac{d^2}{dy^2}\dfrac{y\mathcal{F}_\nu(y^2)}{1-\nu^2}\right|_c$; and $D_\nu = \left|\dfrac{d^2}{d\nu^2}\dfrac{y\mathcal{F}_\nu(y^2)}{1-\nu^2}\right|_c$.

In these equations, some of the important terms are defined as follows

$c(\tilde{\varpi})$ = velocity dispersion of star ,

$K(\tilde{\varpi})$ = epicyclic frequency of star,

$\Omega(\tilde{\varpi})$ = angular frequency of star,

Ω_p = wave pattern speed ,

$\nu = \dfrac{m(\Omega_p-\Omega)}{K} = \dfrac{\omega - m\Omega}{K}$, relative wave frequency ,

$\sigma_*(\tilde{\varpi}), \sigma_g(\tilde{\varpi})$ = stellar and gaseous surface density respectively ,

$X(\tilde{\varpi})$ = complex wavenumber at corotation ,

$\varepsilon(\tilde{\varpi}) = \dfrac{c(\tilde{\varpi})}{\tilde{\varpi}K(\tilde{\varpi})}$, amplitude of epicycle , and

$\mathcal{F}_\nu(y)$ = reduction factor defined in (A5).

The rest is defined in Mark's paper (1975a). The right hand side of (D1) contains terms which are of $O(\varepsilon)$ (epicyclic approximation) smaller than the term $\mathcal{L}(y,\nu,\varpi)$ on the left. The term $\eta(y,\frac{3}{2}+i\nu,\varpi)$ contains information on resonant stars. In our derivation we do not consider the resonant stars effect for reason mentioned in the main text. The following derivation is equally good for axisymmetric and nonaxisymmetric waves since m (the number of spiral arms) appears only in ν. For axisymmetric wave we merely set m = 0 (i.e., $\nu=\frac{\omega}{\kappa}$). Note that there is no corotation region in this case. We remind the readers that corotation region is defined as the place where the wave pattern speed Ω_p is equal to the angular speed of the stars (i.e., $\nu=0$ or $\Omega_p=\Omega$) In the neighborhood of the corotation region (m $\neq$ 0) we expand $\mathcal{L}$ and $\mathcal{M}$ of (D1) in power of ν and of $\delta y=\delta k\,\varpi\varepsilon+k_c\,\delta(\varpi\varepsilon)$ with $\delta k=k-k_c$. Here, k is the wavenumber far away from corotation while k_c is the common wavenumber of the wave at corotation. Note that we have allowed the gradient term of $Q_*(\sim\delta(\varpi\varepsilon))$ in the expansion. Since $\mathcal{M}$ is $O(\varepsilon)$ smaller than $\mathcal{L}$, we keep terms in $\mathcal{L}$ to two orders and in $\mathcal{M}$ the lowest order. The results are

$$\mathcal{M}(y,\nu,\varpi)=\mathcal{M}(y,\nu,\varpi)\Big|_{\varpi=r_c}$$

$$\simeq i\,\varepsilon_c\frac{y_T}{y_c}r_c\frac{d}{d\varpi}\left[\left(1-\frac{1}{Q_*}\right)+\frac{y_c}{y_T}D_y\,\delta y\right]$$

$$\mathcal{L}(y,\nu,\varpi)=-\mathcal{L}(y,\nu,\varpi)\Big|_{r_c}+\nu\frac{\partial\mathcal{L}}{\partial\nu}\Big|_{r_c}+(y-y_c)\frac{\partial\mathcal{L}}{\partial y}\Big|_{r_c}+\frac{\nu^2}{2}\frac{\partial^2\mathcal{L}}{\partial\nu^2}\Big|_{r_c}+$$

$$+\nu(y-y_c)\frac{\partial^2\mathcal{L}}{\partial y\,\partial\nu}\Big|_{r_c}+\frac{1}{2}(y-y_c)^2\frac{\partial^2\mathcal{L}}{\partial y^2}\Big|_{r_c}$$

$$=\left[2y_T\left(1-\frac{1}{Q_*}\right)\right]_{r_c}-\nu^2D_\nu+\Big\{(\delta k)^2(\varpi\varepsilon)^2+2k_c(\varpi\varepsilon)\,\delta(\varpi\varepsilon)\,\delta k+$$

$$+k_c^2\left[\delta(\varpi\varepsilon)\right]^2\Big\}D_y \qquad (D5)$$

with $D_y = \left|\frac{\partial^2}{\partial y^2}\frac{y\mathcal{J}_\nu(y)}{1-\nu^2}\right|_c$ and $D_\nu = \left|\frac{\partial^2}{\partial \nu^2}\frac{y\mathcal{J}_\nu(y)}{1-\nu^2}\right|_c$. Here ,the subscript "c" denotes $\tilde{\omega} = r_c$.

After further reduction, (D1) becomes

$$i(\tilde{\omega}\varepsilon)_c(\tilde{\omega}\varepsilon)D_y\frac{d}{d\tilde{\omega}}\delta k - (\tilde{\omega}\varepsilon)^2 D_y(\delta k)^2 + \left[i\tilde{\omega}_c\varepsilon_c D_y\frac{d}{d\tilde{\omega}}(\tilde{\omega}\varepsilon) - 2k_c(\tilde{\omega}\varepsilon)\delta(\tilde{\omega}\varepsilon)\right]\delta k + $$
$$+\Big\{D_\nu\nu^2 - k_c^2[\delta(\tilde{\omega}\varepsilon)]^2 D_y - \left[2y_T\left(1-\frac{1}{Q_*}\right)\right]_c + i(\tilde{\omega}\varepsilon)\frac{y_T}{y_\cdot}\frac{d}{d\tilde{\omega}}\left[\left(1-\frac{1}{Q_*}\right)\right] +$$
$$+ i(\tilde{\omega}\varepsilon)_c k_c D_y\frac{d}{d\tilde{\omega}}(\delta(\tilde{\omega}\varepsilon))\Big\} = 0 \qquad (D6)$$

We express (D6) in terms of (δk) and obtain

$$i\frac{d}{d\tilde{\omega}}\delta k - \delta k^2 + iP\delta k + R = 0 \qquad (D7)$$

where $P = [\cdots]$ and $R = \{\cdots\}$ in (D6). Moreover, we transform (D7) to the following form

$$w''(\tilde{\omega}) + \left[R - \frac{1}{4}P^2 - \frac{1}{2}P'\right]w(\tilde{\omega}) = 0 \qquad (D8)$$

by the following two transformations

$$\delta k = \frac{1}{iu}\frac{du}{d\tilde{\omega}}$$

and

$$u = w\,e^{-\frac{1}{2}\int P\,d\tilde{\omega}}$$

The coefficient of w in (D8) reduces to

$$R - \frac{1}{4}P^2 - \frac{1}{2}P' = \frac{1}{(\tilde{\omega}\varepsilon)_c^2 D_y}\left\{D_\nu\nu^2 + \left[\frac{2y_T}{Q_*}(1-Q_*)\right]_c + i\frac{y_T(\tilde{\omega}\varepsilon)}{y_c}\frac{d}{d\tilde{\omega}}\left(-\frac{1}{Q_*}\right)\right\} \qquad (D9)$$

Thus, the final form of (D8) becomes

$$w''(\tilde{\omega}) + \frac{1}{(\tilde{\omega}\varepsilon)_c^2 D_y}\left\{D_\nu\nu^2 + \frac{2y_T}{Q_*}(1-Q_*) + i\frac{y_T}{Q_*}\frac{(\tilde{\omega}\varepsilon)_c}{y_c Q_{*c}}\frac{dQ_*}{d\tilde{\omega}}\right\}w(\tilde{\omega}) = 0 \qquad (D10)$$

This is the turning point equation describing the neighborhood of the corotation region ($m \neq 0$). For axisymmetric wave, we set $m = 0$.