ON THE DISSIPATION OF SEISMIC ENERGY FROM SOURCE TO SURFACE

by

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E R R A T A, PhD Thesis by Sven Treitel, Dept. of Geology and Geophysics, M.I.T., June 1958.

"On the Dissipation of Seismic Energy from Source to Surface"

CHAPTER III

1. The second paragraph following equation (3.33) should read:

"For small attenuation coefficients the elastic displacement u_p is small compared to the elastic displacement u (Knopoff and MacDonald, 1958). As a result, we may expect the coupling effect given by the fifth term of (3.31b) to be negligible. The sixth and seventh terms, etc."

2. Equation (3.36b) should read:

$$f_2 = \frac{1}{\rho_0 c_\varepsilon}\left[(\lambda_v + 2\mu_v)\left(\frac{\partial^2 \tilde{u}}{\partial x \partial t}\right) + (\lambda_E + 2\mu_E)\frac{\partial^2 u_p}{\partial x \partial t}\frac{\partial u}{\partial x}\right]$$

3. The term

$$+\left[\frac{M_E}{\rho_0 c_\varepsilon}\mu_c M_E u_x^2 + M_E \varphi_c \left|\frac{\dot{u}_x}{u_x}\right| u_x^2\right]$$

should be added to the right member of eq. (3.72).

4. Equation (3.75) should read

$$\int_0^{2\pi}[f_2(\theta_1)\cos\theta_1]\,d\theta_1 = \frac{4 M_v \varphi_c M_E^2 \sigma^2 A^2}{3\rho_0 c_\varepsilon}\left[\varphi_c\,\omega^2 + (\mu_c + 1/M_v)\,\omega\right]$$

5. Equation (3.90) should read

$$\alpha_2 \approx 10^{-13}\left\{\omega^2 + 10^{-15}\omega^3\right\}$$

None of these corrections alters the conclusions of the original study in any way whatsoever.

Sven Treitel
Havana, Cuba, November 1959

"On the Dissipation of Seismic Energy from Source to Surface"

by

Sven Treitel

Submitted to the Department of Geology and Geophysics on May 19, 1958, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Geophysics.

The progressive decay of a seismic disturbance is traced from its inception as a large amplitude shock front to its attenuation to small amplitude stress waves and ultimate conversion into thermal energy. It is assumed that excess stress accumulates over prolongued periods of time in certain parts of the earth's crust, and that sudden release of such stress at local points of weakness can give rise to a shock wave that will propagate radially outwards from the source zone.

The hydrodynamic equations of supersonic flow are well known, but the dominant effect of their non-linear terms has made it impossible to find exact solutions for shock wave propagation through solids and fluids. A more fruitful approach to this problem can be made through consideration of the Rankine-Hugoniot equations, which relate conditions across an infinitely thin shock front. Combination of these expressions with the Birch-Murnaghan equation of state permits one to perform dissipation calculations in the shock zone. This shock zone is here defined to be that region surrounding the source of the disturbance in which the excess pressure across the shock discontinuity exceeds the yield stress of the rock, $\mathcal{S}$. The small amplitude zone will then be the region in which the excess pressure has decayed to magnitudes of the order of or less than $\mathcal{S}$.

It is shown that enormous amounts of energy are injected into the shock zone by the rapidly decaying front, but that attenuation in the small amplitude zone is quite negligible in comparison. The familiar Gutenberg-Richter earthquake energy formula is based on observations of small amplitude ground motion at the surface. If near-focal shock waves are generated as a result of an earthquake, the total energy estimates of Gutenberg and Richter may be too conservative, perhaps by a factor of ten. The theory of shock wave decay presented in this thesis also suggests that near-source dissipation in seismically active regions over periods of only several hundred thousand years can accumulate sufficient heat in localized areas to cause vulcanism or emplacement of abyssal igneous bodies.

Knopoff and MacDonald (1958, in press) have demonstrated that no solid model of the small amplitude zone, describable by linear differential equations with constant coefficients, can lead to a frequency independent specific dissipation function, $1/Q$. Yet this is exactly what has been observed for rocks and glasses both from seismological and laboratory measurements. Most attenuation treatments in the literature do not take the effect of a finite thermal conductivity into account. Strictly speaking, no dissipation model that neglects associated heat flow is tenable from the thermodynamic viewpoint. Knopoff and MacDonald have derived a theory based on permanent, plastic strain as well as recoverable, elastic strain. In this thesis their work is generalized to take thermal as well as their coupling effects into account. The resulting equations of motion and temperature contain small non-linearities, but solutions can be established by the method of first approximation of Kryloff and Bogoliuboff. It is found that damping in such a medium is describable in terms of two attenuation coefficients, only one of which is a function of the thermal conductivity. This "thermal" attenuation coefficient is probably small compared to the other, which is identical to that of Knopoff and MacDonald, and wh ch leads to a $1/Q$ independent of frequency.

Two linear models are also considered. The first of these is a solid with finite thermal conductivity, and the second a similar medium with viscous damping as well. Both models are shown to lead to damping mechanisms that are not in agreement with observation.

Finally, it is demonstrated that Zener's concept of relaxation by thermal diffusion is inapplicable to seismic wave attenuation, although the theory has yielded good agreement with experiment in the case of many metals. Zener's work is based upon the assumption that the wave length is of the same order of magnitude as the diameter of a crystallite of the medium; this hypothesis cannot be upheld for ordinary seismic frequencies.

It is suggested that experimental work on shock wave propagation through solids will serve to clarify many points that cannot be settled from theoretical considerations alone. In view of the results of Knopoff and MacDonald and the present writer, further work on linear dissipation models does not appear promising.

Thesis Supervisor: Dr. Gordon J.F. MacDonald

Title: Associate Professor of Geology

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N O T A T I O N

An effort has been made to avoid usage of symbols for more than one quantity throughout the text, but this has not always been possible. The following tabulation lists the symbols used together with their principal meaning; in any event, duplication of symbols previously used occurs only in Section 3 of Chapter IV.

SYMBOL	MEANING
P_0	Hydrostatic Pressure
$P - P_0$	Excess over hydrostatic pressure
ρ	Density at pressure P
ρ_0	Density at pressure P_0
V	Specific Volume at pressure P
V_0	Specific Volume at pressure P_0
$\mathcal{S}$	Solid yield stress
d	Width of shock front
v	Particle velocity behind shock front
v_0	Particle velocity ahead of shock front
E	Specific internal energy behind shock
E_0	Specific internal energy ahead of shock
U	Shock velocity
T	Absolute Temperature
R	Radial distance
β	Coefficient of thermal expansion at pressure P
β_0	Coefficient of thermal expansion at pressure P_0
k	Bulk modulus at pressure P
k_0	Bulk modulus at pressure P_0
$\mathcal{K}$	Compressibility

SYMBOL	MEANING
n_1, m_1	Exponents in generalized Birch equation of state
s	Specific entropy
γ	Thermal conductivity
y	Density ratio = (ρ/ρ_0)
V_P	P Wave Velocity
t	Time
x	Distance
C_0	Acoustic Velocity
m	Dimensionless Radial Distance
T	Dimensionless Travel Time
a	Radius of source sphere
δ, n	Shock wave damping coefficients
E_{cum}	Total energy dissipated in shell of thickness m
A	Wave amplitude
B	Temperature amplitude
c	General elastic wave velocity
u	Displacement
σ	Complex wave number = $\nu + j\alpha$
j	$\sqrt{-1}$
ν	Real part of
α	Imaginary part of σ = attenuation coefficient
e	Total strain
ε	Elastic strain
ω	Circular frequency
M_E	Elastic modulus

SYMBOL	MEANING
M_v	"Viscous" modulus
λ_v, μ_v	Viscous parameters
$\lambda_\varepsilon, \mu_\varepsilon$	Elastic parameters
w^i_p	Permanent (plastic) displacement vector
w^i	Elastic displacement vector
ε^i_j	Elastic strain tensor
d^i_j	Total rate of deformation tensor
c^i_j	Rate of permanent deformation tensor
δ^i_j	Kronecker Delta
p^i_j	Total stress tensor
v^i	Total velocity vector
x^i	Position vector
τ^i_j	Thermoelastic stress tensor
$\mathcal{D}/\mathcal{D}t$	Derivative "following the motion"
$I_{(i)}$	i^{th} invariant of elastic stress tensor
$\dot{I}_{(i)}$	i^{th} invariant of time rate of change of elastic stress tensor
μ_c, ψ_c	Plastic constants
κ	Thermal diffusivity
c_ε	Specific Heat at constant strain
φ, ψ	Phase angles
θ_1	$= \sigma x - \omega t + \varphi(t)$
θ_2	$= \sigma x - \omega t + \psi(t)$
$\tilde{M}_T$	$= k/\beta$
$\bar{w}$	Total displacement Vector
$\tilde{\alpha}_1, \tilde{\alpha}_2$	Time attenuation coefficients

SYMBOL	MEANING
α_1, α_2	Distance attenuation coefficients
P	Tensile stress
τ_ε	Relaxation time of stress at constant strain
τ_σ	Relaxation time of strain at constant stress
M_R	Relaxed modulus
M_U	Unrelaxed modulus
$\mathcal{M}$	Complex modulus
Δ	Angle of lag of strain behind stress
$\bar{\tau}$	Geometric mean of τ_ε and τ_σ .
$\bar{M}$	Geometric mean of M_R and M_U .
$1/Q$	Specific Dissipation Function
$\mathcal{D}$	Diffusion distance
λ	Wave length
k_T	Isothermal bulk modulus
$V_{P,T}$	Isothermal P wave velocity
$V_{P,S}$	Adiabatic P wave velocity
M_T'	$\frac{\beta k_T}{\rho_0}$
$\bar{q}$	Specific shell energy
T_{AM}	Ambient rock temperature
M	Earthquake magnitude

C H A P T E R   I

I N T R O D U C T I O N

The propagation of seismic waves through the earth's crust is usually treated with the aid of classical elastic theory alone. Yet it is well known that no physical medium behaves like a perfectly elastic substance. Any disturbance that arises in the medium will eventually be damped to zero amplitude, and the input energy will ultimately appear as heat. If the amplitude of the disturbance is small, that is, if the describing equations of motion are linear or only slightly non-linear, the departures from perfect elasticity are not considerable, and elastic theory may be used with confidence. The observed attenuation of seismic waves is very small. One usually studies damping in a medium by considering an attenuation coefficient α ,

$$A(x) = A_0 e^{-\alpha x} \tag{1.1}$$

where A_0 is the initial amplitude of the disturbance, x the distance from the source, and A(x) the amplitude at the distance x. Gutenberg (1951) has estimated the average value of α for the transmission of compressional waves through the earth to be of the order of 10^{-4}/Km. Studies of seismic surface wave attenuation as well as extensive laboratory work on silicates yield similarly small values of the coefficient α . Much of the available empirical data on silicates has recently been reviewed by Knopoff and MacDonald (1958, in press).

In the immediate neighborhood of the source of a major disturbance, such as an earthquake or a large subterranean blast, the small amplitude assumptions cannot be upheld. The sudden and concentrated release of major amounts of energy gives rise to pulses of large finite amplitudes, which are known as shock fronts, or shock waves. The fronts are formed in solids when the pressure exceeds the yield stress of the medium, $\mathcal{S}$. For rocks, $\mathcal{S}$ is of the order of 10^9 dynes/cm^2 = 10^3 bars. Enormous gradients exist across these fronts; as a result, the shock wave must decay very quickly as it propagates, with consequent rapid injection of large amounts of dissipated energy into a small volume surrounding the source. Shock wave phenomena are thus of considerable interest in the study of conditions existing near the focus of an earthquake, or near the site of an underground explosion.

The problem is of considerable interest to the exploration geophysics industry, since the mechanism of seismic wave generation by explosives is not at all well understood. A considerable amount of work along these lines has been reported in the literature, but most of it is of an empirical nature and of little value to the formulation of a more general theory (Leet, 1951; Habberjam and Whetton, 1952). The experimental difficulties involved are quite formidable, since it is extremely hard to build strain gauges that can withstand the enormous pressures developed near the source of the disturbance. Morris (1950) has recognized that the detonation of an explosive in rock creates a shock wave, which spreads out spherically. As the disturbance travels outwards, the stresses decrease until the yield stress of the rock is reached. From that point on, the wave is transmitted as a small amplitude disturbance. Morris does not attempt to place his statements on a more rigorous mathematical basis, nor did he study the dissipation mechanisms that must act while the disturbance is still a shock wave. W.I. Duvall (1953) has reported experimental work performed by the Bureau of Mines near the sites of major rock blasts. He found that the shock amplitude decayed according to a $1/R^n$ law, where n ranged in value between 1.6 to 2.5 for various rock types and explosives.

The propagation of shock waves in water has been extensively studied during the Second World War. Most of this work has been summarized in a book by R.H. Cole, "Underwater Explosions" (1948). As we shall see in the next chapter, much of this theory can be very conveniently adapted to the study of shock wave propagation in solids.

From time to time major rock blasts have been set off in many parts of the world for various purposes, and in several instances the resulting disturbances have been recorded by seismographs up to a distance of several hundreds of miles from the detonation site. Unfortunately no strain gauges were placed in the rock in the immediate neighborhood of the source, so that no shock wave observations could be made. Willmore (1949) has written a detailed report of seismic measurements made in connection with the blasting of German fortifications on the island of Helgoland in 1946. Several thousand tons of dynamite were detonated simultaneously, whose total energy was estimated at 1.3×10^{20} ergs. The energy appearing in the form of small amplitude seismic waves was calculated by Willmore to be of the order of 10^{17} ergs. In 1921, 4500 tons of dynamite were set off at Oppau, Germany. Jeffreys (1952) calculated that the energy liberated by this explosion was about 6×10^{19} ergs, while only 5×10^{16} ergs appeared as small amplitude waves. In both these cases only 0.1% of the input energy went into small amplitude stress waves. While it is undoubtedly true that a substantial amount of this input energy was imparted to the air, a considerable fraction must also have been dissipated near the source, where the disturbance was still a rapidly decaying shock wave. When major blasts are detonated far underground, on the other hand, there will be no loss into the atmosphere, and all the input energy will then be imparted to the surrounding rock.

On September 19, 1957, a 1.7 Kiloton[1)] atomic bomb was detonated in a tunnel under a mountain at the Nevada A.E.C. Test Site. A preliminary report containing some declassified data about this explosion (OPERATION PLUMBBOB) has been published recently (Johnson et Al, 1958). The near-source observations of the blast, insofar as they have been made available, will be discussed in Chapter V of this thesis.

It thus appears appropiate to examine the theory of shock wave propagation in rocks more closely at this time, and in particular to investigate the dissipation mechanisms

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1) 7.1×10^{19} ergs total energy release

that may be expected to hold for these waves. We shall investigate this question in considerable detail in Chapter II of this thesis, and discuss its seismological implications in Chapter V.

Once the shock front has decayed to pressure levels considerably below the yield stress of the solid, recourse may be taken to linear and slightly non-linear perturbation theory in order to study the propagation of the resulting small amplitude wave. Knopoff and MacDonald (1958) have made an exhaustive survey of observational and experimental data available for the attenuation of small amplitude waves in silicates, and find that the attenuation coefficient α is a linear function of the circular frequency ω in the range $10^{-2} \leq \omega \leq 10^{7}$ rad/sec. They then proceed to show that no <u>linear</u> dissipation model can yield an attenuation coefficient that is proportional to an odd power of ω , and as a result conclude that recourse must be taken to permanent strain mechanisms in order to develop a theory in better agreement with observation.

Any compressional wave travelling in a medium of finite thermal conductivity γ will suffer damping. This occurs because the propagation process is only isentropic and reversible for a medium of zero thermal conductivity. Such a medium is, of course, physically impossible. As a result, all propagation models that do not take thermal phenomena into account are, strictly speaking, thermodynamically incorrect. However, since γ is quite small for silicates (of the order of 0.005 cal/cm-sec-deg.C), such "thermo-elastic" damping, as it will be termed here, is quite small. Nevertheless, thermal terms cannot be neglected in any rigorous development of the equations of motion of small amplitude waves.

We shall accordingly concern ourselves with a model exhibiting permanent strain in Chapter III. This problem has been solved by Knopoff and MacDonald (1958) in the absence of thermal terms. In this thesis, their work is generalized to take thermal phenomena into account. The assumed model, which involves both permanent as well as recoverable strain, leads to non-linear equations of motion. Solutions to these equations can be found by the method of Kryloff and Bogoliuboff (Minorsky, 1947) provided that the non-linear terms are small compared to the linear ones.

In Chapter IV we investigate two linear models in the presence of thermal terms. The literature dealing with the propagation of small amplitude stress waves in solids that exhibit a finite thermal conductivity is quite extensive, but in a rather confusing state. Much of the work that has been done suffers from serious flaws in thermodynamic arguments; and even many of the papers that use a correct and rigorous thermodynamic approach fail to express the final formulae in a form amenable to quantitative examination of resulting attenuation coefficients. The first of the linear models to be investigated in this thesis is an ordinary elastic solid of finite, non-zero thermal conductivity, while the second takes viscous dissipation into account as well. We shall find that neither of these models gives results that agree with observational evidence for silicates, although the former may be applicable to propagation in the megacycle frequency range. Such frequencies are, of course, of no seismic interest.

Chapter V summarizes the results obtained in the three previous chapters, and discusses the geological implications of the work reported there. The propagation of a discontinuity is traced from its origin as a large amplitude shock pulse to its eventual decay to a small amplitude acoustic wave.

We shall, then, first proceed to a detailed study of shock wave phenomena in solids. This will be accomplished in the next chapter.

C H A P T E R I I

SHOCK WAVE DECAY NEAR THE SOURCE

1. Shock Wave Propagation Theory in Solids

Although the literature dealing with the generation and propagation of shock fronts in physical media is quite extensive, most of the treatments available restrict themselves to the study of these phenomena in gases. During the Second World War considerable effort was devoted to the study of shock waves generated by underwater detonations. The results of this work are admirably presented and summarized in R.H. Cole's book, "Underwater Explosions" (1948). Unfortunately, a substantial part of this war-time research has not yet been declassified and is therefore unavailable. There is little doubt in the writer's mind that restricted work on shock wave propagation in solids has been done both here and abroad in connection with the study of energy liberated in atomic and nuclear explosions. The release of results of such investigations would obviously be of great interest to seismology.

The main reference work in this field is the well known book by R. Courant and K.O. Friedrichs, "Supersonic Flow and Shock Waves" (1948). The unlinearized hydrodynamical equations and approximation techniques for their solution are presented in considerable detail. The Rankine-Hugoniot expressions, which relate conditions across a travelling shock front (see below) are also derived from basic principles. However, the discussion of shock phenomena in solids is very brief and sketchy. A similarly short and rather heuristic discussion of shocks in solids may be found in H. Kolsky's "Stress Waves in Solids" (1953), pp. 178-182.

The pressures required to maintain a propagating shock discontinuity in a solid are far above the material's yield stress. It is therefore possible to treat the solid as a fluid for such phenomena, since the shear modulus is bound to lose its significance at these high pressures, (Kolsky, 1953), (Gilvarry and Hill, 1956). The strength of rocks as established by laboratory measurements is usually taken to be of the order of 10^9 dynes/cm^2 (Birch et Al, 1942). In this work we shall define any travelling pressure discontinuity of magnitude greater than the solid's strength to constitute a shock wave.

Walsh and his coworkers (Walsh and Christian, 1955; Walsh, Rice and Yarger, 1957) have carried out extensive experimental work with shock wave propagation at the Los Alamos A.E.C. laboratory. Their measurements have enabled them to find the equations of state that describe the pressure-volume-temperature relationships of twenty-seven different metals. Goranson et Al, (1955) have performed work of a somewhat similar nature on duralumin in the pressure range from 0.15 to 0.33 megabars. They distinguish between isentropic and isothermal equations of state, and succeed in fitting their experimental data to an empirical equation of state of the form,

$$P_s = K_s\left(\frac{\rho-\rho_0}{\rho_0}\right) + K_s'\left(\frac{\rho-\rho_0}{\rho_0}\right)^2$$

where P_s = isentropic pressure, k_s = isentropic bulk modulus, k_s' = second order coefficient, ρ_0 = density at zero pressure, and ρ = density at pressure P_s. Shock wave measurements on metals are also being undertaken by G.E. Duvall and associates at the Poulter laboratories of the Stanford Research Institute (Duvall and Zwolinski, 1955; Drummond, 1957). The only laboratory work with shock waves in rocks has recently been reported by Hughes and McQueen (1957). They have succeeded in measuring the density of two gabbro and one dunite specimens in the pressure range from 0.15 to 0.75 megabars. These large pressures were attained across shock fronts generated by high explosives. Dunite was compressed from $\rho_0 = 3.25$ grs/cm^3 to $\rho = 4.93$ grs/cm^3 at 0.72 megabars, and gabbro from $\rho_0 = 3$ to $\rho = 5$ grs/cm^3 at 0.75 megabars. Both gabbros so tested showed evidence of polymorphic phase transition at a pressure between 0.1 and 0.35 megabars to a more dense and less compressible phase. The theory to be developed in the following pages does not take the possibility of such phase transitions into account. We moreover restrict ourselves to isothermal equations of state (see Section 3 of the present chapter). Even though such idealizations are not strictly correct, they should be adequate to provide us with orders of magnitude of shock wave phenomena.

2. The Formation of a Shock Front

Let P_0 = hydrostatic pressure, $P-P_0$ = the excess over the hydrostatic pressure, ρ_0 = density at pressure P_0, and ρ = density at pressure P. Within the elastic limit, $P-P_0 \ll \mathcal{Y}$, where $\mathcal{Y}$ = yield stress of the solid, and one has

$$\frac{P-P_0}{\rho-\rho_0} = \frac{\Delta P}{\Delta \rho} \approx \text{const.} \tag{2.1}$$

so that all elastic strains are propagated at the same speed. When $P-P_o \gg \mathcal{G}$, however, the quantity $\Delta P/\Delta \rho$ will either decrease asymptotically toward zero, or increase with increasing P. In the former case, a plastic wave with a velocity of propagation less than that of the elastic wave will be produced; in the latter, the larger strains will be propagated faster than the smaller ones, so that such disturbances travel through the medium at super-sonic speeds. Sonic speeds are here assumed to be those that correspond to ordinary elastic waves. The formation of a steep shock front may be schematically illustrated by Figure 2.1 below:

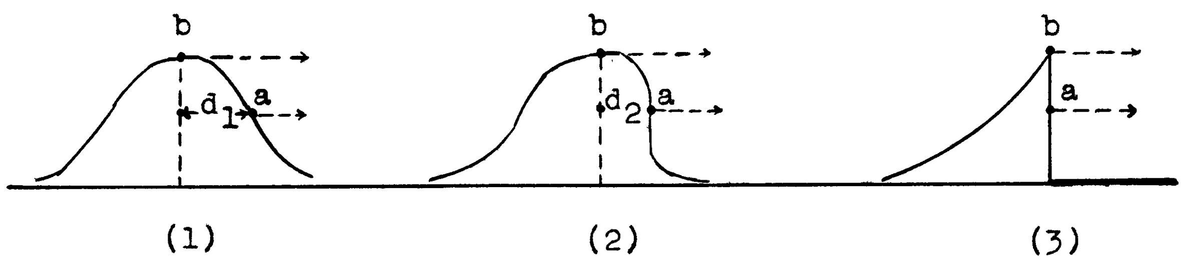

Fig. 2.1----Formation of a Shock Front in a Wave of Finite Amplitude (adapted from Cole, 1948).

Let us assume that the pressure at b is greater than that at a, $P_b > P_a$, and that both P_a and $P_b \gg \mathcal{G}$. Then the disturbance at b will travel faster than at a, so that the distance d diminishes as the pulse travels toward the right. The pulse front will become steeper and steeper, and would ultimately become infinitely steep (Fig. 2.1, (3)), so that $d \approx 0$. This ultimate condition cannot be attained physically, since the differences in pressure and temperature of the material in the disturbed region relative to the undisturbed medium ahead of the pulse become larger and larger as the front steepens, that is to say, the gradients of these quantities approach infinitely high values. In this situation, however, considerable amounts of energy will be dissipated, and the pulse front will only approach, but not actually reach infinite steepness. A pulse that approximates the idealized state illustrated in Fig. 2.1 (3) is known as a shock front. The interval required by a pulse to reach its maximum steepness is called its rise time. So far as is known from experimental measurements, such rise times are exceedingly small, of the order of microseconds in many instances (Cole, 1948). The equations of state that have been found to describe the behavior of rocks in the earth's crust (see below, Section 3 of this chapter) show that ρ increases with increasing P, so that shock waves, rather than plastic waves must form when an earthquake occurs.

In order to make the mathematical analysis of shock phenomena at all tractable, it is necessary to make a number of idealizations. It has turned out, fortunately, that measurements agree very well with theory in spite of the great simplifications that must be made. The region of greatest interest for shock wave behavior lies in the immediate neighborhood of the source of a large disturbance, but it is just in this region where measurements cannot be made, since the best pressure gauges have upper endurance limits far below pressures that appear to be developed near the source. Thus pressures are recorded as close to the origin as is feasible, and the results extrapolated to smaller source distances. This method is used in work with underwater detonations, and has also been employed in the study of disturbances caused by rock blasts (W.I. Duvall, 1953).

The fundamental equations that describe the shock wave are the so called Rankine-Hugoniot relations, (hereafter referred to as the R.H. relations). Since their derivation can be found in basic reference works (see e.g. Courant and Friedrichs, 1948), we merely state them here without proof. They are obtained by a consideration of the zones immediately ahead and behind the actual discontinuity. If the wave front becomes infinitely steep ($d = 0$; See Fig. 2.1, (3)), the pulse has zero width. Now let
U = velocity of shock front relative to the fixed origin O,

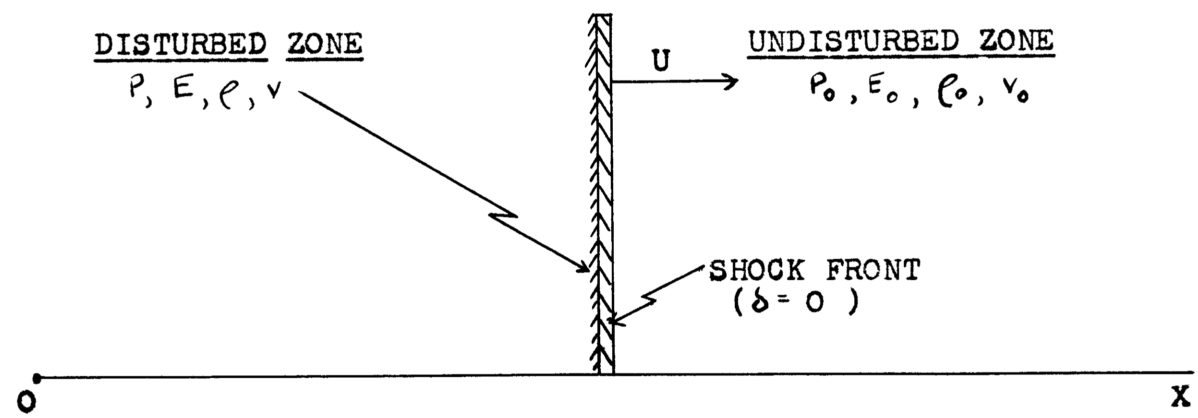

Fig. 2.2-----A shock front propagating into an undisturbed zone from left to right at velocity U relative to a fixed origin O.

and let P_0, ρ_0, E_0, and v_0 be the hydrostatic pressure, density, internal energy, and particle velocity in the undisturbed zone ahead of the shock and P, ρ, E, and v the corresponding quantities immediately behind the shock. Since, by definition, no excitation has occurred in the undisturbed zone prior to the passage of the shock front, $v_0 = 0$. Application of the laws of conservation of mass, momentum, and energy to both sides of the shock front yields the equations

$$\left.\begin{aligned} \rho(U-v) &= \rho_0 U \\ P-P_0 &= \rho_0 U v \\ E-E_0 &= \frac{1}{2}(P+P_0)\left(\frac{1}{\rho_0}-\frac{1}{\rho}\right) \end{aligned}\right\} d=0 \qquad (2.2\ a,b,c)$$

These are the R.H. relations. It should be pointed out again that these expressions were derived subject to the condition $d = 0$, a situation which can only be approached physically, since $d=0$ corresponds to infinitely large gradients across the discontinuity.

Little work has been published on the actual thickness of the shock zone-----or transition zone, as some writers chose to call it. The quantity d is undoubtedly a function of the dissipative forces which become increasingly important as the gradients grow larger. In fluids d is of the order of one molecular mean free path (Kolsky, 1953), but no results are available on the probable thickness of this zone in solids. Nevertheless, agreement between observation and theory is so good that one may safely assert that this restriction on the R.H. relations is not serious.

Conditions in the disturbed zone after the passage of the first shock front are extremely complicated, and involved hysteresis effects must probably be taken into account. Moreover, such shock waves can be reflected from boundaries just as in the case of ordinary elastic waves, so that complex interactions between incident and reflected shock pulses must arise. In this work we must assume,(a) that no reflection of the shock wave occurs within the area of interest and (b) that, at least as far as earthquakes and rock blasts are concerned, the shock phenomena can be adequately described by the passage of a single shock front of infinitesimal width.

3. The Equation of State

If equations (2.2 a,b) be solved simultaneously for the shock velocity U and particle velocity v, one has

$$U = \sqrt{\frac{\rho}{\rho_0}\left(\frac{P - P_0}{\rho - \rho_0}\right)} \tag{2.3}$$

$$v = \frac{\rho - \rho_0}{\rho} U = \sqrt{\frac{\rho - \rho_0}{\rho \rho_0}(P - P_0)} \tag{2.4}$$

If the equation of state of the medium, $P = f(\rho, T)$, where T is the absolute temperature, is known, it becomes possible to express the shock velocity U, the particle velocity v, and the internal energy difference $E-E_0$ across the discontinuity as a function of the excess pressure $P-P_0$, or density increase $\rho - \rho_0$ alone. In particular, if the behavior of $P-P_0$ as a function of the distance from the source, R, be known, then the dependence of

$\rho - \rho_0$ upon R can be found from the equation of state, and thus the functions $U(R)$, $v(R)$, and $E-E_0(R)$ established uniquely. Knowledge of these functions permits detailed calculations of supersonic flow and associated energy losses that one might expect to find near earthquake focii or large rock blasts, such as underground atomic explosions. Obviously, the larger the initial pressure difference $P-P_0$ i.e., the larger the quantity $(P-P_0)_{R=0}$, the greater will be the volume of material around the source in which shock phenomena take place. We recall here that P_0 is the hydrostatic pressure, and $P-P_0$ the excess over this hydrostatic pressure as referred to the disturbed and undisturbed sides of the advancing shock front. In seismological applications, P_0 will of course itself be a function of the depth below the surface.

A successful attack on this problem therefore hinges on two factors:

(1) Knowledge of the equation of state,

(2) Knowledge of the pressure distance decay law.

Now (2) can only be established explicitly if the exact solutions of the non-linear hydrodynamical equations that describe the motion of a shock pulse are known. In particular, the decay law may not only be a function of the equation of state itself, but also of such mechanisms as viscosity and heat conduction. Here we only consider (1), and return to the problem of the hydrodynamical equations in the next section.

Birch (1938,1947,1952) has made an exhaustive study of the behavior of rocks at high pressures, based on the finite strain theory of Murnaghan (1937,1951). Birch's equation of state, which is independent of T, is

$$P = \frac{3}{2} k_0 \left[(V_0/V)^{7/3} - (V_0/V)^{5/3} \right] \tag{2.5}$$

Here k_0 = bulk modulus (or incompressibility) corresponding to zero pressure, V_0 = specific volume at zero pressure, and V = specific volume at pressure P.

Gilvarry (1957) writes (2.5) in the more general form

$$P = (n_1 - m_1)^{-1} k_0 \left[\left(\frac{V_0}{V}\right)^{n_1} - \left(\frac{V_0}{V}\right)^{m_1} \right] \tag{2.6}$$

He calls this equation the generalized form of Birch's isothermal equation of state. Eq. (2.6) reduces to (2.5) by setting $n_1 = 7/3$ and $m_1 = 5/3$. Gilvarry has also lifted the isothermal restriction on Birch's equation, and finds that in this case the equation of state is given to first order by

$$\begin{aligned} P = P(T_0) + \frac{k_0}{n_1 - m_1} \Bigg[& n_1 \left(\frac{V_0}{V}\right)^{n_1} - m_1 \left(\frac{V_0}{V}\right)^{m_1} \\ & - \eta_0 \left\{ \left(\frac{V_0}{V}\right)^{n_1} - \left(\frac{V_0}{V}\right)^{m_1} \right\} \Bigg] \int_{T_0}^{T} \beta_0 \, dT \end{aligned} \tag{2.7}$$

where

$$P(T_0) = \frac{k_0}{n_1 - m_1} \left[\left(\frac{V}{V_0}\right)^{n_1} - \left(\frac{V}{V_0}\right)^{m_1} \right]$$

is the generalized isothermal Birch equation (2.6), T_0 = initial temperature, T = final temperature, β_0 = coefficient of thermal expansion at temperature T_o and zero pressure, and

$$\eta_o = -k_o \beta_o^{-1} \left(\frac{\partial \beta}{\partial P}\right)_T$$

Equation (2.7) permits the calculation of the temperature rise, $T-T_o$, that corresponds to a pressure increase of $P-P_o$ in a solid describable by such an equation of state. Again, the behaviour of the parameters β_o and η_o in a shock zone is not known, so that the use of (2.7) in preference to the isothermal equation (2.5) does not appear to be warranted at this time.

A question of equal importance is the upper pressure limit below which (2.5) can be assumed to be applicable. Birch (1952) presents a curve for iron that he has computed from (2.5) up to a pressure of $\sim 7 \times 10^6$ bars, which corresponds to a density of ~ 15. Walsh and Christian (1955) have measured shock pressures in metals up to $\sim 5 \times 10^5$ bars. It is of course difficult to speculate about the magnitude of $P-P_o$ across the shock front generated by an underground atomic blast or an earthquake, and in the following pages we shall assume that this pressure difference does not exceed 10^6 bars. In this case we are probably well within the region of validity of (2.5). Above pressures of 10^7 bars it appears likely that the equation of state must be found on the basis of quantum-mechanical, rather than the elastico-plastical considerations that have led to Murnaghan's theory. In the former case, the solid is treated as an electron gas. Calculations along such lines have been made by a number of workers (Feynman, Metropolis, and Teller, 1949), but their results hold only for pressures greater than 10^7 bars (10^{13} dynes/cm^2). It is extremely unlikely that pressures of this magnitude are ever developed across shock fronts in rocks;[1)] accordingly, we shall restrict ourselves here to a consideration of the Birch-Murnaghan isothermal equation alone.

4. The Hydrodynamical Equations and Shock Decay

It was pointed out in the previous section that the exact form of the pressure-distance decay law is not only a function of the assumed equation of state, but also of dissipative mechanisms such as viscosity and heat conduction. Since a shock front is actually a very large finite amplitude pulse, the classical linearized hydrodynamical equations cannot be used to describe the propagation of shock waves through any physical medium. Where the deviations from linearity are not considerable, as in the case of the small amplitude waves treated in Chapter III of this thesis, perturbation techniques applied to known solutions of the corresponding linearized equations yield very satisfactory results. In the present case, however, the non-linear terms are so large that any such approximation method breaks down completely. Thus the equations of continuity and motion in one dimension in the absence of viscosity and heat conduction can be written in their

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(1): at least, in the case of earthquakes and non-nuclear blasts.

Eulerian form as

$$\rho_t + (\rho v)_x = 0$$
$$(\rho v)_t + (\rho v^2 + P)_x = 0 \qquad (2.8\ a,b)$$

The subscripts x and t denote differentiation with respect to the space and time variables, and v = particle velocity in the x direction. The isothermal equation of state, $P = f(\rho)$, could be used to eliminate P from the above system, so that it might in theory be solved for u(x) and ρ(x). But, as was pointed out above, the non-linear terms of (2.8) become so large for a shock wave that linear perturbation techniques are not applicable. The system (2.8) is amenable to an exact solution by the "method of characteristics" (Courant and Friedrichs, 1948, p. 38 ff.) in the case of gases, for which simple linear equations of state $P = f(\rho)$ hold. For solids, however, this is not the case at pressure levels that must exist across the shock front. Since the exact solution of (2.8) is not known, only cumbersome numerical iteration methods can be used to attack the problem. Unfortunately these iterative procedures are strongly dependent upon initial conditions in the immediate neighborhood of the source, and it is exactly here where adequate data, either experimental or theoretical, is almost totally lacking. Calculations of this nature have been made for underwater explosions (Cole, 1948), but in this instance some empirical data from near source measurements was at least available. Since up to this time little or no such data exists for rock blasts, and is obviously unattainable directly in the case of earthquakes, such iterative calculations appear to be rather futile.

Generalizations of equations (2.8) are given by Courant and Friedrichs (1948, p. 134) for a viscous and heat conducting fluid. In this case one has for one dimensional flow

$$\begin{aligned} &\rho_t + (\rho v)_x = 0 \\ &(\rho v)_t + (\rho v^2 + P - 4/3\,\lambda_v v_x)_x = 0 \\ &\rho T S_t + \rho v T S_x = 4/3\,\lambda_v v_x^2 + (\gamma T_x)_x = 0 \end{aligned}$$

(2.9 a,b,c)

Here S = specific entropy, λ_v = bulk viscosity (see Ch. III), and γ = thermometric conductivity. The left side of (2.9 c) represents the heat acquired by a unit volume in unit time; the first term on the right represents the heat generated by viscous friction, and the second the contribution due to heat conduction directly. Again, Gilvarry's generalized equation of state (2.7) could be used as an additional relation in conjunction with (2.9), but such computations are subject to the same difficulties as explained above.Moreover, it is by no means certain that viscosity and heat conduction represent meaningful concepts when one deals with processes occurring in the shock front itself.

We must therefore conclude that shock amplitude decay laws cannot be found from consideration of the hydrodynamical equations because

(1) Their non-linear terms are dominant

(2) The meaning of viscosity and thermal conductivity in shock fronts is as yet obscure.

Under these circumstances the only remaining avenue of approach lies in the postulation of certain shock decay laws, and to establish the physical implications to which such assumptions lead. One then hopes that considerations of this nature will at least shed some light on the problem of energy dissipation and supersonic flow in the neighborhood of large sudden disturbances in solids, and yield an idea of the order of magnitude of such quantities. The theory to be presented below is of a rather general nature, since it does not require knowledge of the exact shock decay mechanism until the final stages of the calculation are reached. The chief advantage of this approach lies in the fact that any number of decay laws may be tested in this way, and their physical feasibility established.

5. The R.H. Relations for the Birch Isothermal Equation of State

It was already pointed out in Section 3 that combination of the R.H. relations with a suitable equation of state permitted the unique calculation of shock velocity U, particle velocity v, and energy difference across the shock discontinuity $E-E_0$ as a function of the quantity $\rho - \rho_0$ alone. Here

$\rho =$ density of material at a hydrostatic pressure P, and

$\rho_o =$ density of the material ahead of the shock front, which is at a hydrostatic pressure P_o .

We select the Birch isothermal equation of state (hereafter termed the Birch equation), which can be written in the more convenient form

$$P - P_o = \frac{3}{2} k_o \left(\frac{\rho}{\rho_o}\right)^{5/3} \left[\left(\frac{\rho}{\rho_o}\right)^{2/3} - 1\right] \tag{2.10}$$

as our fundamental relation. Here we set $\rho = 1/V$ and $\rho_o = 1/V_o$ When $\rho = \rho_o$, $P - P_o = 0$, or $P = P_o$. Since rarefaction shock waves in solids and liquids cannot arise (Lamb, 1932), $V_o > V$ and $\rho > \rho_o$ always, so that $\rho/\rho_o \gg 1$.

Substituting (2.10) into (2.3), one has

$$U = \sqrt{\frac{3 k_o}{2(\rho - \rho_o)} \left(\frac{\rho}{\rho_o}\right)^{8/3} \left[\left(\frac{\rho}{\rho_o}\right)^{2/3} - 1\right]} \tag{2.11}$$

Consider for a moment (2.3):

$$U = \sqrt{\left(\frac{\rho}{\rho_o}\right) \frac{P - P_o}{\rho - \rho_o}} \tag{2.3}$$

if we expand P in a Taylor series in $(\rho - \rho_o)$ about P_o,

$$P = P_o + P_1 \left(\frac{\rho - \rho_o}{\rho}\right) + P_2 \left(\frac{\rho - \rho_o}{\rho}\right)^2 + \ldots\ldots\ldots \tag{2.12}$$

The coefficients P_o, P_1, P_2,.......are actually functions of temperature, but since we are dealing here with the isothermal case, these coefficients may be considered to represent elastic constants. If the quantity $\left(\frac{\rho-\rho_0}{\rho}\right)^2$ is small compared to $\frac{\rho-\rho_0}{\rho}$, we need only retain the first term in the expansion (2.12), and thus have

$$P - P_0 = P_1 \frac{(\rho-\rho_0)}{\rho}$$

It is shown in Slater's "Chemical Physics" (1939, p. 203) that $P_1 \approx k_0$, where k_0 = isothermal incompressibility. In this case,

$$P - P_0 = \frac{k_0}{\rho}(\rho-\rho_0) \tag{2.13}$$

For small $\rho-\rho_0$, we can set roughly $\rho \approx \rho_0$. Then, substituting (2.13) into (2.3),

$$U = \sqrt{\frac{\rho\, k_0\, (\rho-\rho_0)}{\rho(\rho-\rho_0)\,\rho_0}} = \sqrt{\frac{k_0}{\rho_0}} = c_0 \tag{2.14}$$

where we recognize c_0 to be the ordinary acoustic velocity. We have thus shown that an acoustic pulse is actually a weak shock. Mathematically, the transition from (2.3) to (2.14) is difficult to establish. One gets around this ambiguity by arbitrarily defining the pressure excess, $P-P_0$, below which the pulse may be considered to be acoustic, i.e., adequately represented by (2.14). If $\mathcal{S}$ = yield stress of solid, we shall term the pulse acoustic when $P-P_0 < \mathcal{S}$. Thus any decaying shock will eventually decay to an ordinary acoustic pulse.

For convenience in later work, we set

$$y = \rho/\rho_0 \tag{2.15}$$

which implies

$$\rho - \rho_0 = \rho_0 (y-1) \tag{2.16}$$

Equations (2.10) and (2.11) may then be written

$$p - p_0 = \frac{3k_0}{2} y^{5/3} \left(y^{2/3} - 1\right) \tag{2.17}$$

and

$$U = \sqrt{\frac{3}{2} \frac{k_0}{\rho_0} \frac{y^{8/3}\left(y^{2/3}-1\right)}{y-1}} = c_0 \sqrt{\frac{3y^{8/3}\left(y^{2/3}-1\right)}{2\,(y-1)}} \tag{2.18}$$

by (2.14). These relations may also be expressed in the convenient dimensionless form

$$\frac{p-p_0}{k_0} = \frac{3}{2} y^{5/3} \left(y^{2/3} - 1\right), \quad y \geq 1$$

$$\frac{U}{c_0} = \sqrt{\frac{3y^{8/3}\left(y^{2/3}-1\right)}{2\,(y-1)}}, \quad y > 1$$

(2.19 a,b)

For $y > 1$, it is easy to see that both $\frac{p-p_0}{k_0}$ and U/c_0 are monotonically increasing functions of y. When y=1, application of L'Hôpital's rule to (2.19b) shows that

$$\lim_{y \to 1} \frac{U}{c_o} = 1$$

If (2.19a) be substituted in (2.2c), one gets a relationship between the difference in internal energy across a shock front, $E-E_o$, as a function of y and the constants ρ_o, P_o, and k_o :

$$E - E_o = \frac{y-1}{2y\rho_o}\left[\frac{3}{2} k_o y^{5/3}\left(y^{2/3} - 1\right) + P_o\right] \qquad (2.20)$$

Similarly, the particle velocity, v, is given by

$$v = \frac{y-1}{y} U \qquad (2.21)$$

Duvall and Zwolinski (1955) have investigated the problem of entropy increase in a medium through which a shock front is propagating. It has been shown by Rayleigh (1910) that a pressure discontinuity can be maintained in an ideal fluid only if the entropy of the fluid increases as the shock passes through it. Courant and Friedrichs (1948, p. 142) have furthermore shown that the entropy change across a shock front for "weak" shocks (see below) is of third order in $\frac{\rho - \rho_o}{\rho}$. Duvall and Zwolinski make use of the results of Rayleigh and Courant and Friedrichs, and give the following expression for entropy increase across a weak shock:

$$S - S_o = \frac{V_o^3}{12 T_o}\left(\frac{d^2 P_i}{dV^2}\right)_{V_o}\left(\frac{\rho - \rho_o}{\rho}\right)^3 \qquad (2.22)$$

Here s = specific entropy behind shock front, s_0 = specific entropy in undisturbed region ahead of shock, T_0 = absolute initial temperature, and P_i = isothermal pressure. Duvall (personal communication, 1958) has informed the writer that (2.22) holds for compressions up to approximately 15%, at whatever initial pressure P_0 these may take place. A "weak" shock is thus one across which $\frac{\rho - \rho_0}{\rho} \leq 15\%$. Duvall has furthermore pointed out that the entropy increase thus calculated is only that given by reversible thermodynamics. This is due to the fact that Courant and Friedrichs (1948), upon whose work Duvall and Zwolinski's derivation is based, do not consider dissipative mechanisms in that part of their analysis.

Combination of the Birch equation (2.10) with (2.22), and use of (2.15) yields the dimensionless relationship

$$\frac{\rho_0 T_0 (s - s_0)}{k_0} = \frac{5}{9}\left[y^{11/3}\right]\left[\frac{7}{4}y^{2/3} - 1\right]\left[\frac{y-1}{y}\right]^3 \qquad (2.23)$$

However, since this equation has been derived on the basis of reversibility, and since in any event it is limited to compressions less than 15%, it cannot be used for dissipation computations. For this purpose we shall make use of (2.20), as will be shown in the next section of this chapter.

It should be remarked here that the terms "weak shock" and "infinitely weak shock" are very loosely used in the literature. Courant and Friedrichs (1948, p. 131) define an "infinitely weak shock" as an ordinary sound wave. We shall adhere to this convention in the present work.

6. The Decay of Shock Amplitude with Distance and Associated Energy Dissipation

We have seen that the dominance of the non-linear terms in the hydrodynamic equations does not permit us to find solutions in the case of large amplitude shocks. It was also pointed out that iteration techniques near the origin were equally futile as far as these phenomena in solids are concerned, since almost no empirical data is available for the region in the immediate neighborhood of the shock source. We accordingly proceed to derive energy relationships under the assumption that the relation of pressure to distance, P(R), is known. Methods somewhat similar in nature have been applied to the study of underwater shock propagation (Brinkley and Kirkwood, 1947; Arons and Yennie, 1948; Cole, 1948).

It is well known that the attenuation factor of seismic waves that propagate in the small amplitude regions, far from the focus of the disturbance, is very small (see Chapter I). One may then conclude that the zone of significant energy dissipation must be restricted to that volume around the source in which

$$\frac{U}{c_0} > 1$$

In other words, a seismic pulse may be expected to suffer its greatest rate of attenuation, and consequently impart a large proportion of its energy to the surrounding medium, in that region in which the pulse is still a shock wave. When $U/c_o \approx 1$, the pulse has degenerated into an acoustic wave, and its propagation will be governed by small amplitude theory (see Chapters III and IV).

Let us assume that the pressure across a propagating shock discontinuity decays according to some inverse power of the distance from the origin of the disturbance. Let us assume further that the source region may be represented by a sphere of radius a, which at time $t = t_o$ suddenly expands and imparts an ideal, infinitely steep compressive pulse of zero width and magnitude $P-P_o$ to the surrounding medium. At successive times $t = t_1, t_2, \ldots\ldots$ the shock front may thus be considered to be representable in space by an expanding sphere concentric with the sphere $R = a$. We may thus write

$$(P-P_o)_R = (P-P_o)_a \left(\frac{a}{R}\right)^n, \quad \begin{matrix} R \geq a \\ n \geq 1 \end{matrix} \qquad (2.24)$$

where R = radial distance from surface of source sphere, $(P-P_o)_R$ = pressure difference across discontinuity at distance R, $(P-P_o)$ = original pressure difference at $R = a$, and n = arbitrary exponent, greater than one. A detailed discussion of this decay law and its implications will be relegated to the end of this section.

Once the pressure distance decay law (2.24) has been postulated, it becomes possible to express the quantities U/c_o and $E-E_o$ as functions of R and the initial amplitude of the shock pulse, $(P-P_0)_a$.

Elimination of y between (2.19 a,b) yields

$$\frac{P-P_0}{k_0} = \frac{3}{2}\left[\frac{\left(\frac{U}{c_0}\right)^2}{\left(\frac{U}{c_0}\right)^2 - \frac{P-P_0}{k_0}}\right]^{5/3}\left\{\left[\frac{\left(\frac{U}{c_0}\right)^2}{\left(\frac{U}{c_0}\right)^2 - \frac{P-P_0}{k_0}}\right]^{2/3} - 1\right\} \qquad (2.25)$$

This relation could be solved for U/c_0 in terms of $\frac{P-P_0}{k_0}$, but the algebra required is rather formidable. Instead, we prefer to use (2.19 a,b) directly as a pair of parametric equations in y ($y = \rho/\rho_0$). Plots of the two dimensionless quantities $\frac{P-P_0}{k_0}$ and $\frac{U}{c_0}$ are shown as functions of y in Figure (2.3). Thus knowledge of the magnitude of ρ/ρ_0 at any point R permits us to find the shock/acoustic velocity ratio at this point. For values of y higher than 4, Table I in the appendix should be consulted.

Equation (2.24) may also be written in the more convenient dimensionless form,

$$\left(\frac{P-P_0}{k_0}\right)_R = \left(\frac{P-P_0}{k_0}\right)_a \left(\frac{a}{R}\right)^n , \quad R \geqslant a \qquad (2.26)$$

If suitable values of the quantities $\left(\frac{P-P_0}{k_0}\right)_a$ and n be assumed, U/c_0 may be found as a function of a/R with the aid of Fig. 2.3 or Table I of the Appendix. In order to render the numerical computations as general as possible, we shall work with the dimensionless distance m, where $m = R/a$, $m \geqslant 1$. Then (2.26) becomes

$$\left(\frac{P-P_0}{k_0}\right)_m = \left(\frac{P-P_0}{k_0}\right)_a m^{-n} \tag{2.27}$$

Once the function $\frac{U}{c_0}(m)$ is known, we may proceed to the calculation of shock front travel time curves in the neighborhood of the source region. Let a spherical shock front of initial amplitude $(P-P_0)_a$ leave the surface of the source sphere at time $t_0 = 0$. Then at any subsequent time t one has

$$t = \int_R \frac{dR}{U(R)} \tag{2.28}$$

In terms of m and the quantity U/c_0, (2.28) may be written

$$T = \frac{c_0 t}{a} = \int_1^m \left[\frac{U}{c_0}\right]^{-1}(m)\, dm \tag{2.29}$$

where $T = \frac{c_0 t}{a}$ is a dimensionless time. This function must be evaluated by numerical integration, since the algebraic solution of (2.25) is so intractable.

It is a matter of considerable interest to compare the travel time curve of a shock front with that of an ordinary P wave that has left the surface of the source sphere $R = a$ at the same time t_0 as the shock. Whether both types of waves are generated at this time, or whether the P wave observed at large focal distances from the earthquake is merely a degenerate shock pulse, is a question that cannot be settled without adequate experimental evidence. However, let us assume here that a shock pulse and a P wave are simultaneously generated at time t_o. S waves cannot be produced in this idealized source model. If the initial magnitude of $P-P_o$ is sufficiently high, the shock pulse will be propagated at a greater velocity than the P wave. At a subsequent time t, however, the shock will have decayed to $m^{-n}(P-P_o)_a$. As it loses amplitude it approaches the sonic velocity c_o, where

$$c_o = \sqrt{\frac{k_o}{\rho_o}}$$

The P wave is travelling at a velocity c given by

$$c = \sqrt{\frac{k_o + 4/3\,\mu_E}{\rho_o}}$$

where μ_E = ordinary elastic rigidity modulus of the medium.

Thus two situations may arise:

(1) The initial shock velocity U is greater than the P wave velocity, c. In this case the shock will first lead the P wave, but at a later time t will have decayed sufficiently so that the P wave will catch up and overtake it.

(2) The initial shock velocity U is equal to or less than c. In either case the P wave will lead the shock pulse for all $t \geq t_o$. We notice that the shock wave leads the P wave only as long as

$$U > \sqrt{\frac{k_o + 4/3\,\mu_E}{\rho_o}}$$

The shock which leaves the source sphere at $t=t_o$ thus decays to an acoustic wave, or "an infinitely weak shock". It may thus be considered to degenerate into a P wave.

At distances far from the source of a disturbance it might therefore be possible to observe two <u>direct</u> P wave phases, where one corresponds to the degenerate shock pulse and the other to the compressional wave that left the source sphere simultaneously with the shock. Which phase arrives first depends upon satisfaction of initial conditions (1) or (2) above.

We must mention that possible interaction between the two pulses at their points of intersection is not considered here. The objection may also be raised that it is meaningless to speak of the rigidity, μ_E, in the near-source region. It must be borne in mind, however, that the solid behaves as a liquid only at pressures developed across a shock front, and that one can therefore not neglect rigidity in discussing the passage of small-amplitude disturbances even through the near-source region.

A somewhat related point is the variation of the bulk modulus, or incompressibility, k, with pressure. Birch (1952) has investigated the dependence of the compressibility, $\mathcal{K}$, ($\mathcal{K} = 1/k$) upon pressure, and found that application of Murnaghan finite strain theory yielded the following results

y	$\left(\frac{\partial \mathcal{K}_T}{\partial P}\right)_T$
1.000	4.00
1.315	3.31
1.656	3.03
2.024	2.87

$\mathcal{K}_T$ = isothermal compressibility

<u>Table</u> 2.1: $\left(\frac{\partial \mathcal{K}_T}{\partial P}\right)_T$ as a function of compression.
Source: Birch (1952), p. 246.

Since this variation is negligible compared to the change of $(P-P_o)$ in the near-source region, use of the isothermal bulk modulus at zero pressure, k_o, is justified for the rough calculations presented here.

The solid curves in Figure 2.4 represent travel time paths for shocks whose <u>initial</u> amplitude is given by

$$\left(\frac{P-P_o}{k_o}\right)_a = \quad 1,5,10, \text{ and } 100,$$

and which have been calculated from equation (2.29). The value of n in (2.27) has been taken to be <u>two</u> (see below, p.55 ff.). The dashed curves represent possible travel-time paths of the P wave pulse that has left the source sphere together with the shock. For a radially outward travelling P wave, we thus have

$$R = V_p t$$

or,

$$\frac{R}{a} = \frac{V_p t}{a} = \frac{V_p}{a}\frac{a T}{c_o} = \frac{V_p}{c_o} T$$

and therefore

$$T = \frac{c_o}{V_p} m = s m \qquad (2.30)$$

where $s = c_o/c$. Since both pulses are assumed to originate at $m = 1$, the equation of the P wave travel time curve in the near source region is

$$T = \frac{c_o}{V_p}(m-1) = s(m-1) \qquad (2.31)$$

in the (T,m) plane. These P wave travel-time curves have been plotted in Figure 2.4 for $s = 1$, 0.9, 0.8, 0.6, 0.4, and 0.2. When $V_p = c_o$, $s = 1$, and (2.31) gives $T = (m-1)$, which yields the travel time path of the infinitely weak shock that travels at constant acoustic velocity c_o. Such a wave can, of course, never actually be produced in a solid, since the acoustic pulse will travel at velocity V_p, where V_p is a function of the rigidity, μ_ε, as well as of k_o and ρ_o.

Intersections of the solid shock curves with the dashed P wave curves will then give the particular values of T and m beyond which the shock will trail the P wave. If the P curve is tangent to the shock curve at $m = 1$, $U_{initial} = c_{initial}$, and the P wave will lead the shock for all $T \geqslant 0$. If $U_{initial} < c_{initial}$, the final lead of the P wave over the shock pulse will be correspondingly greater still. As the shock gradually becomes an acoustic pulse, its travel time curve will tend to become parallel to the curve $T = s(m-1)$.

Now it may be argued that the final velocity of the shock pulse, c_o, at which it travels once $P-P_o \ll \mathcal{S}$, should be given by

$$c_o = \sqrt{\frac{k_o + \frac{4}{3}\mu_\varepsilon}{\rho_o}}$$

rather than by

$$c_o = \sqrt{\frac{k_o}{\rho_o}}$$

since the propagation velocity of an infinitely weak shock, or ordinary acoustic pulse, should be a function of the solid rigidity

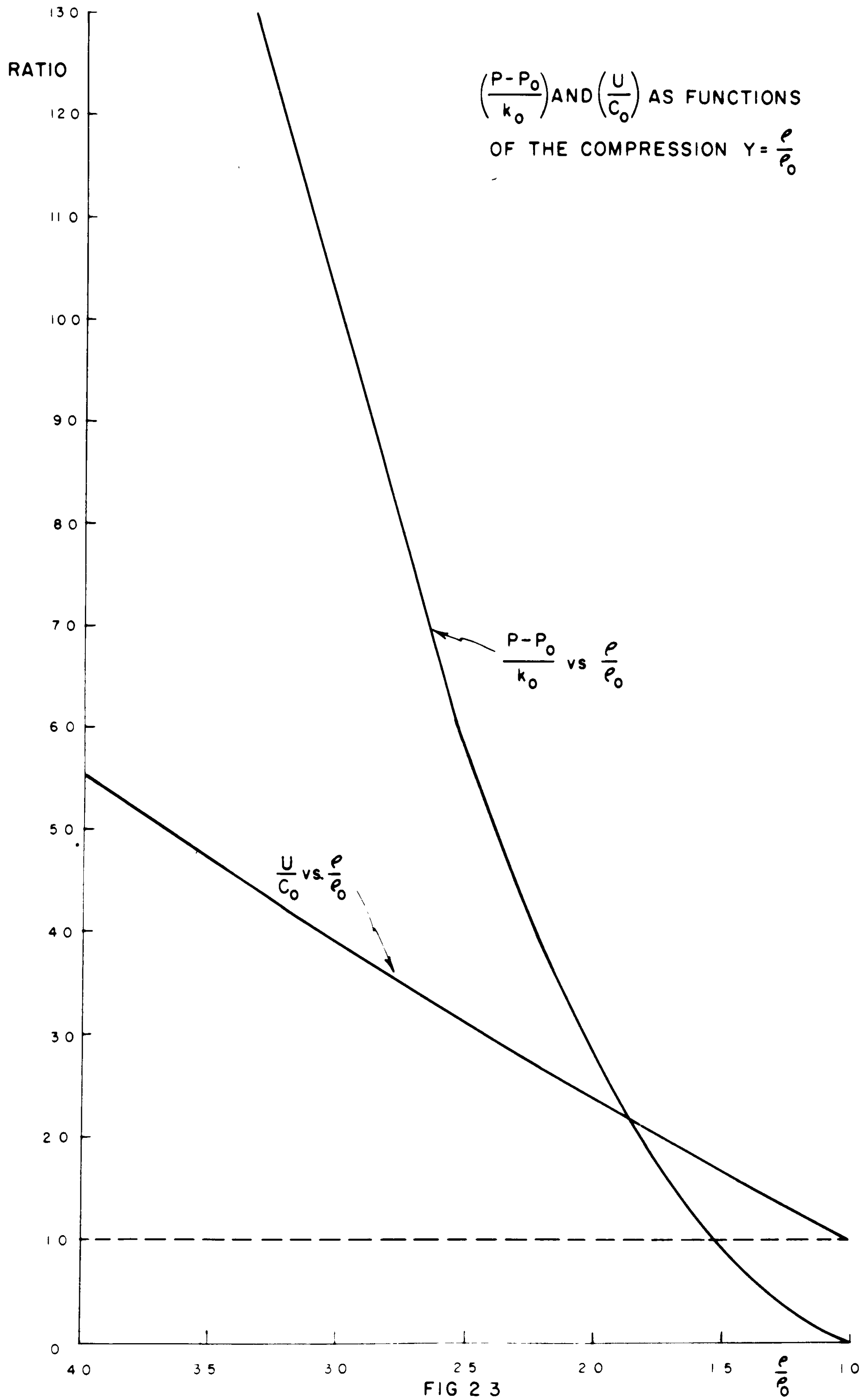
RATIO
$\left(\frac{P-P_0}{k_0}\right)$ AND $\left(\frac{U}{C_0}\right)$ AS FUNCTIONS
OF THE COMPRESSION $Y = \frac{\rho}{\rho_0}$
$\frac{P-P_0}{k_0}$ vs $\frac{\rho}{\rho_0}$
$\frac{U}{C_0}$ vs. $\frac{\rho}{\rho_0}$
13 0
12 0
11 0
10 0
9 0
8 0
7 0
6 0
5 0
4 0
3 0
2 0
1 0
0
4 0
3 5
3 0
2 5
2 0
1 5
1 0
$\frac{\rho}{\rho_0}$

FIG 2 3

μ_E as well. However, it is not at all clear at this time at what point one must cease to treat the solid as a liquid as far as the reaction of the medium to a shock wave is concerned. Experimental work is required to clarify this question.

Let us consider an actual example illustrating the use of Fig. 2.4. Assume that $\left(\frac{P-P_0}{k_0}\right)_a = 100$, and that $s = c_0/V_p = 0.6$. In this case the shock and P travel time curves intersect at $T_1 \sim 13.5$ and $m_1 \sim 23.5$. For $T_1 < 13.5$ and $m_1 < 23.5$, the shock will lead the P wave; for $T_1 > 13.5$ and $m_1 > 23.5$, the converse will be true. The dimensionless times and distances may be converted to their standard equivalents, t and R, if the appropiate values for a and m be substituted into the expressions

$$t = \frac{a}{c_0} T$$

$$R = m a \tag{2.32 a,b}$$

Thus, if in the present case we take $a = 1$ Km and $c_0 = 2$ Km/sec, $t_1 = \frac{1}{2} \times 13.5 = 6.8$ seconds and $R_1 = 23.5 \times 1 = 23.5$ Km. If we take the bulk modulus, k_0 , as 10^{10} dynes/cm^2, the shock pulse has an initial amplitude given by $(P-P_0)_a = 10^2 \cdot 10^{10} = 10^{12}$ dynes/cm^2 = 10^6 bars. This example illustrates the flexibility of plotting the shock and P wave travel time curves in the (T,m) plane.

The analysis outlined so far enables us to make estimates about the actual size of the zone around a source sphere in which shock phenomena may be expected to play a significant role. Moreover, knowledge of near-source travel time curves would permit us not only to arrive at a $\frac{P-P_o}{k_o}(m)$ function by a reversion of the procedure presented here, but also to establish the exact values of the constants of the equation of state (2.6) or (2.7). Work somewhat along these lines has been done by Walsh and co-workers (Loc. Cit., p. 20) on metals, and by Hughes and McQueen (1957) on rocks.

We now proceed to the discussion of a method that will provide us with a quantitative estimate of the actual amounts of energy dissipated in the neighborhood of the source sphere. For this purpose we return to eq. (2.20), which can be written in the form,

$$\frac{2\rho_o}{k_o}\Delta E = \frac{y-1}{y}\left[\frac{3}{2}y^{5/3}(y^{2/3}-1) + \frac{2P_o}{k_o}\right] \quad (2.33)$$

where $\Delta E = E - E_o$. This formula gives the difference in internal energy, ΔE, between the disturbed part of the medium immediately behind the shock discontinuity, and the undisturbed medium ahead. (See Fig. 2.2). Without loss of generality, we may take E_o and P_o to be zero, since we are only interested in the internal energy decrease, ΔE, across the shock discontinuity as a function of distance from the source sphere. Eq. (2.33) thus becomes

$$\frac{2\rho_o}{k_o}\Delta E = \frac{y-1}{y}\left[\frac{3}{2}y^{5/3}(y^{2/3}-1)\right] = \frac{y-1}{y}\frac{P-P_o}{k_o} \quad (2.34)$$

where (2.19 a) has been used to eliminate the term in the square brackets.

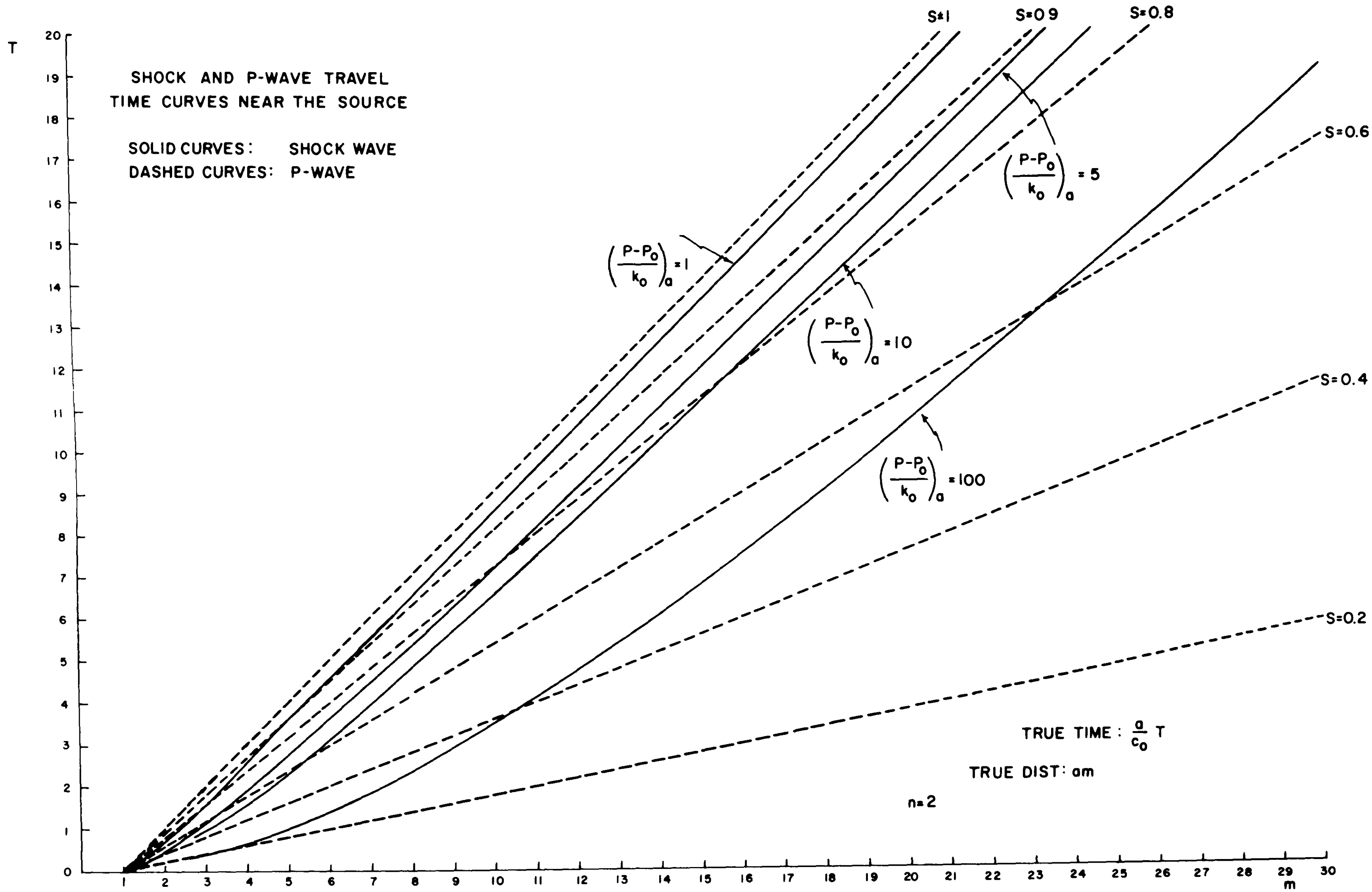
SHOCK AND P-WAVE TRAVEL
TIME CURVES NEAR THE SOURCE
SOLID CURVES: SHOCK WAVE
DASHED CURVES: P-WAVE
T
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
m
S=1
S=0.9
S=0.8
S=0.6
S=0.4
S=0.2
$\left(\frac{P-P_o}{k_o}\right)_a = 1$
$\left(\frac{P-P_o}{k_o}\right)_a = 5$
$\left(\frac{P-P_o}{k_o}\right)_a = 10$
$\left(\frac{P-P_o}{k_o}\right)_a = 100$
TRUE TIME : $\frac{a}{c_o}$ T
TRUE DIST: am
n=2

FIG. 2.4

For an assumed decay law (2.24), $\Delta E(R)$ is easily calculable from (2.34). Before passage of the shock front, the region ahead of it is assumed to be undisturbed. Therefore the decrease in internal energy across the discontinuity, ΔE, may be assumed to represent the amount that is "leaking" from the shock front into the medium. If we then sum all the increments $\Delta E(R, R+\Delta R)$ over a succession of spherical shells of thickness ΔR, we shall have arrived at the total amount of energy dissipated within a spherical shell of thickness $R-a$, in whose geometric center is embedded the source sphere of radius a. Eq. (2.34) expresses the principle of conservation of energy across the shock front. At time $t=0$, all the energy is contained in the shock pulse; as this pulse decays, it loses energy to the medium through which it is travelling. As $t \to \infty$, the pulse will have decayed to zero amplitude, and <u>all</u> its original energy will then have been imparted to the medium.

The energy transferred from the shock pulse to the medium in a spherical shell of thickness ΔR is $\Delta E(R, R+\Delta R)$. Consequently the amount of energy transferred to the medium in a spherical shell of radius $R-a$, E_{cum}, is given by

$$E_{cum} = 4\pi\rho_0 \int_a^R \Delta E(R, R+\Delta R) R^2 \, dR \qquad (2.35)$$

or, in terms of the dimensionless distance m,

$$E_{cum} = 4\pi\rho_0 a^3 \int_1^m \Delta E(m, m+\Delta m) m^2 \, dm \qquad (2.36)$$

For the assumed decay law of form (2.24), the quantity $\frac{2\rho_0}{k_0}\Delta E$ will be known as a function of m from (2.34). Accordingly one writes

$$\frac{2\,E_{cum}}{k_0 a^3} = 4\pi \int_1^m \frac{2\rho_0}{k_0}\,\Delta E\,(m, m+\Delta m)\,m^2\,dm \qquad (2.37)$$

(2.37) is evaluated by numerical integration with data computed from (2.34).

Before proceeding to an examination of the results of such calculations, it will be fruitful to take a closer look at the assumed decay law (2.24),

$$(p-p_0)_R = (p-p_0)_a \left(\frac{a}{R}\right)^n, \qquad \begin{array}{l} R \geq a \\ n > 1 \end{array} \qquad (2.24)$$

As it stands, (2.24) does not permit us to distinguish between the familiar phenomenon of spherical divergence, a purely geometrical effect, and actual wave attenuation, which results in a transfer and ultimate degradation of energy from the shock pulse to the medium. Spherical divergence reduces the amplitude of a propagating disturbance as the inverse first power of its distance from the source; since it is a purely geometrical phenomenon, all pulses, be they large amplitude shocks or infinitesimal acoustic waves, are affected in a like manner, as long as they are spherical waves. Let us therefore modify (2.24) in such a way that the two effects can be considered separately. We write

$$(p-p_o)_R = (p-p_o)_a \left\{ \frac{a}{R+(R^{\delta}-1)} \right\}, \qquad \delta \geq 0 \tag{2.38}$$

where the exponent δ corresponds to dissipative processes alone. The first term in the denominator of the factor within braces will then correspond to spherical divergence, while the second will account for actual dissipation. If there is no damping, the exponent δ is zero, and (2.38) will reduce to

$$(p-p_o)_R = (p-p_o)_a \left(\frac{a}{R} \right) \tag{2.39}$$

which is a special case of (2.24) with n=1. Eq. (2.39) does therefore not represent a damping law as such, since it only expresses the geometrical spreading effect. We now rewrite (2.38) in the form

$$(p-p_o)_R = \frac{(p-p_o)_a \dfrac{a}{R^{\delta}}}{1 + \left(\dfrac{R-1}{R^{\delta}} \right)} \tag{2.40}$$

Since $\left| \dfrac{R-1}{R^{\delta}} \right| < 1$ for $\delta \geq 1$ and $R \geq 1$, (2.40) becomes

$$(p-p_o)_R = (p-p_o)_a \frac{a}{R^{\delta}} \left[1 - \left(\frac{R-1}{R^{\delta}} \right) + \left(\frac{R-1}{R^{\delta}} \right)^2 - \dots\dots \right] \tag{2.41}$$

$$\approx (p-p_o)_a \frac{a}{R^{\delta}}, \qquad \delta \geq 1$$

Eq. (2.41) is similar to (2.24) except that we chose to express (2.24) in such a form that $(P-P_o)_R = (P-P_o)_a$ at $R = a$. It is evident from (2.41) that for any $\delta > 1$, the error in neglecting spherical divergence is small, this error decreasing rapidly with increasing R. One may therefore conclude that (2.24) is an adequate representation of a damping law involving spherical geometry. In other words, even though we should use (2.40) as the form of our damping law, the error incurred by taking the simpler form (2.24) is not great. Essentially, this approximation is equivalent to the assertion that the exponent n in this equation corresponds exclusively to dissipative damping, and not to geometric spreading.

The actual value of the exponent n, which may itself be a function of R and other parameters of the medium, must be found either from a rigorous solution of the non-linear hydrodynamic equations of motion, or from empirical measurements. However, we have seen that the first of these approaches is futile until the theory of non-linearity is better understood. The second alternative has been used both in underwater explosion studies, as well as in laboratory experiments on metals. W.I. Duvall (1953) has published the results of some experimental work done with rock blasts at the Bureau of Mines (see also Chapter I of this thesis). He found that the decrease of peak stress with distance close to the shot point could be given by a law similar to (2.24), where the exponent n ranged in value from 1.6 to 2.5 for various rock types and explosives. In the present work, however, we are primarily interested in the dissipative processes that occur near earthquake focii. Although it is of course impossible to secure stress measurements of this nature in the case of earthquakes, such data could presumably be gathered from underground atomic blasts. The question arises whether the detonation of such a small volume of material leads to near focus dissipation processes that might be expected to be of a nature similar to those occurring near an earthquake focus. Yet little is known about the mechanics of earthquake generation, so that no definite statements can be made in this connection.

In view of these considerations, we shall:

(1) Postulate that the stress release in a focal region is of a sufficient order of magnitude that an ideal shock front can be assumed to have formed in the interior of a focal sphere of radius a within a few milliseconds after the major stress release has taken place.

(2) Postulate that a representative value of the exponent n in (2.24) is $n = 2$.

The calculations that have been carried out here on the basis of equations (2.24), (2.28), and (2.35) are thus all restricted to the case $n = 2$. Nevertheless, the formulas can easily be evaluated for other values of n, $n > 1$, since the computations, although rather laborious, are straightforward. Because of the dearth of adequate empirical data, more general calculations do not appear warranted. In any event, the particular case $n = 2$ chosen here will serve to provide us with a good feeling for the orders of magnitude of shock velocity

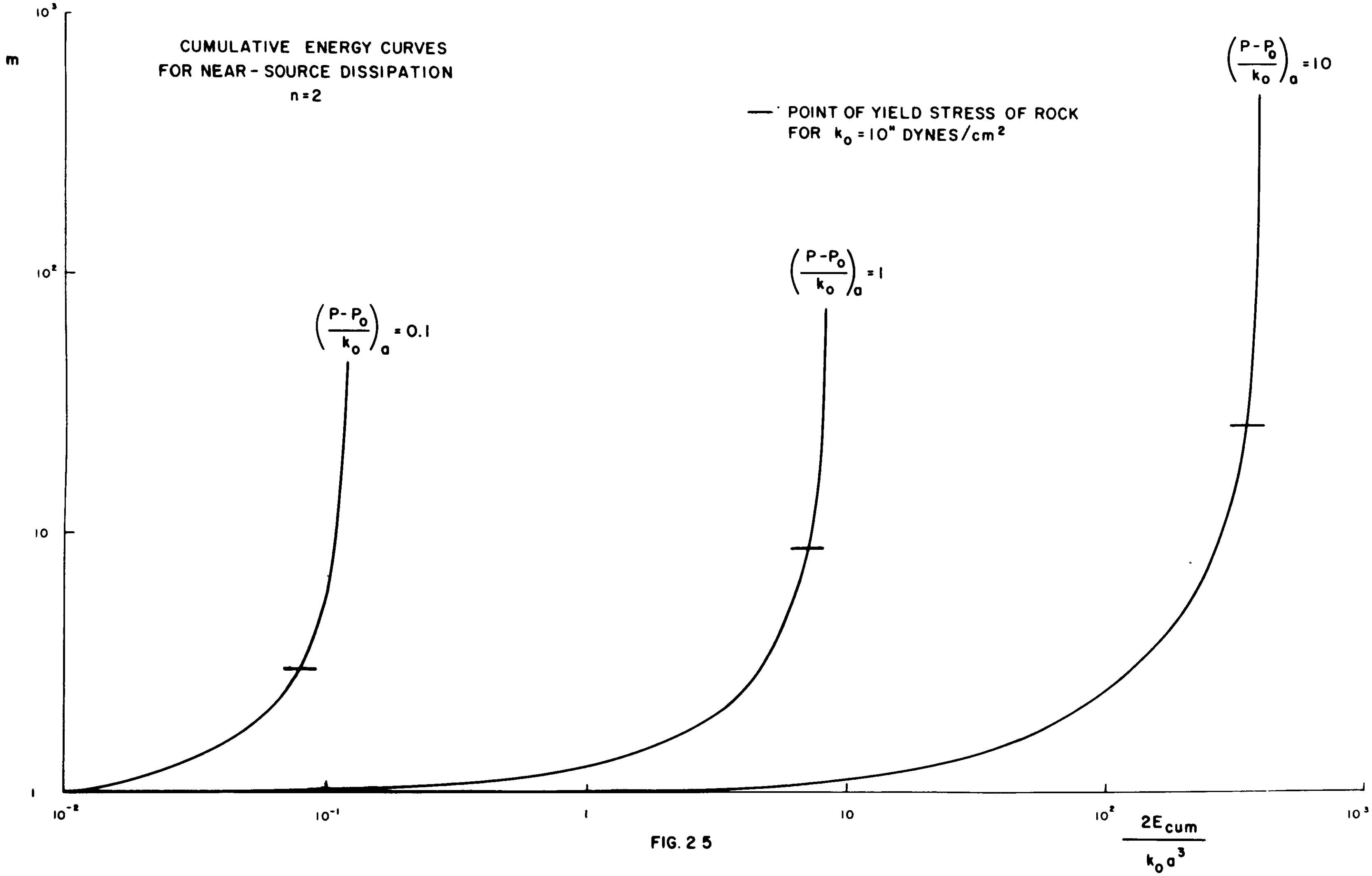
CUMULATIVE ENERGY CURVES
FOR NEAR - SOURCE DISSIPATION
n = 2
— POINT OF YIELD STRESS OF ROCK
FOR $k_o = 10^{11}$ DYNES/cm²
$\left(\frac{P-P_o}{k_o}\right)_a = 0.1$
$\left(\frac{P-P_o}{k_o}\right)_a = 1$
$\left(\frac{P-P_o}{k_o}\right)_a = 10$
m
10^3
10^2
10
1
10^{-2}
10^{-1}
1
10
10^2
10^3
$\frac{2E_{cum}}{k_o a^3}$

FIG. 2 5

and energy dissipation.

The energy calculations have again been carried out in terms of the dimensionless distance m ($m = R/a$) for the cases $\left(\frac{P-P_0}{k_0}\right)_a$ = 0.01, 0.1, 1, and 10. Tables 2.2 to 2.5 present the results of this anlysis, together with corresponding values for the dimensionless time, T, and the shock-to-acoustic velocity ratio, U/c_0 previously computed. The quantity $\frac{2 E_{cum}}{k_0 a^3}$ is plotted against m in Figure 2.5 for the cases $\left(\frac{P-P_0}{k_0}\right)_a$ = 0.1, 1, and 10.

If one wishes to gain a still more quantitative insight into the results of these computations, it is necessary to assume specific values for the bulk modulus, k_0 ; the acoustic velocity, c_0 ; and the radius of the source sphere, a . Tables 2.6 to 2.9 have been prepared by taking:

$$a = 1 \text{ Km}$$
$$k_0 = 10^{11} \text{ dynes/cm}^2$$
$$c_0 = 2 \text{ Km/sec}$$

In addition, Table 2.7 b was calculated for the case a = 10 Km, k_0 and c_0 remaining as above.

The first column of these tables gives R in Km; the second the time t taken by the shock front to reach a point on a spherical surface at a distance R from the surface of the source sphere $R = a$; the third the ratio U/c_0; and the fourth, the cumulative energy, in ergs, transmitted by the pulse to the medium up to that point. The fifth column gives the total volume of the shell, of thickness R, surrounding the source sphere. From entries in the fourth and fifth columns it is possible to calculate the mean energy density, or specific energy, that exists within successive spherical shells of thickness ΔR immediately after the passage of the decaying shock front. The left half of the sixth column gives this specific energy in ergs/cm^3; the right half, in calories/cm^3.

We assume that all the energy that is dissipated during the passage of the shock appears as heat. Thus the specific energies computed here only hold strictly for brief times after the disturbance has traversed the shell. However, since the thermal conductivity of rock is so small, a considerable period of time will be required to conduct the heat so produced away. This problem will be treated in greater detail in Chapter V.

A mathematical difficulty is presented by the question of convergence of the integral (2.37). The small horizontal lines that intersect the energy curves of Figure 2.5 have been drawn at the points at which $(P-P_o)_R$ has reached the value 10^9 dynes/cm^2 for an assumed $k_o = 10^{11}$ dynes/cm^2. Now equation (2.24) shows that $(P-P_o)_R$ can only vanish at $R = \infty$, i.e., at an infinite distance from the source sphere. This means that in practice, no matter how far the shock front may have travelled, energy increments will still contribute to the total value of E_{cum}, eq. (2.37). A little reflection will convince us, however, that this is merely a mathematical, rather than a physical difficulty, for by far the greatest part of the contribution to the integral (2.37) will take place before the shock has degenerated into an acoustic pulse. This can be clearly seen from the three curves plotted in Figure 2.5; in all three cases, the E_{cum} curves rapidly become parallel to the vertical axis for $(P-P_o)_R < \mathcal{G}$.

If the theory developed here is tenable, then it is only the energy that is propagated in the form of acoustic waves that can contribute to the energies measured by seismograph stations at the surface of the earth. Now acoustic propagation can only take place when $(P-P_o)_R \ll \mathcal{G}$; thus the energy dissipated in the shock zone will only be detectable in the form of heat flow at the surface a long period after the earthquake has occurred. We shall return to this question in Chapter V.

In the next two chapters, we shall switch our attention to propagation problems that may be expected to arise in the acoustic region, where the shock front has decayed to a small amplitude pulse, $(P-P_o)_R \ll \mathcal{G}$, ($\mathcal{G} = 10^9$ dynes/cm^2 for rocks). In the final chapter we shall then attempt to take an overall glance at the propagation of the original shock front from the source sphere $R = a$ to its final conversion into a train of acoustic waves of infinitesimal amplitude.

m	$\left(\frac{p-p_0}{k_0}\right)_m$	T	$\frac{2E_{cum}}{k_0 a^3}$	$\frac{U}{c_0}$
1.0	1.00×10^{-2}	0.0	0	1.00
1.2	6.94×10^{-3}	0.2	2.14×10^{-4}	
1.4	5.08	0.4	3.66	
1.6	3.91	0.6	4.66	
1.8	3.09	0.8	5.42	
2	2.50	1.0	6.30	
3	1.11	2	8.45	
4	6.25×10^{-4}	3	9.45	
5	4.00	4	1.01×10^{-3}	
6	2.78	5	1.05	
7	2.05	6	1.08	
8	1.56	7	1.10	
9	1.23	8	1.12	
10	1.00	9	1.14	
20	2.50×10^{-5}	19	1.20	
30	1.11	29	1.22	
40	6.25×10^{-6}	39	1.23	
50	4.00	49	1.235	
∞	0	∞	1.256	

<u>Table</u> 2.2: $\left(\frac{p-p_0}{k_0}\right)_{m=1} = 0.01$; $n = 2$

1	1.00×10^{-1}	0.00	0.000	1.11
2	2.50×10^{-2}	0.99	0.056	1.04
3	1.11	1.99	0.080	1.01
4	6.25×10^{-3}	3.00	0.090	1.00
5	4.00	4.00	0.098	
6	2.78	5.00	0.102	
7	2.04	6.00	0.105	
8	1.56	7.00	0.107	
9	1.23	8.00	0.109	
10	1.00	9.00	0.110	
50	4.00×10^{-5}	49.00	0.120	

<u>Table</u> 2.3: $\left(\frac{p-p_0}{k_0}\right)_{m=1} = 0.1$; $n = 2$

m	$\left(\frac{p-p_0}{k_0}\right)_m$	T	$\frac{2E_{com}}{k_0 a^3}$	$\frac{U}{c_0}$
1.0	1.00	0.00	0.00	1.69
1.1	0.826	0.06	0.35	1.61
1.2	0.694	0.15	0.74	1.54
1.4	0.508	0.27	1.4	1.42
1.6	0.391	0.40	2.1	1.35
1.8	0.309	0.58	2.6	1.28
2.0	0.250	0.79	3.1	1.24
3	0.110	1.6	4.6	1.12
4	0.0625	2.6	5.4	1.07
5	0.0400	3.6	6.0	1.05
6	0.0278	4.6	6.5	1.04
7	0.0204	5.6	6.8	1.03
8	0.0156	6.6	7.0	1.02
9	0.0123	7.6	7.2	1.01
10	0.0100	8.6	7.3	1.01
20	0.0025	18.6	8.0	1.01
40	0.0006	38.6	8.3	↓
60	0.0003	58.6	8.4	

Table 2.4: $\left(\frac{p-p_0}{k_0}\right)_{m=1} = 1$; $n = 2$

1.0	10.00	0.00	0.0	3.88
1.1	8.26	0.04	8.0	3.58
1.2	6.94	0.08	14.5	3.33
1.4	5.08	0.13	30.0	2.96
1.6	3.91	0.20	45.5	2.68
1.8	3.09	0.29	58	2.06
2.0	2.50	0.4	72	1.94
3	1.11	0.9	114	1.74
4	6.25×10^{-1}	1.6	163	1.50
5	4.00	2.3	195	1.35
6	2.78	3.1	220	1.27
7	2.04	3.9	240	1.20
8	1.56	4.8	255	1.17
9	1.23	5.7	269	1.13
10	1.00	6.6	280	1.11
20	2.50×10^{-2}	15.8	340	1.04
30	1.11	25.0	360	1.01
40	6.25×10^{-3}	34.2	370	1.01
50	4.00	43.4	378	↓
100	1.00	89	395	
200	2.50×10^{-4}	181	400	

Table 2.5: $\left(\frac{p-p_0}{k_0}\right)_{m=1} = 10$; $n = 2$

R (Km)	$(P-P_o)_R$ dyn/cm^2	t sec	$\frac{U}{c_o}$	E_{cum} (ergs)	Shell Vol. (cm^3)	Specific Energy ergs/cm^3	Specific Energy cals/cm^3
1.0	1.00×10^{9}	0.0	1.00	0	0		
1.2	6.94×10^{8}	0.1		1.07×10^{22}	3.06×10^{15}	3.5×10^{6}	8.4×10^{-2}
1.4	5.08	0.2		1.83	7.29	1.8×10^{6}	4.3
1.6	3.91	0.3		2.33	1.30×10^{16}	8.8×10^{5}	2.1
1.8	3.09	0.4		2.77	2.02	6.1	1.5
2	2.50	0.5		3.15	2.93	4.2	1.0
3	1.11	1.0		4.23	1.09×10^{17}	1.4	3.3×10^{-3}
4	6.25×10^{7}			4.73	2.64	3.2×10^{4}	7.7×10^{-4}
5	4.00			5.05	5.19	1.3	3.1×10^{-4}
6	2.78			5.25	9.00	5.3×10^{3}	1.3
7	2.04			5.40	1.43×10^{18}	2.8	6.7×10^{-5}
8	1.56			5.50	2.14	1.4	3.3
9	1.23			5.60	3.05	1.1	2.6
10	1.00			5.70	4.18	8.8×10^{2}	2.1
20	2.50×10^{6}			6.00	3.35×10^{19}	1.0	2.4×10^{-6}
30	1.11			6.10	1.13×10^{20}	13	3.1×10^{-7}
40	6.25×10^{5}			6.15	2.68	3	7.2×10^{-8}
50	4.00			6.18	5.24	1	2.4×10^{-8}
∞	0			6.28	∞		

Table 2.6 $\left(\frac{P-P_o}{k_o}\right)_{R=1\,Km} = 0.01$; $n=2$

For: $a = 1$ Km; $k_o = 10^{11}$ dynes/cm^2

$c_o = 2$ Km/sec.

R km.	$(P-P_o)_R$ (dyn/cm^2)	t (sec)	$\frac{U}{c_o}$	E_{cum} (ergs)	Shell Volume.	Specific Energy ergs/cm^3	Specific Energy cals/cm^3
1	1.00×10^{10}	0.00	1.11	0.0	0.00		
2	2.50×10^{9}	0.49	1.04	2.8×10^{24}	2.93×10^{16}	9.6×10^{7}	2.39
3	1.11	0.99	1.01	4.0	1.09×10^{17}	1.5	0.36
4	6.25×10^{8}	1.50	1.01	4.5	2.64	3.2×10^{6}	7.7×10^{-2}
5	4.00			4.9	5.20	1.6	3.8
6	2.78			5.1	9.00	5.3×10^{5}	1.3
7	2.04			5.25	1.43×10^{18}	2.8	6.7×10^{-3}
8	1.56			5.35	2.14	1.4	3.4
9	1.23			5.45	3.04	1.1	2.6
10	1.00			5.50	4.19	4.3×10^{4}	1.0
50	4.00×10^{6}			6.00	5.24×10^{20}	9.6×10^{2}	2.3×10^{-5}

Table 2.7 a: $\left(\frac{P-P_o}{k_o}\right)_{R=1\,Km} = 0.1$; $n = 2$

For $a = 1$ Km ; $k_o = 10^{11}$ dynes/cm^2 $c_o = 2$ Km/sec.

R (km)	$(P-P_o)_R$ (dyn/cm²)	t (sec)	$\frac{U}{c_o}$	E_{cum} (ergs)	Shell volume.	Specific Energy ergs/cm³	Specific Energy cals/cm³
10	1.00×10^{10}	0.0	1.11	0.00	0.00		
						9.6×10^{7}	2.39
20	2.50×10^{9}	4.9	1.04	2.80×10^{27}	2.93×10^{19}		
						1.5	0.36
30	1.11	9.9	1.01	4.0	1.09×10^{17}		
						3.2×10^{6}	7.7×10^{-2}
40	6.25×10^{8}	15	1.01	4.5	2.64		
						1.6	3.8×10^{-2}
50	4.00			4.9	5.20		
						5.3×10^{5}	1.3
60	2.78			5.1	9.00		
						2.8	6.7×10^{-3}
70	2.04			5.25	1.43×10^{21}		
						1.4	3.4
80	1.56			5.35	2.14		
						1.1	2.6
90	1.23			5.45	3.04		
						4.3×10^{4}	1.0
100	1.00			5.50	4.19		
						9.6×10^{2}	2.3×10^{-5}
500	4.00×10^{6}			6.00	5.24×10^{23}		

Table 2.7 b: $\left(\frac{P-P_o}{k_o}\right)_{R=10Km} = 0.1$; $n=2$

For $a = 10$ Km; $k_o = 10^{11}$ dynes/cm²

$c_o = 2$ Km/sec.

R (Km)	$(P-P_o)_R$ (dyn/cm^2)	t (sec)	$\frac{U}{c_o}$	E_{cum} (ergs)	Shell Volume.	Specific Energy ergs/cm^3	cals/cm^3
1.0	1.00×10^{11}	0.00	1.69	0.00	0.00	1.27×10^{10}	3.04×10^{2}
1.1	8.26×10^{10}	0.03	1.61	1.75×10^{25}	1.38×10^{15}	1.16	2.77
1.2	6.94	0.08	1.54	3.70	3.06	7.80×10^{9}	1.9
1.4	5.08	0.14	1.42	7.00	7.29	6.10	1.5
1.6	3.91	0.20	1.35	1.05×10^{26}	1.30×10^{16}	3.50	84
1.8	3.09	0.29	1.28	1.30	2.02	2.70	65
2.0	2.50	0.40	1.24	1.55	2.93	9.40×10^{8}	23
3	1.10	0.80	1.12	2.30	1.09×10^{17}	2.60	6.2
4	6.25×10^{9}	1.30	1.07	2.70	2.64	1.10	2.6
5	4.00	1.80	1.05	3.00	5.19	6.60×10^{7}	1.6
6	2.78	2.30	1.04	3.25	9.00	2.80	0.67
7	2.04	2.80	1.03	3.40	1.43×10^{18}	1.40	0.33
8	1.56	3.30	1.02	3.50	2.14	1.10	0.26
9	1.23	3.80	1.01	3.60	3.05	4.40×10^{6}	0.11
10	1.00	4.30	1.01	3.65	4.18	1.20	0.03
20	2.50×10^{8}			4.00	3.35×10^{19}	6.40×10^{4}	1.5×10^{-3}
40	6.00×10^{7}			4.15	2.68×10^{20}	7.90×10^{3}	1.9×10^{-4}
60	3.00			4.20	9.05		

Table 2.8: $\left(\frac{P-P_o}{k_o}\right)_{R=1\ Km} = 1$; $n = 2$

For $a = 1$ Km; $k_o = 10^{11}$ dynes/cm^2

$c_o = 2$ Km/sec.

R (Km)	$(P-P_o)_R$ (dyn/cm^2)	t (sec)	$\frac{U}{c_o}$	E_{cum} (ergs)	Shell Volume.	Specific Energy ergs/cm^3	cals/cm^3
1.0	1.00×10^{12}	0.00	3.88	0.00	0.00	2.9×10^{11}	7.0×10^{3}
1.1	8.26×10^{11}	0.02	3.58	4.00×10^{26}	1.38×10^{15}	2.0	4.8
1.2	6.94	0.04	3.33	7.30	3.06	1.8	4.3
1.4	5.08	0.07	2.96	1.50×10^{27}	7.29	1.4	3.4
1.6	3.91	0.10	2.68	2.30	1.30×10^{16}	8.3×10^{10}	2.0
1.8	3.09	0.15	2.06	2.90	2.02	6.6	1.6
2.0	2.50	0.20	1.94	3.60	2.93	2.6	6.2×10^{2}
3	1.11	0.45	1.74	5.70	1.09×10^{17}	1.6	3.8
4	6.25×10^{10}	0.80	1.50	8.20	2.64	6.3×10^{9}	1.5
5	4.00	1.15	1.35	9.80	5.19	3.2	77
6	2.78	1.55	1.27	1.10×10^{28}	9.00	1.9	46
7	2.04	1.95	1.20	1.20	1.43×10^{18}	1.2	28
8	1.56	2.40	1.17	1.28	2.14	7.7×10^{8}	19
9	1.23	2.85	1.13	1.35	3.05	4.4	11
10	1.00	3.30	1.11	1.40	4.18	1.0	2.4
20	2.50×10^{9}	7.90	1.04	1.70	3.35×10^{19}	1.3×10^{7}	0.31
30	1.11	12.5	1.01	1.80	1.13×10^{20}	3.2×10^{6}	7.7×10^{-2}
40	6.25×10^{8}			1.85	2.68	1.6	3.8
50	4.00			1.89	5.24	2.5×10^{5}	6.0×10^{-3}
100	1.00			1.98	4.19×10^{21}	6.8×10^{3}	1.6×10^{-4}
200	2.50×10^{7}			2.00	3.35×10^{22}		

Table 2.9: $\left(\frac{P-P_o}{k_o}\right)_{R=1\text{ Km}} = 10$; $n=2$

For $a = 1$ Km; $k_o = 10^{11}$ dynes/cm^2; $c_o = 2$ Km/sec.

CHAPTER III

THE ATTENUATION OF NON-LINEAR SMALL AMPLITUDE STRESS WAVES IN SOLIDS

1. Introductory Remarks

Up to this point we have been concerned with the region around the source of a disturbance in which the resulting pulse amplitude is so large that its propagation is subject to shock wave theory. It was shown that as this pulse travelled outward from the source and decayed, it would eventually move at acoustic velocity and thus become an "infinitely weak shock", that is, a simple elastic wave. Clearly, very different physical processes govern the propagation of the wave once it has reached acoustic speeds. In particular, evidence from both exploration and earthquake seismology as well as from laboratory data indicates beyond any doubt that the damping that these waves suffer is extremely small. This is in marked contrast to the situation which exists while the pulse is still a shock, when the gradients across its front are of such magnitude that dissipative processes must be very strong, perhaps much stronger in many instances than the R^{-2} law of decay postulated for the numerical calculations in the last chapter.

When does a shock cease to be a shock and become an ordinary acoustic pulse? This question is perhaps somewhat ambiguous, because this transition point could be defined in various ways, none of which would necessarily lead to unique results. One convenient criterion is the yield stress of the solid, $\mathcal{S}$, which for rocks is about 10^9 dynes/cm^2. We shall use this convention here, and thus consider pressure discontinuities to constitute shocks or sound pulses according to whether $P-P_0 \gg\ll \mathcal{S}$.

The small horizontal lines that intersect the energy curves of Figure 2.5 have been drawn at the points at which $(P-P_o)$ has reached the value 10^9 dynes/cm^2 for an assumed $k_o = 10^{11}$ dynes/cm^2. Obviously, the larger the value of $(P-P_o)_a$, the more distant from the source sphere will this transition point lie. It will be noticed that the energy curves still continue to grow beyond this point, although at a steadily decreasing rate. When the curves become parallel to the m axis, the total energy dissipated up to that point, E_{cum}, remains constant for all m larger than this critical value. But this occurs only at $m = \infty$, where $\frac{dE_{cum}}{dm} = 0$, and where the amplitude of the pulse has decayed to zero. One might be led to conclude that the propagation of the acoustic wave continues to be describable in terms of shock wave theory for $(P-P_o)_a \ll \mathcal{G}$.

Now classical elastic theory predicts that a pulse will be propagated without damping through any solid in which Hooke's law holds. Large amplitude shock pulses, on the other hand, must decay rapidly because of the enormous gradients that exist across their fronts. Neither state of affairs is in agreement with what is known from observation about the damping of small amplitude waves in solids. The Rankine-Hugoniot relations, upon which the analysis of the previous chapter is based, were derived under the assumption that the solid could be treated as a liquid when $(P-P_o) \gg \mathcal{G}$. This condition of course no longer holds when $(P-P_o) \ll \mathcal{G}$.

Let us consider again the exact equation of motion for plane one-dimensional fluid flow (2.8 b),

$$(\rho v)_t + (\rho v^2 + P)_x = 0 \qquad (2.8\ b)$$

The second term of this expression is always non-linear, while the third may or may not be linear, its non-linearity depending upon the relation that exists between P, the total acting pressure, and the strains produced as a result of P. In treating liquids one expresses this relationship in terms of a parameter ρ/ρ_0, rather than in terms of the strains e, as is done in the case of solids. The two methods are equivalent, nevertheless, because ρ/ρ_0 and e are related in simple ways (Birch, 1952). Thus the third term is non-linear if $P(\rho/\rho_0)$, $[\equiv P(y)]$ or, equivalently, if P(e) is a non-linear relationship. In the case of an elastic medium, $P(e) = M_E e$, (where M_E = an elastic modulus) represents a linear equation, and consequently the third term in (2.8 b) is linear. But if, say, P(y) is given by the Birch equation of state (2.17), which is a non-linear relation, then this term will be non-linear also. The strains e are the generalized higher order strains of Murnaghan (1937), and only reduce to the elastic strains, ε, in the infinitesimal theory.

In the case of shock waves one must thus consider non-linearities in both the second and third terms of (2.8 b). However, we have already seen in the previous chapter that <u>both</u> non-linearities are so great in this case that solutions of (2.8 b) cannot be found.

Once the amplitude of the wave has decayed, so that $P-P_0$ becomes of the order of $\mathcal{S}$, the yield stress of the solid, the particle velocity v has decreased sufficiently so that its square can be neglected with slight error. This can easily be appreciated from eq. (2.4),

$$v = \frac{\rho - \rho_0}{\rho} U \tag{2.4}$$

since, when $(P-P_0)$ becomes small, $\rho \rightarrow \rho_0$. As a result, the second term of (2.8 b) can be dropped when $P-P_0 \leqslant O(\delta)$. It is now only necessary to deal with the non-linearity of the third term of (2.8 b). One accomplishes this by considering the possible forms that P(e) or, more generally, $P(e, \dot{e}, T)$, where $\dot{}$ denotes differentiation w.r.t. time, may be expected to have in solids.

It has already been pointed out that the E_{cum} curves of Fig. 2.5 continue to grow beyond the point $P-P_0 = \delta$. Now we shall postulate here that the difference between the value of E_{cum} at $P-P_0 = \delta$, and the value of E_{cum} at $P-P_0 = 0$ is exactly equal to the energy imparted to the medium by the pulse, which has become acoustic for $P-P_0 < \delta$. On the other hand, we have seen that the upper limit of E_{cum} at the point $P-P_0 \doteq 0$ is impossible to establish in any physical situation unless one knows the original energy content of the entire shock front itself. But it has also been shown (see Tables 2.6 to 2.9) that the specific dissipation energies of shells in which $P-P_0$ has fallen to $O(\delta)$ is very small. Accordingly, we shall shift our attention from an attempt to estimate the amount of energy transferred to the medium beyond the point

$P-P_o < \mathcal{S}$, which we know is quite small, to a detailed examination of the forms that a wave attenuation coefficient α ,

$$A = A_{\mathcal{S}} e^{-\alpha x} \tag{3.1}$$

will have in the region $P-P_o \ll \mathcal{S}$. Here, A = amplitude at a distance x from the point $P-P_o = \mathcal{S}$, and $A_{\mathcal{S}}$ = amplitude at point $P-P_o = \mathcal{S}$.

It will be shown in the present as well as in the subsequent chapter that fruitful attacks on this problem can be made both in the linear and in many non-linear cases of great physical interest.

2. <u>The Equations of Small Amplitude Waves in Solids</u>.

The attenuation of stress waves in solids has been the subject of exhaustive investigation by countless workers ever since Stokes wrote his classical treatise on liquid viscosity in 1849. Detailed surveys of the literature have been published at various times, so that no attempt will be made here to duplicate these efforts. The interested reader is referred to Markham, Beyer, and Lindsay (1951), Kolsky (1953), Hunt (1957), and particularly to Knopoff and MacDonald (1958, in press). A distinguishing feature of all the classical treatments is the fact that they are almost all based on linear theory, that is, the equations of motion are linear differential equations with real constant coefficients.

In what follows, we shall investigate the propagation of sinusoidal stress waves in considerable detail. We shift from the study of a single travelling disturbance, such as is constituted by a shock wave, to a consideration of sinusoidal propagation theory. This is done for mathematical convenience, since the introduction of singularity functions at this point would lead to additional complexities. In any event, we shall be here primarily concerned with the frequency dependence of the attenuation coefficient α ; this frequency relationship must be the same for a single pulse as for a continuous train of sinusoidal waves.

Knopoff and MacDonald (1958) have reviewed the experimental data that has been published, and find that for most solids the attenuation coefficient α is proportional to the first power of the circular frequency of vibration ω . Certain ferromagnetic materials and some inorganic plastics do not satisfy this rule, but such substances are of little interest to the seismology of the earth's crust. Data for the damping of stress waves in rocks as established from seismograms also appears to confirm the laboratory evidence. Although no single substance has been investigated over a broad spectrum, Knopoff and MacDonald conclude that α is proportional to the first power of ω for most inorganic solids in the range $10^{-2} \leq \omega \leq 10^{7}$ rad/sec. The work of Zener (1948) has shown that attenuation of stress waves in some metals and glasses is a variable function of ω , with very pronounced absorption peaks. More will be said about this phenomenon in Chapter IV of this thesis.

It is possible to analyze the behavior of <u>any</u> damping mechanism described by a linear differential equation by considering a perturbed form of the one-dimensional wave equation,

$$\frac{1}{C^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sum_{m,n} a_{m,n} \frac{\partial^{m+n} u}{\partial x^m \, \partial t^n} \tag{3.2}$$

where u = displacement, c = velocity of the elastic wave, and a_{mn} = real and constant coefficients. Knopoff and MacDonald have shown that this equation can only lead to an attenuation coefficient which is a function of an even power of ω , if a solution of the form

$$u = A e^{j(\sigma x - \omega t)} \tag{3.3}$$

be assumed for (3.2). The complex wave number σ is given by $\sigma = \nu + j\alpha$. All equations of motion that describe <u>linear</u> mechanical loss mechanisms may be obtained from (3.2) by suitably specializing the coefficients a_{mn}. Thus the classical Visco-elastic, or Kelvin-Voigt solid, which is defined by the <u>linear</u> stress-strain relation, or equation of state

$$P = M_E \varepsilon + M_V \dot{\varepsilon} \tag{3.4}$$

where M_E=elastic modulus, M_V= viscous modulus, and ε infinitesimal strain, yields the equation of motion (Kolsky, 1953)

$$\rho \frac{\partial^2 w}{\partial t^2} = M_E \frac{\partial^2 w}{\partial x^2} + M_V \frac{\partial^3 w}{\partial x^2 \partial t} \tag{3.5}$$

If in (3.2) $c^2 = M_E/\rho$, $a_{21} = M_V/M_E$ and all other $a_{mn} \neq 0$, it can easily be seen that this equation will reduce to (3.5) for these values of the parameters. In this case the attenuation coefficient α can be shown to be

$$\alpha = \mathrm{Im}(\sigma) = \frac{M_V \omega^2}{2 M_E c} \tag{3.6}$$

a well-known result, (see e.g. Kolsky, 1953).

Since experimental evidence for most solids indicates that α is a linear function of ω , while (3.2) can only lead to an attenuation coefficient which depends on an even power of ω , Knopoff and MacDonald were able to deduce that no model described by (3.2) can yield results in agreement with observation. In view of such considerations, these workers were led to investigate the equations of motion that arise if the linearity restriction on the stress-strain relations $P = f(e)$, or, more generally, $P = f(e, \dot{e}, \ddot{e}, \dots T)$ is lifted.

When a perfectly insulated solid element is compressed elastically by an applied stress $P(\varepsilon)$, where $P(\varepsilon)$ is a linear function of the elastic strain ε, it will return to its original state as soon as the stress is removed. Since the element is insulated, there is no outward flux of heat, and the process is therefore adiabatic. Equivalently, ome may say that the thermal conductivity of the medium, γ, is zero. No net entropy has been generated in this process, which is thus thermodynamically reversible. If $\gamma = \infty$, then any heat formed during compression will be conducted away instantaneously, so that there is no net rise of temperature in the element. This process is then isothermal, but no longer isentropic with respect to the element's surroundings. No physical medium has either a zero or an infinite thermal conductivity, and therefore any actual deformation of a solid will involve a net outward flux of heat, and consequently the creation of irreversible entropy. In particular, the entropy thus generated will be in addition to that produced by any viscous or other dissipation mechanisms. When one speaks of an elastic deformation, therefore, one should specify that the thermal conductivity of the medium is zero; for otherwise the process is not reversible, as usually postulated. It is obvious that any rigoroulsy correct damping theory must take thermal phenomena into account. A substance possessing a finite γ , even when subjected to an infinitesimal stress dP, therefore loses a part of the energy of compression in the form of heat that is conducted away----this phenomenon is usually known as thermal damping, and was first attacked by Kirchhoff (1868) for heat-conducting gases. A large number of treatments about this type of damping have been published since then for the case of solids and liquids as well, but a good number of them are based on fallacious thermodynamic arguments. In this and the subsequent chapter an attempt will be made to place the question of thermal damping in solids on a sounder footing.

Although Kelvin was the first investigator to realize that no problem involving deformation could be treated with rigor without recourse to thermodynamics, comparatively few writers have done so. Notable exceptions are Eckart (1940, 1948); Bridgman (1950); and more recently, Synge (1955), and Hunt (1957). Knopoff and MacDonald (1958) have developed a theory of solid deformation in terms of the observable quantities mass, elasticity, permanent deformation, and temperature. Their analysis is patterned after that of Eckart (1948). We shall only outline their method here; for a detailed derivation, the reader is referred to the authors' 1958 paper.

Consider an isotropic, homogeneous, and infinite solid. Let the Cartesian position vector of a point in the solid, $x_i(t)$, be given by

$$x_i(t) = x_0^i + w_p^i(t) + w_i(t) \tag{3.7}$$

where X_0^i = position vector at an initial time t_o, w_p^i permanent, or non-recoverable (plastic) displacement vector, and w^i = elastic displacement vector. The elastic strain tensor ε^i_j can be given as a function of w^i :

$$\varepsilon^i_j = \frac{1}{2}\left(\frac{\partial w^i}{\partial x^j} + \frac{\partial w^j}{\partial x^i}\right) \tag{3.8}$$

The <u>total</u> rate of deformation tensor, d^i_j, is defined by

$$d^i_j = \frac{1}{2}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right) \tag{3.9}$$

where V^i = <u>total</u> velocity vector. In the presence of both elastic strain as well as permanent, non-recoverable plastic strain, one has

$$d^i_j = c^i_j + \frac{d\,\varepsilon^i_j}{dt} \tag{3.10}$$

where c^i_j is the rate of permanent deformation tensor.

The equation of conservation of momentum is

$$\rho_0 \frac{\partial^2 x^i}{\partial t^2} = \frac{\partial P^i_j}{\partial x_j} \tag{3.11}$$

where P^i_j = <u>total</u> stress tensor, and ρ_0 = density in initial, unstrained state. Let Υ^i_j = thermo-elastic stress tensor, given by (Love, 1927, p. 108)

$$\tau^i_j = \lambda_E \varepsilon^h_h \delta^i_j + 2\mu_E \varepsilon^i_j - \beta k \Delta T \delta^i_j \tag{3.12}$$

where:

λ_E, μ_E = elastic constants

β = linear coefficient of thermal expansion

k = bulk modulus

δ^i_j = Kronecker Delta

ΔT = $T - T_0$; T_0 = uniform temperature of initial reference state.

The summation convention is assumed to hold for all repeated indices. In the presence of viscous resistance, the total stress tensor P^i_j will not only be a function of τ^i_j, but also of the total rate of deformation d^i_j :

$$P^i_j = \tau^i_j + \lambda_v d^h_h \delta^i_j + 2\mu_v d^i_j \tag{3.13}$$

where λ_v and μ_v are viscous constants. It should be emphasized that λ_v and μ_v are <u>not</u> the usual viscosities that one associates with the visco-elastic, or Kelvin-Voigt solid, but only reduce to these when the rate of permanent deformation, c^i_j , vanishes, in which case one has from (3.10):

$$d^i_j = \frac{d\varepsilon^i_j}{dt} \tag{3.14}$$

Substitution of (3.12) into (3.13) yields:

$$P^i_j = (\lambda_E \varepsilon^h_h + \lambda_v d^h_h - \beta k \Delta T)\delta^i_j + 2\mu_E \varepsilon^i_j + 2\mu_v d^i_j \tag{3.15}$$

The total displacement vector $\bar{w}_i$ is given by the sum of the permanent and recoverable displacements

$$\bar{w}_i = w^i_p + w^i, \qquad i = 1, 2, 3. \tag{3.16}$$

We may take $x^i_0 = 0$ in (3.7) without loss of generality. Then combination of (3.7), (3.11), and (3.15) yields the three equations of motion

$$\rho_0 \frac{\partial^2 \bar{w}^i}{\partial t^2} = \frac{\partial}{\partial x_j}\left\{(\lambda_E \varepsilon^h_h + \lambda_v d^h_h - \alpha k \Delta T)\delta^i_j + 2\mu_E \varepsilon^i_j + 2\mu_v d^i_j\right\} \qquad i, j = 1, 2, 3. \tag{3.17}$$

Relations (3.16) and (3.17) thus provide us with six equations in the seven unknowns w^i_p, w^i, and T. It is therefore necessary to seek an additional relationship between these quantities before formal solution can be attempted. This may be accomplished by a consideration of the irreversible entropy that is created in any deformation process occurring in a medium of finite thermal conductivity γ. The equation of continuity of entropy (Denbigh, 1951) is

$$\rho \frac{\mathcal{D} s}{\mathcal{D} t} = -\frac{\partial}{\partial x^i}\left(\frac{q^i}{T}\right) + \frac{\mathcal{D} S_{irr}}{\mathcal{D} t} \tag{3.18}$$

where s = specific entropy (entropy per unit mass),

$\frac{D\,S_{irr}}{Dt}$ = rate of generation of irreversible entropy, and

q^i = heat flux vector. The operator $\frac{\mathcal{D}}{\mathcal{D}t}$ is given by

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \tag{3.19}$$

Radiation effects are neglected in (3.18), that is, the heat is assumed to diffuse only by conduction.

From the second law of thermodynamics, $\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t} \geq 0$, where the equality sign holds only if the process is reversible. Knopoff and MacDonald show that

$$\rho \frac{\mathcal{D}s}{\mathcal{D}t} = \frac{\rho\, c_\varepsilon}{T} \frac{\mathcal{D}T}{\mathcal{D}t} + k\beta T \frac{\mathcal{D}\varepsilon^h{}_h}{\mathcal{D}t} \tag{3.20}$$

where c_ε = specific heat at constant strain. The third term of (3.18) may be broken down into two separate parts,

$$\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t} = \left.\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t}\right|_{MECH} + \left.\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t}\right|_{HEAT\ FLOW} \tag{3.21}$$

$\left.\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t}\right|_{MECH}$ is the rate of generation of irreversible entropy due to all <u>mechanical</u> dissipation processes, while $\left.\frac{\mathcal{D}\,S_{irr}}{\mathcal{D}t}\right|_{HEAT\ FLOW}$ is the rate of generation of irreversible entropy due to heat transfer in a medium of finite thermal conductivity.

Now,

$$\left.\frac{D\,S_{irr}}{\mathcal{D}t}\right|_{MECH} = \frac{1}{T}\left\{ p^i_j\, d^j_i - \tau^i_j \left(d^j_i - c_i{}^j\right)\right\} \tag{3.22}$$

(Knopoff and MacDonald, 1958), and

$$\left.\frac{DS_{irr}}{Dt}\right|_{\text{HEAT FLOW}} = \frac{\gamma}{T^2}\left(\frac{\partial T}{\partial x^i}\right)^2 \tag{3.23}$$

(Denbigh, 1951).

Combination of equations (3.18)-(3.23) can be shown to lead to the so-called "temperature" equation,

$$\begin{aligned} \rho c_\varepsilon \frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^{i2}} &+ \left[\lambda_v d^h_h \delta^i_j + 2\mu_v d^i_j\right] d^i_j \\ &+ c^i_j \tau^j_i - k\beta T \frac{\partial \varepsilon^h_h}{\partial t} \\ &- \left[k\beta T \frac{\partial \varepsilon^h_h}{\partial x^i} + \rho c_\varepsilon \frac{\partial T}{\partial x^i}\right] \delta^i_j v^i \end{aligned} \tag{3.24}$$

where we have replaced q^i in (3.18) by

$$q_i = -\gamma \frac{\partial T}{\partial x^i} \tag{3.25}$$

the familiar Fourier heat conduction law for an infinite isotropic medium. Both the equation of motion (3.17) and the temperature equation (3.24) can be expressed in terms of w^i_p and w^i by use of (3.16) and the defining relations (3.8)-(3.10) and (3.12).

Knopoff and MacDonald assume that the rate of permanent deformation tensor c^i_j can be written

$$c^i_j = f(I_1, I_2, I_3; \dot{I}_1, \dot{I}_2, \dot{I}_3) \tag{3.26}$$

where $I_{(i)} =$ i^{th} invariant of the elastic stress tensor and $\dot{I}_{(i)} =$ i^{th} invariant of the time rate of change of the elastic stress tensor (Sokolnikoff, 1952, p. 303). This relation is, in general, non-linear in the stresses, so that the terms involving d^i_j in the equation of motion (3.17) are non-linear in that case. If, however,

$$c^i_j = \mu_c \,\tau^i_j \tag{3.27}$$

eqs. (3.26) and (3.17) are linear, and the latter is solvable by familiar techniques (see Chapter IV of this thesis). A model described by (3.27) is known as a Maxwell solid; the constant $1/\mu_c$ is called the Maxwellian viscosity.

Consider now the temperature equation (3.24). The first and fourth terms of the right member are always linear, while the remaining terms are always non-linear, irrespective of the functional form of c^i_j. In the absence of viscosity and permanent deformation, and neglecting the term in v^i (which is equivalent to setting $\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t}$, a valid step for small deformations in solids), (3.24) reduces to

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^{i\,2}} - k\beta T \frac{\partial \varepsilon^h_h}{\partial t} \tag{3.28}$$

where $\kappa = \gamma/\rho c_\varepsilon$ is the thermal diffusivity of the medium. Relation (3.28) resembles the standard Fourier heat flow equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^{i\,2}} \qquad (3.29)$$

except for the last term. It is usually assumed that the strains produced by a thermal gradient in a solid are negligible; in this case (3.29) gives quite satisfactory results. When one investigates thermo-elastic or thermo-plastic phenomena, however, the third term of (3.28) must be retained.

We are now in a position to recognize the formal similarity of (3.11) and (2.8 b), since the second term of the latter can be neglected in the case of small amplitude waves. The form of P is, of course, different in each case------for shock waves, we have chosen to use the Birch equation of state, while for small amplitude waves we adopt the plastic theory described in this section. The chief difference between both approaches lies in the degree of non-linearity of the describing equations; as we have seen, the shock wave equations cannot be solved satisfactorily with any techniques available to us at this time. The equations of small amplitude motion, however, can be solved by linear perturbation methods. The assumption must be made that the non-linearities involved are small with respect to some parameter, because only in this case is the existence of such solutions assured.

3. Solutions of the Small Amplitude Equations in Solids

In this section we shall investigate in some detail solutions of the system of non-linear equations given by

(3.16), (3.17), and (3.24). In order to simplify the mathematics, we rewrite these equations in their one-dimensional form, and consider the propagation of a compressional wave through the medium. The results may then be suitably specialized for the case of shear waves; in this instance, the terms modulated by δ^i_j in (3.17) and (3.24) vanish, since for $i \neq j$, $\delta^i_j = 0$. Accordingly we write (3.16) in the form,

$$\bar{u}^1 = \bar{u}^1_p + u^1 , \qquad \bar{u}^2 = \bar{u}^3 = 0$$
$$u^2_p = u^3_p = 0$$
$$u^2 = u^3 = 0$$

or simply,

$$\bar{u} = u_p + u \tag{3.30}$$

Equations (3.17) and (3.24) then become:

$$\rho_0 \frac{\partial^2 \bar{u}}{\partial t^2} = (\lambda_E + 2\mu_E) \frac{\partial^2 u}{\partial x^2} + (\lambda_V + 2\mu_V) \frac{\partial^3 \bar{u}}{\partial x^2 \partial t} - \beta k \frac{\partial T}{\partial x} \tag{3.31 a}$$

$$\frac{\partial T}{\partial t} = \left[\kappa \frac{\partial^2 T}{\partial x^2} \right] - \left[\frac{k \beta T}{\rho_0 c_\varepsilon} \frac{\partial^2 u}{\partial x \partial t} \right] + \left[\frac{\lambda_V + 2\mu_V}{\rho_0 c_\varepsilon} \left(\frac{\partial^2 \bar{u}}{\partial x \partial t} \right)^2 \right] + \frac{1}{\rho_0 c_\varepsilon} \left[\frac{\partial^2 u_p}{\partial x \partial t} \right] \left[(\lambda_E + 2\mu_E) \frac{\partial u}{\partial x} \right] - \left[\frac{\beta k \Delta T}{\rho_0 c_\varepsilon} \frac{\partial^2 u_p}{\partial x \partial t} \right] - \left[\frac{k \beta T}{\rho_0 c_\varepsilon} \frac{\partial^2 u}{\partial x^2} \right] \frac{\partial \bar{u}}{\partial t} - \left[\frac{\partial T}{\partial x} \frac{\partial \bar{u}}{\partial t} \right] \tag{3.31 b}$$

where

$$d'_1 = \frac{\partial^2 \bar{u}}{\partial x\, \partial t}\ ; \quad c'_1 = \frac{\partial^2 u_p}{\partial x\, \partial t}\ ; \qquad \varepsilon = \frac{\partial u}{\partial x} \tag{3.32}$$

From (3.30) one has that

$$\frac{\partial^2 \bar{u}}{\partial x\, \partial t} = \frac{\partial^2 u_p}{\partial x\, \partial t} + \frac{\partial^2 u}{\partial x\, \partial t} \tag{3.33}$$

Eq. (3.31 b) is the generalized heat flow equation. The first term on the right hand side represents heat flow due to the thermal gradient itself; the second yields the contribution of elastic straining; while the third (which can be split into three separate terms by eq. 3.33), gives the contribution due to viscous and permanent deformation, as well as the coupling between these two effects. The remaining terms of (3.31 b) represent coupling between permanent and elastic strain, temperature gradient and permanent strain, total particle velocity and elastic displacement, and total particle velocity and thermal gradient, respectively.

For small attenuation factors, the permanent displacement u_p is small compared to the elastic displacement u . (Knopoff and MacDonald, 1958). As a result, we may expect the coupling effects given by the fourth and fifth terms of (3.31 b) to be negligible. The sixth and seventh terms will also be vanishingly small, because for small amplitude waves in solids, the approximation

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \approx \frac{\partial}{\partial t}$$

i.e., $v_i \approx 0$ certainly holds. Consequently we shall only retain the first three terms of (3.31 b) in what follows. <u>The non-linear term in the square of the total rate of deformation, $\left(\frac{\partial^2 \bar{w}}{\partial x \partial t}\right)^2$, can of course not be similarly neglected.</u>

Now equations (3.30) and (3.31 a,b) are still expressed in terms of the variables $\bar{w}, w_p, w$, and T. However, w_p, and, as a result, $\bar{w}$, are actually assumed to be functions of the elastic stress and the rate of change of elastic stress, along with suitable constants (see eqs. 3.26 and 3.27). In the one-dimensional case, (3.26) reduces to

$$c_1' = \frac{\partial^2 w_p}{\partial x \partial t} = g(I_1, \dot{I}_1)\, \tau_1' \qquad (3.34)$$

where g is a scalar function of zeroth order in stress. This particular form of g is assumed because it is found experimentally that for small amplitudes, attenuation is independent of amplitude.

We next rewrite (3.31 a,b) in the form,

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = (\lambda_E + 2\mu_E)\frac{\partial^2 w}{\partial x^2} + (\lambda_v + 2\mu_v)\frac{\partial^3 w}{\partial x^2 \partial t} - \beta k \frac{\partial T}{\partial x}$$

$$+ f_1\left[\frac{\partial^3 w_p}{\partial x^2 \partial t}; \frac{\partial^2 w_p}{\partial t^2}\right] \qquad (3.35\text{ a})$$

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} - \frac{k\beta T_0}{\rho_0 c_\varepsilon}\frac{\partial^2 w}{\partial x \partial t} +$$

$$+ f_2\left[\frac{\partial^2 \bar{w}}{\partial x \partial t}; \left(\frac{\partial^2 \bar{w}}{\partial x \partial t}\right)^2; \frac{\partial^2 w_p}{\partial x \partial t}; \frac{\partial w}{\partial x}\right] \qquad (3.35\text{ b})$$

where we have assumed that T may be approximated by its equilibrium value T_o in the second term of the r.h.s. of (3.35 b), and where:

$$f_1 = (\lambda_v + 2\mu_v)\left(\frac{\partial^3 u_p}{\partial x^2 \partial t}\right) - \rho_0 \frac{\partial^2 u_p}{\partial t^2} \qquad (3.36\text{ a})$$

$$f_2 = \left[\frac{\lambda_v + 2\mu_v}{\rho_0 c_\varepsilon}\left(\frac{\partial^2 \bar{u}}{\partial x \partial t}\right)^2\right] \qquad (3.36\text{ b})$$

The functions f_1 and f_2 thus incorporate the entire non-linearities of (3.35 a,b). As explained in connection with (3.34), one can treat (3.35 a,b) as a system of non-linear partial differential equations in the variables u and T. By hypothesis, the non-linearities expressed by f_1 and f_2 are small------in other words, we assume that the permanent, plastic strains are small compared to the elastic strains. If this requirement be upheld, solutions of the system (3.35 a,b) can be found by a technique which will be developed in the present section.

The approach is a generalization of the theory of first approximation of Kryloff and Bogoliuboff (Minorsky, 1947). Essentially, the method assumes that wave amplitude and phase are slowly varying functions of the time t, so that they may be approximated by a constant mean value in some interval $(t, t+\tau)$, where τ = period of oscillation. This assumption can be shown to convert the original non-linear equation into two subsidiary relations, one in amplitude, and one in phase. These, although still non-linear in the general case, are always integrable in terms of elementary functions. Space does not permit a detailed description of the method here; the reader is referred to the reference cited above.

Our point of departure is the system (3.35 a,b). In order to forestall a mathematical difficulty which will become evident later, we differentiate both members of (3.35 a,b) w.r.t. time. The system to be solved is then

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = (\lambda_E + 2\mu_E)\frac{\partial^2 w}{\partial x^2} + (\lambda_v + 2\mu_v)\frac{\partial^3 w}{\partial x^2 \partial t} - \beta k \frac{\partial T}{\partial x} + f_1$$

$$\frac{\partial^2 T}{\partial t^2} = K \frac{\partial^3 T}{\partial x^2 \partial t} - \frac{k\beta T_0}{\rho_0 c_\varepsilon}\frac{\partial^3 w}{\partial x \partial t^2} + \frac{\partial f_2}{\partial t}$$

$$\frac{\partial f_2}{\partial t} \neq 0 \qquad (3.37\ a,b)$$

We assume solutions of this system in the form

$$w = A(t) \sin\left[\sigma x - \omega t + \phi(t)\right] = A(t) \sin \theta_1$$

$$T = B(t) \sin\left[\sigma x - \omega t + \psi(t)\right] = B(t) \sin \theta_2$$

$$(3.38\ a,b)$$

If $f_1 = f_2 = 0$, (3.37 a,b) would be linear, and solutions (3.38 a,b) can be found by standard methods. In this case, both the amplitudes A,B and the phase angles ϕ, ψ are constants independent of time. We now assume that solutions of type (3.38 a,b) can be found such that (3.37 a,b) be satisfied when f_1 and $f_2 \neq 0$, where the amplitudes A,B and phase angles ϕ, ψ are explicit functions of time. If expressions for these four quantities can be found, their substitution into (3.38 a,b) yields the desired complete solution.

Differentiating (3.38 a,b) w.r.t. time, one has

$$\dot{w} = -\omega A \cos\theta_1 + \dot{A} \sin\theta_1 + A\dot{\phi}\cos\theta_1$$

$$\dot{T} = -\omega B \cos\theta_2 + \dot{B} \sin\theta_2 + B\dot{\psi}\cos\theta_2$$

(3.39 a,b)

where the dot denotes differentiation w.r.t. time. In the linear case, where A, B, ϕ, and ψ are constant, these relations would yield

$$\dot{w} = -\omega A \cos\theta_1$$

$$\dot{T} = -\omega B \cos\theta_2$$

(3.40 a,b)

So that (3.39 a,b) reduce to (3.40 a,b) in the linear case, we accordingly must require that

$$\dot{A}\sin\theta_1 + A\dot{\phi}\cos\theta_1 = 0$$

$$\dot{B}\sin\theta_2 + B\dot{\psi}\cos\theta_2 = 0$$

(3.41 a,b)

Furthermore, remembering that A,B, ϕ, and ψ are not explicit functions of x, one calculates from (3.35 a,b), ($_x = \frac{\partial}{\partial x}$):

$$T_x = +B\sigma\cos\theta_2$$

$$T_{xx} = -B\sigma^2\sin\theta_2$$

(3.42 a,b)

$$w_x = +A\sigma\cos\theta_1$$

$$w_{xx} = -A\sigma^2\sin\theta_1$$

(3.43 a,b)

and from (3.40 a,b):

$$\ddot{w} = -\omega^2 A \sin\theta_1 - \omega \dot{A} \cos\theta_1 + \omega \dot{\phi} A \sin\theta_1$$

$$\ddot{T} = -\omega^2 B \sin\theta_2 - \omega \dot{B} \cos\theta_2 + \omega \dot{\psi} B \sin\theta_2$$

$$\dot{w}_x = -\omega^2 \sigma A \cos\theta_1 + \omega \dot{A} \sigma \sin\theta_1 + \omega \dot{\phi} A \sigma \cos\theta_1$$

(3.44 a,b,c)

From (3.40 b):

$$\dot{T}_x = \omega \sigma B \sin\theta_2$$

$$\dot{T}_{xx} = \omega \sigma^2 B \cos\theta_2$$

(3.45 a,b)

and from (3.40 a):

$$\dot{w}_x = \omega \sigma A \sin\theta_1$$

$$\dot{w}_{xx} = \omega \sigma^2 A \cos\theta_1$$

(3.46 a,b)

If relations (3.40 a,b) and (3.42)-(3.46) be substituted into the system (3.37 a,b), one has

$$\rho_0 \omega \cos\theta_1 \underline{\dot{A}} + \rho_0 \omega A \sin\theta_1 \underline{\dot{\Phi}} = - M_E A \sigma^2 \sin\theta_1 + M_V \omega \sigma^2 A \cos\theta_1 - \tilde{M}_T B \sigma \cos\theta_2 + \rho_0 \omega^2 A \sin\theta_1 + f_1$$

(3.47 a)

$$-\omega \cos\theta_2 \,\underline{\dot{B}} + \omega B \sin\theta_2 \,\underline{\dot{\psi}} + M_T \omega \sigma \sin\theta_1 \,\underline{\dot{A}}$$
$$+ M_T \omega A \sigma \cos\theta_1 \,\underline{\dot{\phi}} =$$
$$K \omega \sigma^2 B \cos\theta_2 + \dot{f}_2 \tag{3.47 b}$$

where:

$$M_E = \lambda_E + 2\mu_E \qquad M_T = \frac{k \beta T_0}{\rho_0 c_\varepsilon}$$
$$M_v = \lambda_v + 2\mu_v \qquad \tilde{M}_T = k\beta \tag{3.48}$$

Relations (3.47 a,b), in conjunction with (3.41 a,b), yield four <u>linear</u> <u>algebraic</u> equations in the unknowns $\dot{A}$, $\dot{B}$, $\dot{\phi}$, and $\dot{\psi}$. These four equations may be written in the more compact form:

$$\begin{aligned}
m_{11}\dot{A} + m_{12}\dot{\phi} & & &= F_1 \\
m_{21}\dot{A} + m_{22}\dot{\phi} &+ m_{23}\dot{B} &+ m_{24}\dot{\psi} &= F_2 \\
m_{31}\dot{A} + m_{32}\dot{\phi} & & &= 0 \\
&+ m_{43}\dot{B} &+ m_{44}\dot{\psi} &= 0
\end{aligned} \tag{3.49}$$

where

$$\begin{aligned}
m_{11} &= -\rho_0 \omega \cos\theta_1 & m_{23} &= -\omega \cos\theta_2 \\
m_{12} &= +\rho_0 \omega A \sin\theta_1 & m_{24} &= +\omega B \sin\theta_2 \\
m_{21} &= +M_T \omega \sigma \sin\theta_1 & m_{31} &= +\sin\theta_1 \\
m_{22} &= +M_T \omega A \sigma \cos\theta_1 & m_{32} &= +A\cos\theta_1 \\
m_{43} &= +\sin\theta_2 & m_{44} &= +B\cos\theta_2
\end{aligned} \tag{3.50}$$

and

$$F_1 = M_V \omega \sigma^2 A \cos \theta_1 - M_E A \sigma^2 \sin \theta_1 - \tilde{M}_T B \sigma \cos \theta_2 + \rho_0 \omega^2 A \sin \theta_1 + f_1 \tag{3.51 a}$$

$$F_2 = K \omega \sigma^2 B \cos \theta_2 + \dot{f}_2 \tag{3.51 b}$$

After much laborious but straightforward algebra, one finds that the solutions of (3.49) are given by

$$\dot{A} = \frac{-\cos\theta_1 \; F_1}{\rho_0 \, \omega} \qquad \dot{\varphi} = \frac{\sin\theta_1 \; F_1}{A \rho_0 \, \omega} \tag{3.52 a,b}$$

$$\dot{B} = \frac{-\cos\theta_2 \; F_2}{\omega} \qquad \dot{\psi} = \frac{\sin\theta_2 \; F_2}{B \, \omega} \tag{3.53 a,b}$$

Relations (3.52 a,b) can be written with the aid of (3.51 a,b) in the form

$$\dot{A} = \frac{-\cos\theta_1}{\rho_0 \, \omega}\left[M_V \omega \sigma^2 \cos\theta_1 A - M_E \sigma^2 \sin\theta_1 A + + \rho_0 \omega^2 \sin\theta_1 A - \tilde{M}_T \sigma \cos\theta_2 B + f_1 \right] \tag{3.54 a}$$

$$\dot{B} = \frac{-\cos\theta_2}{\omega}\left[K \omega \sigma^2 \cos\theta_2 \, B + \dot{f}_2 \right] \tag{3.54 b}$$

Consider now the generalized heat flow equation (3.35 b). Substituting from relations (3.40)-(3.46) into this expression, and solving for B(t), one has

$$B(t) = \frac{f_2 - M_T\,\omega\,\sigma\,\sin\theta_1\,A}{\kappa\sigma^2\sin\theta_2 - \omega\cos\theta_2} \qquad (3.55)$$

This formula may be substituted into (3.54a), to yield:

$$\dot{A} = \frac{-\cos\theta_1}{\rho_0\,\omega}\Bigg[M_V\,\omega\,\sigma^2\cos\theta_1\,A - M_E\,\sigma^2\sin\theta_1\,A + \rho_0\,\omega^2\sin\theta_1\,A - \tilde{M}_T\,\sigma\cos\theta_2\left\{\frac{f_2 - M_T\,\omega\,\sigma\sin\theta_1\,A}{\kappa\sigma^2\sin\theta_2 - \omega\cos\theta_2}\right\} + f_1\Bigg] \qquad (3.56)$$

We have thus been able to express the rate of change of the wave amplitude, $\dot{A}(t)$, as a function of circular frequency ω, wave number σ, and the appropiate moduli of the medium. It is now evident why it was necessary to differentiate (3.35 b) w.r.t. time in order to solve the resulting system (3.37 a,b) for u and T, since this step enables us to use (3.35 b) as a separate relation with which to express $\dot{A}$ in (3.54) as a function of A and the appropiate constants alone. In a similar way, we find that $\dot{\phi}$ and $\dot{\psi}$ are given by

$$\dot{\varphi} = \sin\theta_1 \left[\frac{M_v}{\rho_0} \sigma^2 \cos\theta_1 - \frac{M_E \sigma^2}{\rho_0 \omega} \sin\theta_1 \right.$$

$$\left. + \omega \sin\theta_1 - \frac{\tilde{M}_T \sigma \cos\theta_2}{\rho_0 \omega} \frac{B}{A} + \frac{1}{\rho_0 \omega A} f_1 \right] \qquad (3.57\ a)$$

$$\dot{\psi} = \sin\theta_2 \left[K\sigma^2 \cos\theta_2 + \frac{1}{B\omega} \dot{f}_2 \right] \qquad (3.57\ b)$$

Let us fix our attention on (3.56). This equation contains two non-linear terms,

$$\frac{\cos\theta_1}{\rho_0\, \omega} \left[\frac{\tilde{M}_T \sigma \cos\theta_2}{K\sigma^2 \sin\theta_2 - \omega\cos\theta_2} \right] f_2 \qquad (3.58\ a)$$

and

$$-\frac{\cos\theta_1}{\rho_0\, \omega} f_1 \qquad (3.58\ b)$$

Up to this point, our treatment has been exact. Now the general method of Kryloff and Bogoliuboff assumed that the right hand members of (3.54 a,b) and (3.57 a,b) can be expanded in a Fourier series of period 2π. In particular, the theory of first approximation of Kryloff and Bogoliuboff shows that to first order, these right-hand members are given simply by the first term of the expansion. This is equivalent to averaging the equations over a period, so that higher order terms of the series vanish identically (Minorsky, 1947).

In the present situation, (3.54 a,b) and (3.57 a,b) are functions both of θ_1 and θ_2, where θ_1 and θ_2 are given by

$$\theta_1 = \sigma x - \omega t + \phi(t)$$

$$\theta_2 = \sigma x - \omega t + \psi(t)$$

(3.59 a,b)

Equation (3.56) can be written in the form

$$\dot{A} = -\frac{\cos\theta_1}{\rho_0\,\omega}\left[\tilde{F}_1(\theta_1, \theta_2)\right] \qquad (3.60)$$

where F_1 stands for the expression within square brackets of (3.56). We assume that F_1 can be expanded in a double Fourier series in θ_1, θ_2 (Carslaw and Jaeger, 1947, pp. 158-162):

$$\tilde{F}_1(\theta_1, \theta_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K_{m,n} \sin m\theta_1 \sin n\theta_2$$

$$+ \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} K'_{m,n} \sin m\theta_1 \cos n\theta_2 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} L'_{m,n} \cos m\theta_1 \sin n\theta_2 +$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_{m,n} \cos m\theta_1 \cos n\theta_2 \qquad (3.61)$$

where the $K_{m,n}$; $K'_{m,n}$; $L_{m,n}$; and $L'_{m,n}$ are the appropiate two-dimensional Fourier coefficients. To first order, however, the first three terms of (3.61) vanish, so that one has simply

$$\tilde{F}_1(\theta_1, \theta_2) = L_{0,0} \tag{3.62}$$

where

$$L_{0,0} = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} \tilde{F}_1(\theta_1, \theta_2)\, d\theta_1\, d\theta_2 \tag{3.63}$$

Combining (3.60), (3.62), and (3.63), one derives the relation

$$\boxed{\dot{A} = -\frac{1}{4\pi^2 \rho_0 \omega} \int_0^{2\pi}\int_0^{2\pi} \tilde{F}_1(\theta_1, \theta_2) \cos\theta_1\, d\theta_1\, d\theta_2} \tag{3.64}$$

where $\tilde{F}_1(\theta_1, \theta_2)$ is given by

$$\tilde{F}_1(\theta_1, \theta_2) = \Bigg[M_V \omega \sigma^2 \cos\theta_1 A - M_E \sigma^2 \sin\theta_1 A + \rho_0 \omega^2 \sin\theta_1 A - \tilde{M}_T \sigma \cos\theta_2 \left\{ \frac{f_2 - M_T \omega \sigma \sin\theta_1 A}{K\sigma^2 \sin\theta_2 - \omega \cos\theta_2} \right\} + f_1 \Bigg] \tag{3.65}$$

In a similar way, it can be shown that (3.52 b) and (3.53 a,b) lead to the corresponding <u>first</u> <u>order</u> relations

$$\dot{B} = -\frac{1}{4\pi^2 \omega} \int_0^{2\pi}\int_0^{2\pi} F_2(\theta_1, \theta_2) \cos\theta_2\, d\theta_1\, d\theta_2 \tag{3.66 a}$$

$$\dot{\varphi} = \frac{1}{4\pi^2 A \rho_0 \omega} \int_0^{2\pi}\int_0^{2\pi} F_1(\theta_1, \theta_2) \sin\theta_1 \, d\theta_1 \, d\theta_2 \qquad (3.66\ b)$$

$$\dot{\psi} = \frac{1}{4\pi^2 B \omega} \int_0^{2\pi}\int_0^{2\pi} F_2(\theta_1, \theta_2) \sin\theta_2 \, d\theta_1 \, d\theta_2 \qquad (3.66\ c)$$

where $F_1(\theta_1, \theta_2)$ and $F_2(\theta_1, \theta_2)$ are given by (3.51 a,b). If the exact form of the non-linear terms f_1 and f_2 is known, the double integrals in the above expression are evaluated first, and the resulting differential equations then solved explicitly for A, B, φ, and ψ. These values are finally substituted into the assumed solutions (3.38 a,b).

Let us first study the displacement amplitude equation (3.64). Term by term integration of the right member will involve, among others, integrals of the form:

$$J_1 = \int_0^{2\pi} \sin^n \theta_i \, d\theta_i \begin{array}{l} = 0 \text{ for n odd} \\ \neq 0 \text{ for n even} \end{array} \qquad (3.67\ a)$$

$$J_2 = \int_0^{2\pi} \cos^n \theta_i \, d\theta_i \begin{array}{l} = 0 \text{ for n odd} \\ \neq 0 \text{ for n even} \end{array} \qquad (3.67\ b)$$

$$J_3 = \int_0^{2\pi} \sin^n \theta_i \cos\theta_i \, d\theta_i = 0 \text{ for all n} \qquad (3.67\ c)$$

$$J_4 = \int_0^{2\pi} \cos^n \theta_i \sin\theta_i \, d\theta_i = 0 \text{ for all n} \qquad (3.67\text{ d})$$

where n is an integer $\geqslant 1$, and $i = 1,2$.

Consider now the linear terms of (3.65) that is, all terms which do <u>not</u> involve f_1 and f_2. Then application of (3.67) with $i = 1$ shows that all terms except the first vanish in the integration; the integral of the first term is

$$M_v \omega \sigma^2 A \int_0^{2\pi}\int_0^{2\pi} \cos^2\theta_1 \, d\theta_1 \, d\theta_2 = 2\pi^2 M_v \omega \sigma^2 A \qquad (3.68)$$

Consequently we may write (3.64) in the form,

$$\dot{A} = -\left[\frac{M_v \sigma^2 A}{2\rho_0} - \frac{\tilde{M}_T \sigma}{4\pi^2 \rho_0 \omega}\int_0^{2\pi}\int_0^{2\pi} \frac{\cos\theta_2 \left[f_2(\theta_1)\cos\theta_1 \, d\theta_1\right] d\theta_2}{K\sigma^2 \sin\theta_2 - \omega\cos\theta_2} \right.$$
$$\left. + \frac{1}{4\pi^2 \rho_0 \omega}\int_0^{2\pi}\int_0^{2\pi} f_1(\theta_1)\cos\theta_1 \, d\theta_1 \, d\theta_2\right] \qquad (3.69)$$

In order to evaluate the above double integral, we must consider the explicit forms of the non-linear functions f_1 and f_2 given by (3.36 a,b). Now Knopoff and MacDonald (1958) have shown that the expression for the rate of permanent deformation (3.34) can be simplified if it be assumed that the application of hydrostatic pressure results only in elastic deformation. This assumption has been amply confirmed by experimental high pressure work performed by Bridgman (1949). Under these conditions, C_1' is a function only of the elastic stress, the rate of change of elastic stress, and three constants. For one-dimensional

P wave propagation, (3.34) can thus be written

$$c'_1 = \frac{\partial^2 w_p}{\partial x \, \partial t} = g(I_1, \dot{I}_1)\tau'_1 = \left[\mu_c + \psi_c \left|\frac{\dot{I}_1}{I_1}\right|\right]\tau'_1$$

$$= \left[\mu_c + \psi_c \left|\frac{\frac{\partial^2 w}{\partial x \, \partial t}}{\frac{\partial w}{\partial x}}\right|\right] M_E \frac{\partial w}{\partial x} \qquad (3.70)$$

where μ_c and ψ_c are two constants having the dimensions of inverse viscosity and inverse stress, respectively. When ψ_c vanishes, (3.70) becomes

$$c'_1 = \mu_c \tau'_1 \qquad (3.70')$$

which is seen to be identical to (3.27). Moreover, we have neglected the thermal stress term of τ'_j (see eq. 3.12) in the statement of (3.70), but the coupling terms thus discarded are negligible, as has already been pointed out on page 86 of this chapter.

Combining (3.70) with (3.10), and using dot and subscript notation, one has

$$d'_1 = \dot{\bar{w}}_x = \mu_c M_E w_x + \psi_c M_E \left|\frac{\dot{w}_x}{w_x}\right| + \dot{w}_x \qquad (3.71)$$

Substitution of this relation into (3.66 b) gives

$$f_2 = \left[\frac{M_v}{\rho_0 c_\varepsilon} \quad (\mu_c M_E)^2 (w_x)^2 + 2(\mu_c \psi_c M_E^2) \left| \frac{\dot{w}_x}{w_x} \right| (w_x)^2 \right.$$

$$+ 2\mu_c M_E w_x \dot{w}_x + 2(\psi_c M_E) \left| \frac{\dot{w}_x}{w_x} \right| \dot{w}_x w_x +$$

$$\left. + (\psi_c M_E)^2 \left| \frac{\dot{w}_x}{w_x} \right| w_x^2 + (\dot{w}_x)^2 \right] \tag{3.72}$$

Consider the second term of the right member of (3.69). Direct integration over θ_2 yields

$$\int_0^{2\pi} \frac{\cos\theta_2 \, d\theta_2}{K\sigma^2 \sin\theta_2 - \omega\cos\theta_2} = \frac{-2\pi}{\omega^2 + K^2\sigma^4} \tag{3.73}$$

In order to perform the integration over θ_1, one substitutes from relations (3.43)-(3.46) into (3.72), and enters with the resulting expression into the second term of (3.69). The integral to be evaluated is then

$$\int_0^{2\pi} \left[f_2(\theta_1) \cos\theta_1 \right] d\theta_1 = \frac{M_v}{\rho_0 c_\varepsilon} \int_0^{2\pi} \left[(\mu_c M_E)^2 (A\sigma)^2 \cos^3\theta_1 \right.$$

$$+ 2(\mu_c \psi_c M_E^2)(A\sigma)^2 \omega \cos^3\theta_1 \left| \tan\theta_1 \right|$$

$$+ 2(\mu_c M_E)(A\sigma)^2 \omega \cos^2\theta_1 \sin\theta_1 + 2(\psi_c M_E)(A\omega\sigma)^2 \cos^2\theta_1 \sin\theta_1 \left| \tan\theta_1 \right|$$

$$+ (\psi_c M_E)^2 (A\omega\sigma)^2 \cos^3\theta_1 \left| \tan^2\theta_1 \right|$$

$$\left. + (A\omega\sigma)^2 \sin^2\theta_1 \cos\theta_1 \right] d\theta_1 \tag{3.74}$$

Only those terms involving $|\tan\theta_1|$ and $|\tan^2\theta_1|$ do not vanish in the interval $\theta_1 (0, 2\pi)$, and one finds that to first order,

$$\int_0^{2\pi} \left[f_2(\theta_1) \cos\theta_1 \right] d\theta_1 =$$

$$= \frac{4 M_v \psi_c M_E^2 \sigma^2 A^2}{3 \rho_0 c_\varepsilon} \left[\psi_c \omega^3 + 2\mu_c \omega \right] \qquad (3.75)$$

Combining (3.75) with (3.73), (3.69) becomes

$$\dot{A} = -\left[\frac{M_v \omega^2}{2 \rho_0 V_p^2} A + \left\{ \frac{2 M_v}{3\pi \rho_0 c_\varepsilon} \frac{\tilde{M}_T \psi_c M_E^2}{\rho_0} \right\} \left\{ \frac{\psi_c}{1 + \frac{K^2 \omega^2}{V_p^4}} + \frac{\frac{2\mu_c}{\omega}}{1 + \frac{K^2\omega^2}{V_p^4}} \right\} \frac{\omega^3}{V_p^3} A^2 \right.$$
$$\left. + \frac{1}{2\pi \rho_0 \omega} \int_0^{2\pi} f_1(\theta) \cos\theta_1 \, d\theta_1 \right] \qquad (3.76)$$

where we have integrated the last term over θ_2 directly, since f_1 is independent of θ_2. The wave number σ has been replaced in the above expression by $\sigma = \omega / V_p$, V_p = unperturbed elastic P wave velocity. The quantity

$$1 + \frac{K^2 \omega^2}{V_p^4} = 1 + \left(\frac{K\omega}{V_p^2} \right)^2 \approx 1$$

since $V_p^2 \gg K\omega$. Thus for rocks, $K = 0(10^{-2}$ cm^2/sec) and $V_p = 0(10^5$ cm/sec), so that the above approximation holds provided that $\omega \ll 10^{12}$ rad/sec. Eq. (3.76) then becomes

$$\dot{A} = \left[\frac{M_v \omega^2}{2 \rho_0 V_p^2} \underline{\underline{A}} + \left\{ \frac{2 M_v}{3\pi \rho_0 c_\varepsilon} \frac{\tilde{M}_T \psi_c M_E^2}{\rho_0 V_p^3} \right\} \left\{ \psi_c \omega^3 + 2\mu_c \omega^2 \right\} \underline{\underline{A}}^2 \right.$$
$$\left. + \frac{1}{2\pi \rho_0 \omega} \int_0^{2\pi} f_1(\theta_1) \cos\theta_1 \, d\theta_1 \right] \qquad (3.77)$$

Eq. (3.77) can still not be solved for the amplitude A unless the third term of the right member is integrated. We recall that f_1 is given by (3.36 a), and represents the non-linearity of the equation of motion. Knopoff and MacDonald (1958) have evaluated this integral provided that interaction terms between viscosity and permanent deformation can be neglected. They find that

$$\frac{1}{2\pi\rho_0\omega}\int_0^{2\pi} f_1(\theta_1)\cos\theta_1\, d\theta_1 = \left\{\frac{2}{3}\mu_c\rho_0 V_p^2 + \frac{4}{3\pi}\psi_c\omega V_p^2\rho_0\right\}A \tag{3.78}$$

where $$V_p^2 = M_E/\rho_0$$

Substituting (3.78) into (3.77) gives, after some further algebraic simplification,

$$\dot{A} = \left[\frac{M_v\omega^2}{2M_E}\underline{A} + \left\{\frac{2M_vM_E}{3\pi\rho_0 c_\varepsilon}\ \frac{k\beta\psi_c}{V_p}\right\}\left\{\psi_c\omega^3 + 2\mu_c\omega^2\right\}\underline{\underline{A^2}} + \left\{\frac{2}{3}\mu_c\rho_0 V_p^2 + \frac{4}{3\pi}\psi_c\omega V_p^2\rho_0\right\}\underline{A}\right] \tag{3.79}$$

where we have replaced $\tilde{M}_T$ by $\tilde{M}_T = k\beta$. This equation, <u>although still non-linear</u> in the amplitude A, is nevertheless easily integrable. For convenience, we write (3.79) in the form

$$\frac{dA}{dt} = -\left[\tilde{\alpha}_1 A + \tilde{\alpha}_2 A^2\right] \tag{3.80}$$

where

$$\tilde{\alpha}_1 = \frac{2}{3}\mu_c \rho_0 V_p^2 + \frac{4}{3\pi}\psi_c V_p^2 \rho_0 \underline{\underline{\omega}} + \frac{1}{2}\frac{M_v}{M_\varepsilon}\underline{\underline{\omega}}^2 \tag{3.81 a}$$

$$\tilde{\alpha}_2 = \frac{2 M_v M_\varepsilon}{3\pi\rho_0 c_\varepsilon} \frac{k\beta\psi_c}{V_p}\left\{2\mu_c \underline{\underline{\omega}}^2 + \psi_c \underline{\underline{\omega}}^3\right\} \tag{3.81 b}$$

The solution of (3.80) may be found by separation of variables and is

$$A(x) = \frac{\alpha_1 \dfrac{A_0}{A_0\alpha_2 + \alpha_1} e^{-\alpha_1 x}}{1 - \alpha_2 \dfrac{A_0}{A_0\alpha_2 + \alpha_1} e^{-\alpha_1 x}} \tag{3.82}$$

where we have replaced the time variable t by the distance x, $t = x/V_p$. The **quantities** α_1 and α_2 are thus defined by

$$\alpha_1 = \frac{\tilde{\alpha}_1}{V_p} \qquad [1/\text{distance}]$$

$$\alpha_2 = \frac{\tilde{\alpha}_2}{V_p} \qquad [1/\text{distance}] \tag{3.83 a,b}$$

and A_0 is the wave amplitude at $x = 0$.

An involved calculation similar to the one presented above shows that, to first order, the amplitude phase angle φ is constant and independent of both x and t. Substitution of (3.82) into (3.38 a) yields the solution for the displacement u in the form

$$w = \left[\frac{\alpha_1 \dfrac{A_0}{A_0\alpha_2 + \alpha_1} e^{-\alpha_1 x}}{1 - \alpha_2 \dfrac{A_0}{A_0\alpha_2 + \alpha_1} e^{-\alpha_1 x}} \right] \sin(\sigma x - \omega t + \varphi_0) \quad (3.84)$$

where $\varphi_0 =$ constant.

The quantities α_1 and α_2 define two separate distance attenuation coefficients and are related to the time attenuation coefficients $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ by (3.83 a,b) The coefficient α_1 is a function of three terms, proportional respectively to the zeroth, first and second powers of the circular frequency ω . The zeroth and quadratic factors correspond to linear terms in the equation of motion and represent damping in the classical Maxwell and Kelvin-Voigt (Visco-elastic) solids, respectively. (See e.g. Kolsky, 1953). The factor linear in ω is a direct consequence of the non-linear stress-strain relationship (3.70).

The second attenuation coefficient α_2 contains terms proportional to the second and third powers of ω . The first of these is again a result of the linear terms of the original system (3.35 a,b), while the second is attributable to the non-

$\lambda_\varepsilon \approx \mu_\varepsilon$ and $\lambda_v \approx \mu_v$ (1), a rough calculation gives

$$10^{-8} \ll \omega \ll 10^{9} \text{ rad/sec.} \tag{3.88}$$

The second attenuation coefficient α_2 is given by

$$\alpha_2 = \frac{2}{3\pi} \frac{M_v k \beta \psi_c}{c_\varepsilon} \left\{ 2\mu_c \omega^2 + \psi_c \omega^3 \right\} \tag{3.89}$$

Taking $k = 10^{11}$ dynes/cm^2, $c_\varepsilon = 10^7$ ergs/gram, $\beta = 10^{-5}/{}^\circ C$, and the values (3.86), one has

$$\alpha_2 \approx 10^{-14} \left\{ 10^{-22} \omega^2 + 10^{-14} \omega^3 \right\} \tag{3.90}$$

For these values of the constants (3.86), it is obvious that α_2 will be negligible-----one may then neglect thermal damping with small error and use (3.85), rather than (3.84). However, it must be borne in mind that μ_c and ψ_c are very poorly known; in particular, the magnitude of μ_c is based on a single calculation of Haskell (1935). If more refined experimental work does show an amplitude decay mechanism faster than is reconcilable with (3.85), the thermodynamically more accurate form (3.84) should be useful. We notice that if $\psi_c = 0$, (see eq. 3.70), $\alpha_2 = 0$. This means that thermal damping is

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(1): This assumption is quite a controversial one in the literature, but it is probably justified for rough order of magnitude estimates.

negligible in the Maxwell solid, as was concluded by Knopoff and MacDonald without formal proof.

The equation of motion (3.17) will not involve the temperature T explicitly for the case of a shear wave. As a result, there exists no coupling between the equations of motion and of temperature and, to first order, thermal damping will not arise in S-wave propagation. This problem has been solved for a solid obeying the stress-strain relation (3.70) by Knopoff and MacDonald.

C H A P T E R   I V

THE ATTENUATION OF LINEAR SMALL AMPLITUDE STRESS WAVES IN A SOLID EXHIBITING FINITE THERMAL CONDUCTIVITY.-

1. Small-Signal Thermoelastic Theory

In the previous chapter we have concerned ourselves with the study of solutions of the system (3.35 a,b), of which both equations are non-linear. It was also pointed out that quasi-harmonic solutions of the type (3.38 a,b) exist only when these non-linear terms are small enough so that the resulting equations may be treated by linear perturbation techniques. The theory of first approximation of Kryloff and Bogoliuboff was then shown to be a powerful tool in the attack of the non-linear problem.

We shall here investigate the behaviour of solutions of the system (3.35 a,b) in the absence of non-linearity, $f_1 = 0$, $f_2 = 0$:

$$\rho_0 \frac{\partial^2 w}{\partial t^2} = (\lambda_E + 2\mu_E)\frac{\partial^2 w}{\partial x^2} + (\lambda_v + 2\mu_v)\frac{\partial^3 w}{\partial x^2 \partial t} - \beta k \frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} - \frac{k \beta T_0}{\rho_0 c_\varepsilon} \frac{\partial^2 w}{\partial x \partial t}$$

(4.1 a,b)

Now reference to (3.65) and (3.67 a-d) shows that the linear thermal term in the amplitude equation (3.64) vanishes to first order, since

$$\frac{1}{4\pi^2} \frac{M_{T-} \tilde{M}_{T-}}{\rho_0} \sigma^2 A \int_0^{2\pi}\int_0^{2\pi} \frac{\cos\theta_2 \sin\theta_1 \cos\theta_1}{K\sigma^2 \sin\theta_2 - \omega\cos\theta_2} d\theta_1 d\theta_2 = 0$$

(4.2)

where

$$\frac{M_T \tilde{M}_T}{\rho_0} = \frac{T_0}{c_\varepsilon}\left(\frac{k\beta}{\rho_0}\right)^2 \tag{4.3}$$

from (3.48). Hence the effect of linear thermal terms on wave amplitude decay, although small by (4.2), cannot be studied by suitably specializing the quasi-harmonic solutions found in the previous chapter. We must, then, seek other techniques that will permit us to analyze the effects of thermal linearity explicitly.

Weiner (1957) has recently drawn attention to the fact that the Fourier Heat Conduction equation is an energy balance which neglects the interconvertibility of mechanical and thermal energy, a phenomenon which he calls "thermoelastic coupling". Weiner shows that although such coupling terms are negligible for cases in which heat is supplied from external sources, the same is not true when temperature fluctuations arise because of internal deformations within the medium. The term (4.2) is an example of such thermoelastic coupling which vanishes to first order, but whose influence we shall nevertheless wish to investigate.

The problem of thermoelastic deformation has received considerable attention in the literature,, although most workers have treated static, rather than dynamic situations. Again few of the dynamic treatments take viscosity or permanent deformation into account, and in some instances false thermodynamic premises invalidate results presented. Markham, Beyer, and Lindsay (1951), Truesdell (1953), and Hunt (1957), have made exhaustive studies of sound absorption in fluids in the presence of viscosity and heat conduction. The method to be employed in this chapter for the study of attenuation in solids is in many respects similar to that of Hunt. An important point to keep in mind is that the energy dissipated due to thermoelastic coupling or, what is the same thing, the energy dissipated due to infinitesimal elastic deformations in a medium of non-zero thermal conductivity, must be considered in addition to any energy dissipated by viscous or other attenuation processes. Kasahara (1956), in considering the problem of strain energy in a visco-elastic crust, fails to take this fact into account, so that his calculations are incorrect.

Synge (1955) has derived an equation of motion for a fluid exhibiting both viscosity and non-zero thermal conductivity. His work is based on that of Eckart (1940), and is thus thermodynamically rigorous, since it considers the production of irreversible energy. Synge's equations of motion and temperature are non-linear, but the non-linear terms were dropped before solution of the system was attempted. Solutions of the form

$$T = T_1 e^{j(\sigma x + \omega t)}$$
$$w = w_1 e^{j(\sigma x + \omega t)} \qquad (4.4\ a,b)$$

were assumed, and subsequent substitution of these relations in the equations of motion and temperature yielded a secular equation in the generally complex wave number σ . T_1 and u_1 are real constants that may be found by satisfaction of prescribed boundary conditions.

Synge did not solve his secular equation, which is biquadratic in the complex quantity σ . Lessen (1957) has attempted to solve this equation by approximation procedures. His equations have dimensional inconsistencies, however; nor does he attempt to investigate the frequency dependence of the attenuation coefficient α . (We recall that $\alpha = \mathrm{Im}(\sigma)$; $\mathrm{Im}(\nu + j\alpha)$).

Biot (1955;1956) has developed a theory of thermoelasticity based on irreversible thermodynamics. His arguments lack generality, since he derives the equations of motion and temperature from the reversible forms of the first and second laws. He is therefore unable to arrive at the more general theory of Synge (1955) and of Knopoff and MacDonald (1958). Biot's relations are similar to those of Synge (1955) and Lessen (1957), except that he prefers to express his temperature equation as a function of the specific entropy, rather than of the temperature explicitly, (see below, Section 2 of the present chapter). Biot has not attempted to solve his equations in closed form, nor has he studied attenuation of stress waves in a thermoelastic medium. He also ignores the effect of viscosity and permanent deformation.

Deresciewicz (1957) has made use of Biot's equations in a study of plane wave attenuation in a thermoelastic solid.

He finds that his secular equation is intractable, and then proceeds to show that an assumed ω^2 damping law,

$$\alpha = (\text{const.})\,\omega^2$$

satisfies the secular equation for low frequencies. His approximations are somewhat obscure and, since he does not actually solve the secular equation, his results again lack generality.

A totally different approach to the problem of thermo-elastic dissipation has been made by Zener (1948). He assumes that the non-elastic behavior of a solid can be described by a model which he terms the "standard linear solid", and whose stress-strain relation is given by (See Zener, 1948, p. 43 ff.):

$$p + \tau_\varepsilon \dot{p} = M_R\,(\varepsilon + \tau_p\,\dot{\varepsilon}) \qquad (4.5)$$

where p = tensile stress, ε = tensile strain, τ_ε = relaxation time of stress at constant strain, τ_p = relaxation time of strain at constant stress, and M_R "relaxed modulus". M_R can be identified with the familiar elastic modulus M_E, since for $\dot{\varepsilon}$ and $\dot{p} \to 0$, (4.5) gives

$$p = M_R\,\varepsilon = M_E\,\varepsilon \qquad (4.6)$$

For the sinusoidal steady state, he assumes solutions of (4.5) in the form,

$$p(t) = p_o e^{j\omega t}$$
$$\varepsilon(t) = \varepsilon_o e^{j\omega t}$$
(4.7 a,b)

where p_o and ε_o are real constants. Substitution of (4.7 a,b) into (4.5) yields

$$(1 + j\omega \tau_\varepsilon) p_o = M_R (1 + j\omega \tau_p) \varepsilon_o \qquad (4.8)$$

or

$$p_o = \mathcal{M} \varepsilon_o \qquad (4.9)$$

where the complex modulus $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{1 + j\omega \tau_p}{1 + j\omega \tau_\varepsilon} \qquad (4.10)$$

A convenient measure of internal friction is afforded by the tangent of the angle by which strain lags behind applied stress. Zener defines an angle Δ , such that

$$\tan \Delta = \frac{\mathrm{Im}(\mathcal{M})}{\mathrm{Re}(\mathcal{M})} \qquad (4.11)$$

which, after further manipulation, is shown to be

$$\tan \Delta = \frac{M_U - M_R}{\bar{M}} \frac{\omega \bar{\tau}}{1+(\omega \bar{\tau})^2} \tag{4.12}$$

In this expression $\bar{\tau}$ is the geometric mean of the two relaxation times,

$$\bar{\tau} = (\tau_\varepsilon \tau_\rho)^{1/2}$$

$M_U = \frac{\tau_\rho}{\tau_\varepsilon} M_R$, where M_U is called the "unrelaxed" elastic modulus, and $\bar{M}$ is the geometric mean of the two elastic moduli,

$$\bar{M} = (M_R M_U)^{1/2}$$

The quantity $\tan \Delta$ will be at a maximum when $\omega \bar{\tau} = 1$. It can be shown (Zener, 1948, p. 62 ff.) that

$$\tan \Delta = \frac{1}{2\pi} \frac{\Delta E}{E} \tag{4.13}$$

if Δ is small. Here ΔE = energy dissipated per cycle per unit volume, and E = elastic energy per unit volume when the strain is at a maximum. MacDonald and Knopoff (1958) write the right member of (4.13) in the form

$$\frac{1}{Q} = \frac{T \Delta S_{irr}}{2\pi E} \tag{4.14}$$

Combining (4.13) and (4.14), we have

$$\frac{1}{Q} = \tan \Delta \tag{4.15}$$

The quantity $1/Q$ is called the "<u>specific dissipation function</u>" by Knopoff and MacDonald, while Zener terms the ratio $\Delta E/E$ the "specific damping capacity". It is further shown by Knopoff and MacDonald that $1/Q$ is related to the coefficient of attenuation α by

$$\frac{1}{Q} = \frac{2\alpha c}{\omega} \qquad (4.16)$$

where c = wave propagation velocity. Combination of (4.12), (4.15) and (4.16) gives

$$\alpha = \frac{M_U - M_R}{2Mc} \frac{\omega^2 \bar{\tau}}{1 + (\omega\bar{\tau})^2} \qquad (4.17)$$

Now it was indicated in the last chapter that most available evidence points to the fact that the attenuation coefficient α is a <u>linear</u> function of ω for silicates. It is obvious that a mechanism of the type (4.17) cannot be brought into agreement with what is known empirically in the case of rocks and glasses.

It should be emphasized that up to this point no thermodynamical arguments have been introduced into Zener's theory.

In order to attack the thermo-elastic problem, Zener presents the concept of relaxation by thermal diffusion, (Log. Cit., p. 89 ff.). He states that the time of relaxation for the establishment of temperature equilibrium is given approximately by

$$\tau = \frac{D^2}{\kappa} \tag{4.18}$$

where D is of the order of the distance that heat must flow for thermal equilibrium to be established, and κ= thermal diffusivity. The distance D is equivalent to the "mean diameter of a crystallite" as defined by Mason (1957). Zener next assumes that $D \approx \lambda$, where λ= wave length, and identifies the thermal relaxation time (4.18) with the quantity $\bar{\tau}$ in (4.12) and (4.17). No physical reason for this step is suggested by this worker, although it does lead to a theory of thermo-elastic damping in apparent agreement with experiments performed on many metals (Bennewitz and Rötger, 1936; Randall, Rose, and Zener, 1939). In this instance eq. (4.17) is found to reproduce quite accurately the marked absorption peaks that characterize the frequency dependence of attenuation in metals.

The wave lengths of seismic waves produced by earthquakes or artificial explosions are of course larger by several orders of magnitude than the diameter of a crystallite in the rock, so that the assumption $\lambda \approx D$ can under no circumstances be upheld in seismology. In the megacycle frequency range, however, the wave length may become of the order of D; but even in this case high-frequency measurements on rocks have failed to show the absorption peaks observable in metals.

Zener's theory appears to agree with experiment in the case of many metals, it is not, as has sometimes been stated, based on a rigorous thermodynamic development. In particular, almost the entire theory hinges on (4.17), a relation which has been derived without any recourse to thermodynamics.

2. The Thermo-Elastic Solid.

In this section we shall investigate in detail the solutions of system (4.1 a,b). We will first treat the case $(\lambda_v + 2\mu_v) = 0$, so that we seek to solve

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = (\lambda_\varepsilon + 2\mu_\varepsilon) \frac{\partial^2 u}{\partial x^2} - \beta k \frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{k \beta T_0}{\rho_0 c_\varepsilon} \frac{\partial^2 u}{\partial x \partial t}$$

(4.19 a,b)

A solid describable by (4.19 a,b) will be called a "thermo-elastic solid", while the more general model (4.1 a,b) will be termed "visco-thermo-elastic solid". That model will then be considered in the subsequent section of this chapter. It must be emphasized that only compressional infinitesimal waves give rise to thermal phenomena, since for an infinitesimal shear wave the equation of motion (3.17) reduces to the familiar form,

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \mu_\varepsilon \frac{\partial^2 u}{\partial x^2} \qquad (4.20)$$

while the temperature equation (3.24) reduces to the ordinary heat flow law (3.29), and no thermo-elastic coupling exists.

The general form of the equation of state of a solid may be written in the form

$$dP = \left(\frac{\partial P}{\partial V}\right)_T dV + \left(\frac{\partial P}{\partial T}\right)_V dT$$

$$= V\left(\frac{\partial P}{\partial V}\right)_T dV + \beta k_T \, dT \qquad (4.21)$$

since

$$k_T = -V\left(\frac{\partial P}{\partial V}\right)_T$$

and

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

where k_T and β are the isothermal bulk modulus and thermal expansion, respectively. Equation (4.21) can be shown to reduce to the thermo-elastic stress tensor (3.12), which in one-dimensional form is

$$\tau'_1 = (\lambda_E + 2\mu_E)_T - \beta k_T \, \Delta T \qquad (4.22)$$

where $(\lambda_E + 2\mu_E)_T =$ isothermal elastic modulus.

Similarly the temperature equation (4.19 b) is derivable from a combination of the first and second laws of thermodynamics, which under equilibrium conditions may be written (Allis and Herlin, 1952, p. 105):

$$T_0 \frac{ds}{V_0} = c_\varepsilon \frac{dT}{V_0} + T_0 k_T \beta \frac{dV}{V_0} \tag{4.23}$$

where s = specific entropy. Differentiating (4.23) w.r.t. time and letting $1/V_0 = \rho_0$ and $dV/V_0 = \varepsilon$, one has

$$\frac{\partial T}{\partial t} = \frac{T_0}{c_\varepsilon} \frac{\partial s}{\partial t} - \frac{T_0 k_T \beta}{\rho_0 c_\varepsilon} \frac{\partial \varepsilon}{\partial t} \tag{4.24}$$

Relations (4.22) and (4.24) can be shown to lead to system (4.19 a,b), (Knopoff and MacDonald, 1958). The important point to realize here is that the elastic constants in (4.19 a,b) are actually the <u>isothermal</u> ones, so that we now write

$$\frac{\partial^2 w}{\partial t^2} = \frac{k_T + 4/3\,\mu_\varepsilon}{\rho_0} \frac{\partial^2 w}{\partial x^2} - \frac{\beta k_T}{\rho_0} \frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} - \frac{k_T \beta T_0}{\rho_0 c_\varepsilon} \frac{\partial^2 w}{\partial x \partial t}$$

(4.22 a,b)

where we have replaced $(\lambda_\varepsilon + 2\mu_\varepsilon)_T$ by $k_T + 4/3\,\mu_\varepsilon$. From (4.23) one has also

$$\frac{\partial T}{\partial x} = \frac{T_0}{c_\varepsilon} \frac{\partial s}{\partial x} - \frac{T_0 k_T \beta}{\rho_0 c_\varepsilon} \frac{\partial \varepsilon}{\partial x} \tag{4.23'}$$

which, when substituted into (4.22 a) yields,

$$\frac{\partial^2 w}{\partial t^2} = \left[\frac{k_T + 4/3\, \mu_E}{\rho_0} + \frac{T_0}{c_E}\left(\frac{k_T \beta}{\rho_0}\right)^2 \right] \frac{\partial^2 w}{\partial x^2} - \frac{\beta k_T T_0}{\rho_0} \frac{\partial s}{\partial x} \tag{4.24}$$

The adiabatic bulk modulus, k_s is related to the isothermal modulus by

$$k_s = k_T + \frac{T_0}{\rho_0 c_E} \left(\beta k_T\right)^2 \tag{4.25}$$

(Bullen, 1952, p. 26), so that (4.24) may be written

$$\frac{\partial^2 w}{\partial t^2} = \frac{k_s + 4/3\, \mu_E}{\rho_0} \frac{\partial^2 w}{\partial x^2} - \frac{\beta k_T T_0}{\rho_0} \frac{\partial s}{\partial x} \tag{4.26}$$

The equations of motion (4.22 a) and (4.26) express the <u>same</u> relationship, except that the former has temperature and the latter specific entropy as an independent variable.

Let $\frac{\partial T}{\partial x} = 0$ in (4.22 a). In this case one has,

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{V_{P,T}^2} \frac{\partial^2 w}{\partial x^2} \tag{4.27}$$

where

$$V_{P,T} = \sqrt{\frac{k_T + 4/3\, \mu_E}{\rho_0}} \tag{4.28}$$

defines the isothermal P wave velocity. Now let $\frac{\partial s}{\partial x} = 0$ in (4.26). Then one writes

$$\frac{\partial^2 w}{\partial t^2} = \frac{1}{V_{P,S}^2} \frac{\partial^2 w}{\partial x^2} \tag{4.29}$$

where

$$V_{P,S} = \sqrt{\frac{k_s + 4/3\,\mu_E}{\rho_0}} \tag{4.30}$$

defines the adiabatic P wave velocity. Since $k_S > k_T$ by (4.25), $V_{P,S}$ is always greater than $V_{P,T}$. The solutions of equations (4.27) and (4.29) are well known; in particular, neither model can give rise to attenuation, since neither contains dissipative terms. The isothermal case corresponds to an infinite, and the adiabatic to a zero thermal conductivity of the medium, (see Chapter 3, Section 1).

Bullen (1952, p. 83) states that thermodynamical conditions during the propagation of a seismic wave are very nearly adiabatic. Some controversy exists in the literature about this point, but as far as geophysical applications are concerned, the problem is largely academic. Jeffreys (1931) estimates that the discrepancy between the velocities (4.28) and (4.30) is only of the order of 1% in the earth, which is certainly well below present observational error.

Here we are primarily concerned with the propagation of waves in a medium described by (4.22 a,b) i.e., one in which the thermal conductivity is neither zero nor infinite. In this case thermal attenuation occurs, as has already been pointed out in the previous chapter. Again, let

$$M_T = \frac{k_T \beta T_0}{\rho_0 c_\varepsilon}$$

$$M_T' = \frac{\beta k_T}{\rho_0} \qquad (4.31\ a,b)$$

We follow the method of Synge (1955) and substitute solutions of the form (4.4 a,b) into (4.22 a,b). This leads to the simultaneous algebraic system in u and T,

$$-(M_T \sigma \omega) u + (K\sigma^2 + j\omega) T = 0$$

$$(V_{\rho,T}^2 \sigma^2 - \omega^2) u + (j\sigma M_T') \quad T = 0 \qquad (4.32\ a,b)$$

So that non-trivial solutions exist, the determinant of coefficients must vanish,

$$\begin{vmatrix} -M_T \sigma \omega & K\sigma^2 + j\omega \\ (V_{\rho,T}^2 \sigma^2 - \omega^2) & j\sigma M_T' \end{vmatrix} = 0 \qquad (4.33)$$

Expansion of the above determinant then gives the secular equation

$$(V_{\rho,T}^2 K)\sigma^4 + \left[j\omega\left(M_T M_T' + V_{\rho,T}^2\right) - \omega^2 K\right]\sigma^2 - j\omega^3 = 0 \qquad (4.34)$$

But

$$V_{p,s}^2 = M_T M_T' + V_{p,T}^2 = V_{p,T}^2 + \frac{T_0}{C_\varepsilon}\left(\frac{\beta k_T}{\rho_0}\right)^2$$

as can easily be seen from (4.24). Thus (4.34) becomes

$$\left(V_{p,T}^2 K\right)\sigma^4 + \left[j\omega V_{p,s}^2 - \omega^2 K\right]\sigma^2 - j\omega^3 = 0 \qquad (4.35)$$

This equation is biquadratic in the complex wave number σ . Its solution can be found by standard, although quite laborious algebraic techniques.

Solving (4.35) for σ^2 with the aid of the quadratic formula, one has

$$\sigma^2 = \frac{\omega}{2V_{p,T}^2 K}\left[\left(\omega K - jV_{p,s}^2\right) \pm \sqrt{\omega^2 K^2 - V_{p,s}^4 + 2j\omega K\left(2V_{p,T}^2 - V_{p,s}^2\right)}\right] \qquad (4.36)$$

It is convenient to express the radical of (4.36) in the form

$$\sqrt{\omega^2 K^2 - V_{p,s}^4 + 2j\omega K\left(\Delta V_p\right)} = C_1 + j\mathcal{D}_1 \qquad (4.37)$$

where

$$\left(\Delta V_p\right) = 2V_{p,T}^2 - V_{p,s}^2 \qquad (4.38)$$

Further calculation leads to the expressions

$$C_1 = \left\{ \frac{V_{P,S}^4 - \omega^2 K^2}{2} \left[\sqrt{1 + \left(\frac{2 \omega k \, \Delta V_p}{V_{P,S}^4 - \omega^2 K^2} \right)^2} - 1 \right] \right\}^{1/2}$$

$$\mathcal{D}_1 = \left\{ \frac{V_{P,S}^4 - \omega^2 K^2}{2} \left[\sqrt{1 + \left(\frac{2 \omega k \, \Delta V_p}{V_{P,S}^4 - \omega^2 K^2} \right)^2} + 1 \right] \right\}^{1/2}$$

(4.39 a,b)

Thus (4.36) may be written

$$\sigma^2 = \frac{\omega}{2 V_{P,T}^2 K} \left[\omega K \pm C_1 + j \left(\pm \mathcal{D}_1 - V_{P,S}^2 \right) \right] \qquad (4.40)$$

Since

$$\sigma^2 = \nu^2 - \alpha^2 + 2 j \nu \alpha \qquad (4.41)$$

combination with (4.40) and subsequent separation into real and imaginary parts leads to the system

$$\nu^2 - \alpha^2 = \frac{\omega (\omega K \pm C_1)}{2 V_{P,T}^2 K}$$

$$\nu \alpha = \frac{\omega (\pm \mathcal{D}_1 - V_{P,S}^2)}{4 V_{P,T}^2 K} \qquad (4.42)$$

Solving for α^2, one has

$$\alpha^2 = \frac{\omega(\pm C_1 + \omega k)}{4 V_{p,T}^2 k} \left[\sqrt{1 + \left(\frac{\pm \mathcal{D}_1 - V_{p,s}^2}{\pm C_1 + \omega k} \right)^2} - 1 \right] \quad (4.43)$$

This relation expresses the attenuation coefficient α in terms of the quantities ω, k, $V_{p,s}$, and $V_{p,T}$. Numerical calculations based on (4.43) are obviously quite laborious but, as will be shown presently, approximations can be made that reduce (4.43) to a much more tractable form. It is to be noted that (4.43), in conjunction with (4.39 a,b), constitutes the <u>exact</u> solution of damping in the thermo-elastic solid, subject to no approximations of any kind. These results are thus more general than those of Lessen (1957) and Deresciewicz (1957).

Consider now (4.38) and (4.39 a,b). Since $V_{p,s} \approx V_{p,T}$, we may write

$$(\Delta V_p) \approx V_{p,s}^2$$

Moreover,

$$\frac{2 \omega k \, \Delta V_p}{V_{p,s}^4 - \omega^2 k^2} = \frac{\dfrac{2 \omega k}{V_{p,s}^2}}{1 - \dfrac{\omega^2 k^2}{V_{p,s}^4}} \quad (4.44)$$

When $V_{p,s}^4 \gg \omega^2 k^2$ i.e., when $V_{p,s}^2 \gg \omega k$, (4.44) becomes

$$\frac{2 \omega k \, \Delta V_p}{V_{p,s}^4 - \omega^2 k^2} = \frac{2 \omega k}{V_{p,s}^2} \left[1 + \frac{\omega^2 k^2}{V_{p,s}^4} - \dots \right] \approx \frac{2 \omega k}{V_{p,s}^2} \quad (4.45)$$

For silicates, $V_{p,s} = O(10^5 \text{ cm/sec})$, $\kappa = O(10^{-2} \text{ cm}^2/\text{sec})$, and (4.45) will hold as long as

$$\omega \ll 10^{12} \text{ rad/sec} \tag{4.46}$$

which is certainly within any frequency range of physical interest. Inserting (4.45) into (4.39 a,b), one has

$$C_1 = \left\{ \frac{V_{p,s}^4 - \omega^2\kappa^2}{2} \left[\sqrt{1 + \left(\frac{2\omega\kappa}{V_{p,s}^2} \right)^2} - 1 \right] \right\}^{1/2}$$

$$D_1 = \left\{ \frac{V_{p,s}^4 - \omega^2\kappa^2}{2} \left[\sqrt{1 + \left(\frac{2\omega\kappa}{V_{p,s}^2} \right)^2} + 1 \right] \right\}^{1/2} \tag{4.47 a,b}$$

To the approximation (4.46),

$$\sqrt{1 + \left(\frac{2\omega\kappa}{V_{p,s}^2} \right)^2} = 1 + \frac{2\omega^2\kappa^2}{V_{p,s}^4} + \ldots\ldots\ldots \tag{4.48}$$

so that (4.47 a,b) become

$$C_1 = \left\{ \frac{\omega^2\kappa^2 (V_{p,s}^4 - \omega^2\kappa^2)}{V_{p,s}^4} \right\}^{1/2}$$

$$D_1 = \left\{ V_{p,s}^4 - \omega^2\kappa^2 \right\}^{1/2} \tag{4.49 a,b}$$

However, C_1 can be written

$$C_1 = \left\{ \omega^2 K^2 \left(1 - \frac{\omega^2 K^2}{V_{P,S}^4} \right) \right\}^{1/2}$$

which is to the approximation (4.46),

$$C_1 = \omega K$$

Thus we have the simple relations

$$C_1 = \omega K$$

$$D_1 = \left\{ V_{P,S}^4 - \omega^2 K^2 \right\}^{1/2} \qquad (4.50\ a,b)$$

Inserting (4.50 a,b) into (4.43),

$$\alpha^2 = \frac{\omega^2}{2 V_{P,T}^2} \left\{ \sqrt{1 + \left[\frac{(V_{P,S}^4 - K^2\omega^2)^{1/2} - V_{P,S}}{2\omega K} \right]^2} - 1 \right\} \qquad (4.51)$$

where we have only used the positive values of C_1 and D_1, since negative values of C_1 would cause the expression under the radical sign of (4.43) to become infinite.

The expression within square brackets under the radical sign of (4.51) can be written

$$\left[\frac{(V_{P,S}^4 - K^2\omega^2)^{1/2} - V_{P,S}^2}{2\omega K} \right] = \frac{\left(1 - \frac{K^2\omega^2}{V_{P,S}^4}\right)^{1/2} - 1}{\frac{2\omega K}{V_{P,S}^2}}$$

$$\approx -\frac{K\omega}{4 V_{P,S}^2} \qquad (4.52)$$

again to approximation (4.46). Substitution of (4.52) into (4.51) gives

$$\alpha^2 = \frac{\omega^2}{2V_{P,T}^2}\left\{\sqrt{1 + \left(\frac{K\omega}{4V_{P,S}^2}\right)^2} - 1\right\} \qquad (4.53)$$

Expanding the radical of (4.53), one has to approximation (4.46),

$$\alpha^2 = \frac{\omega^2}{2V_{P,T}^2}\left\{1 + \frac{K^2\omega^2}{32V_{P,S}^4} + \ldots\ldots - 1\right\}$$

that is,

$$\alpha = \frac{1}{8}\frac{K\omega^2}{V_{P,T}V_{P,S}^2}$$

or simply

$$\boxed{\alpha = \frac{1}{8}\frac{K\omega^2}{V_P^3}}\;, \quad \omega \ll \frac{V_P^2}{K} \qquad (4.54)$$

since $V_{P,S} \approx V_{P,T} = V_P$ for brevity. We have thus deduced the important result that the thermo-elastic attenuation coefficient is proportional to the square of the circular frequency ω for all ω of seismic interest.

In order to gain an idea of the order of magnitude of thermo-elastic attenuation in rocks, we take:

$O(V_P)$ 5×10^5 cm/sec

$O(K)$ 10^{-2} cm^2/sec

The following table is then easily computed from eq. (4.54):

ω (rad/sec)	Log_{10} (cm^{-1})	Log_{10} (Km^{-1})
10^{-2}	-24	-19
10^{-1}	-22	-17
1	-20	-15
10	-18	-13
10^{2}	-16	-11
10^{3}	-14	- 9
10^{4}	-12	- 7
10^{5}	-10	- 5
10^{6}	- 8	- 3
10^{7}	- 6	- 1
10^{8}	- 4	- 1
10^{9}	- 2	+ 3
10^{10}	0	+ 5

Table 4.1 : Values of the thermo-elastic attenuation coefficient α as a function of frequency.

These values indicate clearly that thermoelastic attenuation in rocks is significant only at very high frequencies. Thus Gutenberg (1951) estimates the average value of α for the transmission of longitudinal waves through the interior of the earth to be of the order of 10^{-4}/Km. It is evident from the above table that thermoelastic damping can yield an $\alpha = 10^{-4}$/Km only for

$\omega \gg 10^5$ rad/sec; accordingly one cannot expect that the observed damping of earthquake waves is explainable in terms of thermo-elastic attenuation. Moreover, we saw in the last chapter that all available empirical evidence for rocks points to a first power of ω damping law, a condition which is not satisfied by the present model.

Mason (personal communication to Knopoff and MacDonald, 1958) has observed that glasses exhibit an attenuation coefficient which depends on the square of the frequency for $\omega \sim 2\text{-}3 \times 10^6$ cps. It is difficult to ascertain at this point whether this behavior is evidence of true visco-elastic or true thermo-elastic damping, or whether it may not be a combination of both.

Before closing this section, it should be emphasized again that thermoelastic damping must exist in any medium possessing a finite and non-zero thermal conductivity. Whereas viscosity in solids may or may not correspond to an actual physical phenomenon, the damping mechanism discussed here is subject to no such restrictions. It is quite conceivable that what has been regarded as evidence of visco-elastic attenuation in solids actually corresponds to thermo-elastic losses.

3. The Visco-Thermoelastic Solid

We now turn our attention to the system (4.1 a,b), which we seek to solve in the presence of the viscous term in the equation of motion. The technique to be followed is identical to the one used in the previous section. Accordingly, we substitute solutions of the form (4.4 a,b) into (4.1 a,b), which yields the algebraic system

$$-(M_T \sigma \omega) w + (K\sigma^2 + j\omega) T = 0$$

$$(V_{P,T}^2 \sigma^2 + j r^2 \sigma^2 \omega - \omega^2) w + (j\sigma M_T') T = 0$$

(4.55 a,b)

where

$$r^2 = \frac{\lambda_v + 2\mu_v}{\rho_0} \quad (4.56)$$

and the elastic parameters are again the isothermal ones. System (4.55 a,b) differs from (4.32 a,b) only in the presence of the term $(+j r^2 \sigma^2 \omega)$ in the coefficient of the first term of (4.55 b). The secular equation corresponding to (4.55 a,b) is

$$\left[V_{P,T}^2 K + j\omega K r^2\right]\sigma^4 + \left[j\omega V_{P,S}^2 - \omega^2(K + r^2)\right]\sigma^2 - j\omega^3 = 0 \quad (4.57)$$

Solution of the bi-quadratic (4.57) is again straightforward, although extremely laborious because of the presence of a complex coefficient in the first term. Solving for σ^2, one gets,

$$\sigma^2 = \frac{\omega}{2(1 + s\omega^2)}\left\{\left[C_1(\omega) - a\omega\right] + j\left[D_1(\omega) - fg\omega^2 - b\right]\right\} \quad (4.58)$$

where

$$C_1^2(\omega) = \frac{A\omega^4 + B\omega^2 + E}{2}\left\{\sqrt{1 + \left[\frac{2\omega(F\omega^2 + G)}{A\omega^4 + B\omega^2 + E}\right]} - 1\right\}$$

$$D_1^2(\omega) = \frac{A\omega^4 + B\omega^2 + E}{2}\left\{\sqrt{1 + \left[\frac{2\omega(F\omega^2 + G)}{A\omega^4 + B\omega^2 + E}\right]} + 1\right\} \qquad (4.59\ a,b)$$

and

$$\frac{r^2\left(V_{P,S}^2 - V_{P,T}^2\right) - V_{P,T}^2 K}{V_{P,T}^4 K} = a \qquad (4.60\ a)$$

$$\frac{V_{P,S}^2}{V_{P,T}^2 K} = b \qquad (4.60\ b) \qquad\qquad \frac{r^4}{V_{P,T}^4} = s \qquad (4.60\ d)$$

$$\frac{r^2}{V_{P,T}^4 K} = f \qquad (4.60\ c) \qquad\qquad \frac{1}{V_{P,T}^2 K} = b' \qquad (4.60\ e)$$

and

$$A = f^2 g^2 - 4sf$$
$$B = 2bfg - 4f - a^2$$
$$E = b^2$$
$$F = afg + 2sb'$$
$$G = ab + 2b'$$
$$(4.61\ a\text{-}e)$$

Since the wave number σ is complex, $\sigma = \nu + j\alpha$, (4.58) may be separated into real and imaginary parts. The resulting two simultaneous algebraic equations may be solved for the attenuation coefficient α ,

$$\alpha^2 = \frac{\omega\left[C_1(\omega) - a\omega\right]}{4(1+s\omega^2)}\left\{\sqrt{1 + \left(\frac{\left[D_1(\omega) - \{b + fg\omega^2\}\right]}{\left[C_1(\omega) - a\omega\right]}\right)^2} - 1\right\}$$

(4.62)

Eq. (4.62), in conjunction with relations (4.59) - (4.61), again constitutes the <u>exact</u> solution for attenuation in a visco-thermoelastic medium. In the absence of viscosity, $\eta^2 = 0$, relations (4.59) can be easily shown to reduce to (4.39 a,b), and (4.62) to (4.43).

When $K = 0$, it is necessary to return to the secular equation (4.57), since relations (4.60) become indeterminate in this case. For zero thermal diffusivity, one thus derives from (4.57) the secular equation of the standard visco-elastic solid,

$$\left[j\omega V_{p,s}^2 - \omega^2\eta^2\right]\sigma^2 - j\omega^3 = 0 \qquad (4.63)$$

Separation into real and imaginary parts leads to the system,

$$V_{p,s}^2\left[\nu^2 - \alpha^2\right] - 2\omega\eta^2\nu\alpha - \omega^2 = 0$$

$$\omega\eta^2\left[\nu^2 - \alpha^2\right] + 2V_{p,s}^2\nu\alpha = 0$$

(4.64 a,b)

These equations, when solved for ν and α, yield the familiar attenuation and dispersion formulae in the classical visco-elastic, or Kelvin-Voigt solid, see e.g. Kolsky (1953, p. 117) and this thesis, (Ch. III, p. 75 ff). This visco-elastic attenuation coefficient is, as we have seen, given by

$$\alpha = \frac{1}{2V_p} \frac{(\lambda_v + 2\mu_v)}{(\lambda_E + 2\mu_E)} \omega^2 \tag{4.65}$$

However, no physical medium has a vanishing thermal **diffusivity**, so that this model is open to serious criticism on thermodynamic grounds. In particular, it clearly violates the criteria of Weiner (1957; this thesis, Chapter IV, p.111).

The exact expression for attenuation in a thermo-viscoelastic medium, eq. (4.62), is again extremely ponderous. Fortunately, simplifying approximations can be made that reduce the formulae to more tractable form. Knopoff and MacDonald (1958) have shown that for silicates the viscosity μ_v has as an upper limit a value of 10 dyne sec/cm^2, while μ_E is of the order of 10^{11} dynes/cm^2. If $\lambda_v \approx \mu_v$, and $\lambda_E \approx \mu_E$, the viscous term of the equation of motion (4.1 a) can be treated as a perturbation of the ordinary thermo-elastic equation of motion. Thus, assuming that

$$r^2 = \frac{\lambda_v + 2\mu_v}{\rho_0}$$

is small, and that

$$V_{P,T}^2 \gg \omega k$$

in analogy to the thermo-elastic case, relations (4.60) can be shown to reduce to

$$a = \frac{1}{V_{P,T}^{2}} \qquad s = \frac{r^4}{V_{P,T}^{4}}$$

$$b = \frac{1}{K} \tag{4.66}$$

and

$$fg = \frac{r^4}{V_{P,T}^{4} K}$$

while (4.59 a,b) become

$$C_1(\omega) = \frac{\omega\left(F\omega^2 + G\right)}{\left(A\omega^4 + B\omega^2 + E\right)^{1/2}}$$

$$D_1(\omega) = \left(A\omega^4 + B\omega^2 + E\right)^{1/2} \tag{4.67 a,b}$$

where

$$A = \frac{r^8}{V_{P,T}^{8} K^2} \qquad E = \frac{1}{K^2} \qquad G = \frac{1}{V_{P,T}^{2} K}$$

$$B = \frac{2r^4}{V_{P,T}^{4} K} \qquad F = \frac{r^4}{K V_{P,T}^{6}} \tag{4.68}$$

Insertion of relations (4.67 a,b), and use of relations (4.68), permits us to write (4.62) in the form

$$\frac{\omega^2}{2V_{P,T}^{2}}\left\{\sqrt{1 + \left[\frac{V_{P,T}^{2}}{2K\omega}\left(1 + \frac{r^8}{V_{P,T}^{8}}\omega^4 + \frac{2r^4}{V_{P,T}^{4}}\omega^2\right)^{1/2} - \left(1 + \frac{r^4\omega^2}{V_{P,T}^{4}}\right)\right]^2} - 1\right\} \tag{4.69}$$

This expression can be further simplified by reapplication of the conditions τ^2 = small and $V_{P,T}^2 \gg \omega k$. The final result of the calculation is then

$$\boxed{\alpha = \frac{(\lambda_v + 2\mu_v)^4}{4 V_p^7 k \rho_0} \omega^4} \tag{4.70}$$

where we have written $V_{P,T} = V_P$ for brevity. We have thus derived the result that the attenuation in a thermo-viscoelastic medium is proportional to the fourth power of the circular frequency for $\lambda_v + 2\mu_v$ small and $V_p^2 \gg \omega k$. Relation (4.70) does NOT reduce to (4.65) because the classical visco-elastic theory does not take finite thermal diffusivity or, equivalently, finite thermal conductivity into account. Since the classical visco-elastic solid is derived on the basis of incorrect thermodynamic assumptions, (4.65) can obviously not be derived as a special case of (4.69) or (4.70).

In any event, the ω^4 frequency dependence of the attenuation coefficient α is again not in conformity with a linear frequency damping mechanism. Consequently the visco-thermoelastic model, even though it is based on a more rigorous thermodynamic footing, cannot serve as a theoretical interpretation of observed internal losses in silicates.

C H A P T E R V

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK.-

1. The Shock Zone and its Surrounding Regions

In the course of the past three chapters, we have traced the propagation of a disturbance from its inception as a shock front of arbitrary amplitude to its decay into an infinitesimal wave and to its final conversion to heat. From all that has been said so far, it is evident that no single propagation mechanism can be used to describe the progressive decay of the shock throughout its entire path. Instead, we will find it convenient to speak of two separate regions which surround the source of the disturbance. The first of these may conveniently be termed the "shock zone"; the second we will then call the "small amplitude zone". As has been pointed out in Chapter III, no clear-cut boundary between these regions exists, but one can arbitrarily specify that the shock zone is that region surrounding the source in which $(P-P_o) \gg \mathcal{G}$; the small amplitude zone will then begin when $(P-P_o) = O(\mathcal{G})$. Energy dissipation in the former may be treated by techniques developed in Chapter II, and in the latter by the methods of Chapter III or IV. Evidently, the larger the magnitude of $(P-P_o)$ at $t = 0$, the larger will be the volume of the shock zone surrounding the source.

Bullen (1953; 1953 a; 1955) assumes that the strained region prior to a major earthquake can be represented by a sphere of rock of minimum radius 25 Km, and maximum 50 Km. Earthquake shock waves, however, are probably generated well within the interior of this strained region. Accumulated stress may not be uniformly distributed, but will probably tend to concentrate at certain points inside the source sphere. A shock wave can then be formed as soon as such a localized stress accumulation is suddenly relieved. As it moves into the strained region, the front first builds up energy, in a way perhaps somewhat similar to a detonation wave propagating inside an explosive charge. As soon as it reaches rock under <u>normal</u> hydrostatic stress alone, the shock will begin to decay, and the dissipation mechanisms described in Chapter II may then be expected to become operative. Accordingly, the calculations in that chapter were carried out for a source sphere radius $a = 1$ Km, and in one instance (Table 2.7 b) for $a = 10$ Km. There is, of course, no a priori reason for selecting such magnitudes of source sphere radii; but these values seem reasonable when compared to Bullen's estimates of the total volume of the strained region prior to the occurrence of an earthquake.

If $\left(\frac{P - P_0}{k_0}\right)_{R=a} \leqslant 10^9$ dynes/cm^2 $= \mathcal{S}$ for rocks, then obviously no shock wave can be generated, and the propagation of the resulting wave can be treated by small amplitude stress wave theory alone.

If the sudden release of localized stress accumulation simultaneously produces a shock wave as well as an ordinary P wave that leave the surface of the source sphere $R = a$ at time $t = 0$, then the travel time curves of Chapter II clearly show that at least two separate and <u>direct</u> P wave phases should be observable on a seismogram. One of these will be the degenerate shock wave, which at sufficiently large distances from the focus has decayed into an ordinary acoustic disturbance, while the other will be the P wave that was generated at $R = a$ together with the shock front.

2. Energy Dissipation in the Shock Zone.

The problem of energy dissipation in the shock zone has already been treated in great detail in the latter part of Chapter II. Here we return to the results of that discussion, insofar as its influence on a number of seimological problems is concerned.

Tables 2.6 - 2.9 present results of energy dissipation calculations in the shock zone that have been carried out for the cases

$$\left(\frac{p - p_0}{k_0}\right)_{R=a} = 0.01,\ 0.1,\ 1,\ \text{and } 10.$$

It was assumed furthermore that

$$a = 1 \text{ Km}$$
$$k_0 = 10^{11} \text{ dynes/cm}^2$$
$$c_0 = 2 \text{ Km/sec},$$

except for Table 2.7 b, whoch was calculated for the case $a = 10$ Km. We recall that the specific energies recorded in the last column of the tabulations are those which exist in the shock zone immediately after the passage of the pulse. At all subsequent times, the heat so produced will of course diffuse radially outward, away from the source sphere. It will be noted that for the value of k_0 assumed here, significant heat production in the shock zone will only occur for the cases

$$\left(\frac{P-P_o}{k_o}\right)_{R=a} = \quad 1 \text{ and } 10.$$

As a matter of fact, the second value above corresponds to an initial $(P-P_o)_{R=a} = 10^{12}$ dynes/cm^2 = 10^6 bars. It is open to question whether this magnitude of stress accumulation prior to an earthquake is possible at given points of the earth's crust. For such pressures, moreover, the validity of the Birch equation of state may also be somewhat in doubt, since quantum mechanical effects may have to be taken into account at that point. In the case of underground atomic blasts, on the other hand, these high stresses do appear to be developed.

On September 19, 1957, a small atom bomb was detonated in a tunnel under a mesa in the Nevada A.E.C. Test Site, (Operation PLUMBBOB). A preliminary report containing some data declassified to date has been published recently (Johnson et al, 1958). The total energy released by the device was about 7.1 x 10^{19} ergs. A rough estimate of the shock pressure as a function of radial distance from the cavity (whose original diameter was eight feet) is given by Johnson et al to be of the order of 6 x 10^6 bars. At a point some 200 feet from the cavity, the shock pressure is estimated to have fallen to 1 x 10^3 bars = 10^9 dynes/cm^2. Since $\mathcal{G}$ = O(10^9 dynes/cm^2) for rocks, the diameter of the shock zone in this instance is about 400 feet.

Regrettably the report does not explain how these pressures were calculated, except to state that they are based on initial energy densities; neither is any mention made of the magnitudes of these energy densities nor of the techniques of their measurement. Accelerometer readings were made in the neighborhood of the detonation site, but the data has not been declassified. This is rather unfortunate, since the availability of both pressure and velocity data as a function of radial distance from the detonation cavity would enable one to calculate the equation of state of the rock by the methods of Chapter II. This, in turn, should help to settle the question of pressure ranges within which the various equations of state may be expected to hold satisfactorily.

Table 2.8 probably furnishes a good estimate of the order of magnitude of energy dissipation in the shock zone for an earthquake, while Table 2.9 may be more applicable to the situation arising for a major underground nuclear blast. Thus for $k_0 = 10^{11}$ dynes/cm^2,

$$\begin{aligned} (P-P_0)_{R=a} &= 10^5 \text{ bars} \qquad \text{(Table 2.8)} \\ (P-P_0)_{R=a} &= 10^6 \text{ bars} \qquad \text{(Table 2.9)} \end{aligned} \qquad (5.1)$$

The value of $a = 1$ Km is excessively large for an atomic blast; $a = 10$ meters is much more realistic. The specific energies remain unchanged in this case, but the values of E_{cum} (Table 2.9) must be reduced by a factor of 10^{-6}, and the R entries by a factor of 10^{-2}. (Compare also Tables 2.7 a and 2.7 b in this connection.)

All present computations have been carried out for

$k_o = 10^{11}$ dynes/cm^2. Bullen (1953, p. 220) has calculated that the bulk modulus increases rapidly with depth in the earth, and estimates its value at a depth of 33 Km to be already 1.16×10^{12} dynes/cm^2. A bulk modulus of the order of 10^{12} dynes/cm^2 would increase the values of $(P-P_o)_a$ and E_{cum} of Tables 2.6-2.9 by a factor of 10. This problem cannot be solved from theoretical considerations alone; further shock wave work in rocks along the lines of the recently reported investigation of Hughes and McQueen (1957) is necessary to settle the question.

The specific energies afford a convenient method to estimate mean initial temperatures within each shell immediately after the passage of the shock front. If $\bar{q}$ = specific energy, in calories/cm^3, then

$$T - T_{AM} = \Delta T = \frac{\bar{q}}{\rho_o c_\varepsilon} \tag{5.2}$$

where T_{AM} = ambient temperature in rock prior to passage of front. The quantity ΔT is thus immediately calculable and will yield an estimate of the <u>mean initial temperature</u> rise in each shell surrounding the source sphere $R = a$. Computations of this nature have been carried out for the specific energy distributions (5.1), and are tabulated in Table 5.1. It is to be emphasized that the initial temperatures thus computed are <u>mean</u> values for each shell; the continuous temperature-radial distance curve could be found by performing such calculations over successively thinner shells, but such refinement is unwarranted in view of the uncertainties of the values of the various parameters involved.

The mean initial temperatures for the case $\left(\frac{P-P_0}{k_0}\right)_{R=a} = 10$ are enormous near the source sphere. Physically, of course, such high temperatures signify that fusion must occur in this region. Indeed, exactly such a phenomenon has been observed in rock surrounding the original cavity of the "Operation Plumbbob" underground atomic blast. Johnson et al report that a shell of fused tuff rock, 10 cm thick, was formed at a distance of 50 feet from the source. These workers also estimate that about 7×10^8 grams of rock reached an initial temperature in the range between 1200 to 1500°C. Latent heats of fusion have not been taken into account for the calculations of $\left(\frac{P-P_0}{k_0}\right)_{R=a} = 10$ in Table 5.1. The computed temperatures have accordingly been bracketed in order to indicate that they should merely be considered to represent orders of magnitude.

The temperatures calculated for the case $\left(\frac{P-P_0}{k_0}\right) = 1$ would not indicate that the heat developed in the rock is sufficient to melt it. However, we recall that these entire calculations are based on a $1/R^2$ decay law. If the decay rate near the source sphere is greater, correspondingly larger amounts of energy will be dissipated per unit shell thickness traversed by the shock wave, and in this event fusion of rock may occur even in the case of earthquakes. This question cannot be settled without empirical data, whose procurability is certainly a mute point at present. An alternate fusion mechanism will be discussed in Section 4 of the present chapter.

Up to this stage we have been concerning ourselves only with the temperature distribution in the shock zone immediately

	$\left(\frac{p-p_0}{k_0}\right)_{R=a}=1$		$\left(\frac{p-p_0}{k_0}\right)_{R=a}=10$		
R (km)	$\bar{q}\left(\frac{cal}{cm^3}\right)$	ΔT (°C)	$\bar{q}\left(\frac{cal}{cm^3}\right)$	ΔT (°C)	REMARKS
1.0					
	300	750°	7000	(17,500°)	$\rho_0 = 2$ grs/cm^3
1.1					
	280	700°	4800	(12,000°)	$c_\varepsilon = 0.2$ cal/gr°C
1.2					
	190	480°	4300	(10,800°)	$k_0 = 10^{11}$ dyne/cm^2
1.4					
	150	370°	3400	(8,500°)	
1.6					n = 2
	80	200°	2000	(5,000°)	
1.8					
	65	160°	1600	(4,000°)	
2.0					
	20	50°	620	(1,600°)	
3					
	6	15°	380	(1,000°)	
4					
	3	8°	150	350°	
5					
	2	5°	80	200°	
6					
	0.7	2°	50	125°	
7					
	0.3	0.8°	30	75°	
8					
			20	50°	
9					
			10	25°	
10					
			2.4	6°	
20					

Table 5.1: Mean Initial Shell Temperatures in Shock Zone (Based on specific energies of Tables 2.8 and 2.9).

after the passage of the pulse. The heat evolved by the progressive decay of the shock will eventually be conducted away: however, because the thermal diffusivity of rocks is so small, very long periods of time will elapse before heat produced in the focal region of an earthquake appears at the surface.

The time, t, required for heat to diffuse through a shell of thickness R is given by

$$t = \frac{R^2}{4\kappa} \qquad (5.3)$$

where κ = thermal diffusivity (Carslaw and Jaeger, 1947, p. 33).

R (Km)	t (years)	R (Km)	t (years)
1	8.0×10^3	100	8.0×10^7
2	3.2×10^4	200	3.2×10^8
3	7.3	300	7.3
4	1.3×10^5	400	1.3×10^9
5	2.0	500	2.0
10	8.0	600	2.9
20	3.2×10^6	700	3.8
30	7.3		
40	1.3×10^7		
50	2.0		

Table 5.2: Thermal diffusion times a a function of radial distance R, $\kappa = 10^{-2}$ cm^2/sec.

Table 5.2 has been computed for a number of radial distances with $\kappa = 10^{-2}$ cm^2/sec, the usual value taken for rocks.

We note at once that even for an earthquake whose focus is merely 5 Km deep, 130,000 years will have to elapse before the heat generated by the shock wave reaches the surface.

During the course of geologic time, the earth has undergone a number of large-scale tectonic revolutions. These periods were undoubtedly characterized by increased seismic activity in the major orogenic belts. It is thus quite conceivable that anomalously high values of heat flow may be detectable in regions which have experienced tectonic upheavals in the past. In particular, it would be interesting to compare heat flow measurements on island arc systems with readings in less disturbed areas of the world. Such measurements have not yet been made extensively. Admittedly, the separation of heat flow due to primary heat, vulcanism, and radioactivity may be difficult to effect, but anomalous values over island arc systems might indicate that part of the total flow is attributable to past earthquakes. A related problem, vulcanism, caused by dissipation in the shock zone, will be treated in detail in section 4 of this chapter.

The heat from deep focus earthquakes may take billions of years before it arrives at the surface. Even if radiative transfer cannot be neglected, as has recently been suggested by Clark (1957), enormous times will have to elapse for the heat generated by the shock wave to diffuse away.

3. The Small Amplitude Zone and Related Problems

We have already seen in Chapter III that energy dissipation beyond the shock zone is quite negligible in comparison to the large amounts of heat that are evolved while the pulse is still a shock. On the other hand, the observed damping of small amplitude seismic waves cannot be explained in terms of a pure elastic theory. Knopoff and MacDonald (1958, in press) first showed that an attenuation coefficient proportional to the first power of the circular frequency is irreconcilable with any linear model treated in the literature. They then demonstrated that a model characterized by the stress-strain relation (3.26) did lead to an attenuation coefficient proportional to the first power of ω . In the present work their technique was generalized to take the effect of thermal terms into account. It was shown that these considerations led to the displacement relation (3.84), but that the second "thermal" attenuation coefficient α_2 (eq. 3.89) is probably quite small in comparison to α_1 (eq. 3.81 a). However, the question cannot be settled without recourse to experiment.

In Chapter IV we then proceeded to study two linear damping models in order to investigate whether the thermal terms might still bring thermodynamically more rigorous linear theory into agreement with observation. However, we found that neither the "thermo-elastic" nor the "thermo-viscoelastic" solids yielded attenuation coefficients that checked with empirical measurements, although it is possible that thermo-elastic damping may become important in the megacycle range. Finally, it was shown in that chapter that the work of Zener is not applicable to rocks, although it has given good agreement with observation in the case of many metals.

If the theory of energy dissipation in the shock zone surrounding the source of an earthquake is tenable, then the total seismic energies computed from the well-known Gutenberg-Richter formulae are in all likelihood much to small. This is evidently so because Gutenberg and Richter calculate the total seismic energy release from observed ground motion amplitude at the surface. In other words, only the energy that is not dissipated in the shock zone will contribute to ground motion at the surface; and, as we have seen, attenuation beyond the shock zone is quite negligible.

Let us thus consider an earthquake that may be described by the example computed in Table 2.8, for which $a = 1$ Km. Some 4.3 seconds after the generation of the shock, the rapidly decaying front has reached a point 10 Km from the center of the source sphere. At this position, the magnitude of $(P-P_o)_R$ is 10^9 dynes/cm^2, and from here outwards the disturbance becomes essentially a small amplitude stress wave, subject to only slight further attenuation (see Chapters III and IV). Now Gutenberg and Richter calculate the total energy of an earthquake only from observed ground motion at the surface, and assume that dissipation can be neglected. This assumption is undoubtedly true for the small amplitude zone, but it breaks down completely in the shock region, which in our particular example here is a shell of rock of inner and outer radius 1 and 10 Km, respectively.

Table 2.8 indicates that a total of 3.7×10^{26} ergs have been injected into this shell due to rapid decay of the shock front.

The energy dissipation calculations have been carried beyond the point $(P-P_o)_R = 10^9$ dynes/cm^2; strictly speaking, however, the propagation of the disturbance is no longer describable in terms of the shock wave theory of Chapter III. This means that, to first order, energy dissipation in the small amplitude zone may be neglected. Just what happens when $(P-P_o)_R = O(\mathcal{G})$ is not clear; experimental work is necessary to settle the question. For this reason, the shock decay computations were extended to excess pressures less than 10^9 dynes/cm^2 in Tables 2.6 to 2.9.

Let us then postulate that losses of energy are small for $R > 10$ Km. As a first approximation, we assume further that prior to the occurrence of the earthquake, $\frac{P-P_o}{k_o} = 1$ throughout the source sphere. Taking $k_o = 10^{11}$ dynes/cm^2, the excess stress accumulation at $t \leq 0$ is $(P-P_o)_a = 10^{11}$ dynes/cm^2. The volume of a source sphere of 1 Km radius is 4.2×10^{15} cm^3. Then the total potential energy stored initially in this strained region is roughly 4.2×10^{26} ergs. Assuming that all this energy leaves the source sphere in the shock front and in the P waves generated simultaneously, about $4.2 \times 10^{26} - 3.7 \times 10^{26} = 5 \times 10^{25}$ ergs will appear in the form of small amplitude stress waves beyond the sphere $R = 10$ Km. Since energy transmission may be expected to be radially uniform, roughly half this energy, or 2.5×10^{25} ergs say, will contribute to ground motion observable at surface observatories, (Jeffreys, 1952, p. 101).

The magnitude, M, of such an earthquake, computed on the basis of the formula

$$\log_{10} E = 5.8 + 2.4 M \qquad (5.4)$$

(Gutenberg, 1957), where E = total <u>small</u> amplitude stress wave energy, would be

$$M = \frac{\log_{10}(2.5 \times 10^{25}) - 5.8}{2.4} = 8.2 \qquad (5.5)$$

Thus roughly 10% of the total input energy of 4.2×10^{26} ergs will be detectable at the surface, and the energy releases computed on the basis of the Gutenberg-Richter magnitude formulae are accordingly <u>much</u> <u>too</u> <u>small</u>-----<u>in the present example, by at least a factor of ten</u>.

4. <u>Vulcanism Associated with Near-Source Dissipation</u>

The theory of shock wave decay near the source of a major earthquake may provide a possible explanation for vulcanism. Bullard (1954) has expressed the view that the source of volcanic heat may be sought in the dissipation of energy by friction near the focus of an earthquake. Energy may also be dissipated by plastic distortion and fracturing of rock. In order to illustrate this suggestion quantitatively, he has discussed <u>observed</u> annual energy release in the Japan-Kamchatka area, a region which in present times exhibits an abnormally high seismicity. The total annual seismic energy release calculated from observed ground motion at the surface is roughly 1.7 x 10^{26} ergs per year. The area of the region is about 2 x 10^6 Km^2, and the focii of most of the earthquakes are at an average depth of 60 Km. Assuming a specific heat for rocks of 1 cal/°C cm^3, and that the energy dissipated near the focus equals the energy radiated away as small amplitude stress waves, Bullard has calculated that at the present rate of seismic activity, all rock between depth 20 to 60 Km would be molten within a span of 30 million years.

If the theory of shock wave dissipation expounded in the present thesis is tenable, Bullard's estimate for the time required to melt such a deep layer of rock can be considerably reduced. Consider a slab 10 Km thick, whose upper and lower faces are 50 and 60 Km below the surface, respectively. Let us assume further that the earthquake focii are all located in the interior of the slab. If the area of the horizontal faces is 2 x 10^6 Km^2, the total volume of the slab will be 2 x 10^{22} cm^3. Assume now that, as in the example treated in the previous section, only 10% of the total earthquake energy can be observed at the surface. Then the total annual energy release will be 1.7 x 10^{27} ergs, of which 1.7 x 10^{27} - 0.17 x 10^{27} = 1.5 x 10^{27} ergs will be dissipated in the shock zone. The mean specific energy of the slab will thus be raised by

$$\frac{1.5 \times 10^{27} \text{ ergs}}{2.0 \times 10^{22} \text{ cm}^3} = 7.5 \times 10^{14} \text{ ergs/cm}^3$$

= 1.8 x 10^{-3} cal/cm^3 per year. Taking c_ε = 0.2 cal/gr °C, and ρ_o = 2 gr/cm^3, this would correspond to a mean temperature rise in the slab of 4.5 x 10^{-3} °C/year. From Table 5.2, we note that some 8 x 10^5 years must elapse before the heat generated by shock wave decay will have uniformly diffused through the 10 Km slab. Consequently, provided 1.7×10^{27} ergs are released in the slab every year, the temperature will have risen within 800,000 years to $(8 \times 10^{5}$ years$) \times (4.5 \times 10^{-3}$ ^{o}C/year$) =$ 3600 ^{o}C above the ambient temperature existing at that depth prior to the commencement of seismic activity.

This calculation is admittedly very rough, and is again only meant to suggest orders of magnitude. Nevertheless, it may be possible not only to account for vulcanism in this way, but also for the emplacement of large igneous bodies such as batholiths and laccoliths. Bullard (1954) proposes that current volcanic activity might well indicate seismic activity in the past. The results of the computations performed here certainly support such a hypothesis.

5. Suggestions for Future Work

A considerable amount of experimental research has been reported to date on shock wave propagation in metals and in water, but no work along such lines appears to have been carried out for rocks, except for the recently reported work of Hughes and McQueen (1957). Underground nuclear blasts afford an excellent method to study the propagation of shock waves in the earth, but unless complete and adequate data about such explosions is released to the scientific community at large, the bennefit of these measurements to seismology is limited. Further shock wave measurements on silicates should be carried out in the laboratory, and theory checked with observation.

The shock wave calculations of Chapter II are entirely based on the isothermal Birch-Murnaghan equation of state. It might be fruitful to perform similar computations for equations of state that hold above excess pressures of 10^7 bars, as for example the equation of Feynman, Metropolis, and Teller (1949). As has been pointed out before, these equations of state probably hold at pressures that are developed near an underground nuclear explosion, but <u>not</u> near the focus of an earthquake.

Further theoretical research into linear dissipation models does not appear to be promising in view of the results of Knopoff and MacDonald (1958) and the present work.

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A P P E N D I C E S

APPENDIX I $\frac{P-P_0}{k_0}$ vs. $\frac{U}{c_0}$ vs. $\frac{\rho}{\rho_0}$

$\frac{P-P_0}{k_0}$	$\frac{U}{c_0}$	$\frac{\rho}{\rho_0}$	$\frac{P-P_0}{k_0}$	$\frac{U}{c_0}$	$\frac{\rho}{\rho_0}$
0.00	1.00	1.00	0.40	1.35	1.29
0.01	1.01	1.01	0.50	1.42	1.33
0.02	1.03	1.02	0.60	1.48	1.37
0.03	1.04	1.03	0.70	1.54	1.41
0.04	1.05	1.04	0.80	1.59	1.46
0.05	1.06	1.05	0.90	1.64	1.50
0.06	1.07	1.05	1.00	1.69	1.53
0.07	1.08	1.06	1.1	1.74	1.56
0.08	1.09	1.07	1.2	1.79	1.59
0.09	1.10	1.08	1.3	1.83	1.63
0.10	1.11	1.09	1.4	1.88	1.66
0.11	1.12	1.09	1.5	1.92	1.69
0.12	1.13	1.10	1.6	1.96	1.72
0.13	1.14	1.11	1.7	2.00	1.74
0.14	1.15	1.12	1.8	2.03	1.77
0.15	1.16	1.12	1.9	2.07	1.79
0.16	1.17	1.13	2.0	2.11	1.82
0.17	1.18	1.14	2.1	2.14	1.84
0.18	1.18	1.14	2.2	2.18	1.87
0.19	1.19	1.15	2.3	2.22	1.89
0.20	1.20	1.16	2.4	2.25	1.92
0.21	1.21	1.16	2.5	2.28	1.94
0.22	1.22	1.17	2.6	2.31	1.96
0.23	1.23	1.18	2.7	2.34	1.98
0.24	1.23	1.18	2.8	2.37	2.00
0.25	1.24	1.19	2.9	2.40	2.02
0.26	1.25	1.20	3.0	2.43	2.04
0.27	1.26	1.21	3.1	2.46	2.06
0.28	1.27	1.21	3.2	2.49	2.08
0.29	1.27	1.22	3.3	2.52	2.10
0.30	1.28	1.22	3.4	2.54	2.12
			3.5	2.57	2.14

$\frac{p-p_0}{k_0}$	$\frac{U}{c_0}$	$\frac{\rho}{\rho_0}$	$\frac{p-p_0}{k_0}$	$\frac{U}{c_0}$	$\frac{\rho}{\rho_0}$
3.6	2.60	2.16	6.6	3.27	2.60
3.7	2.63	2.18	6.7	3.29	2.61
3.8	2.65	2.19	6.8	3.31	2.62
3.9	2.68	2.21	6.9	3.32	2.63
4.0	2.70	2.22	7.0	3.34	2.64
4.1	2.73	2.24	7.1	3.36	2.65
4.2	2.75	2.26	7.2	3.37	2.66
4.3	2.78	2.28	7.3	3.39	2.68
4.4	2.80	2.29	7.4	3.41	2.69
4.5	2.83	2.31	7.5	3.42	2.71
4.6	2.85	2.32	7.6	3.44	2.72
4.7	2.87	2.34	7.7	3.47	2.73
4.8	2.89	2.35	7.8	3.49	2.74
4.9	2.91	2.37	7.9	3.51	2.75
5.0	2.94	2.38	8.0	3.53	2.76
5.1	2.96	2.40	8.1	3.55	2.77
5.2	2.98	2.41	8.2	3.57	2.78
5.3	3.01	2.43	8.3	3.59	2.80
5.4	3.03	2.44	8.4	3.61	2.81
5.5	3.05	2.46	8.5	3.62	2.82
5.6	3.08	2.47	8.6	3.64	2.83
5.7	3.10	2.49	8.7	3.66	2.84
5.8	3.12	2.50	8.8	3.67	2.85
5.9	3.14	2.52	8.9	3.69	2.86
6.0	3.16	2.53	9.0	3.71	2.87
6.1	3.18	2.55	9.1	3.72	2.88
6.2	3.20	2.56	9.2	3.74	2.89
6.3	3.22	2.57	9.3	3.76	2.91
6.4	3.24	2.58	9.4	3.78	2.92
6.5	3.26	2.59	9.5	3.79	2.93
			9.6	3.81	2.94
			9.7	3.83	2.95
			9.8	3.84	2.96

$\frac{P-P_0}{k_0}$	$\frac{U}{c_0}$	$\frac{\rho}{\rho_0}$
9.9	3.86	2.97
10.0	3.88	2.98
11.0	4.03	3.10
12.0	4.18	3.18
13.0	4.32	3.27
14.0	4.46	3.35
15.0	4.60	3.44
16.0	4.73	3.52
17.0	4.86	3.59
18.0	4.98	3.66
19.0	5.10	3.73
20	5.21	3.80
30	6.2	4.4
40	7.1	4.9
50	7.9	5.3
60	8.5	5.7
70	9.2	6.1
80	9.7	6.3
90	10.2	6.6
100	10.7	6.8
110	11.2	7.1
120	11.6	7.4
130	12.1	7.6
140	12.6	7.9
150	13.0	8.1
160	13.4	8.3
170	13.7	8.5
180	14.1	8.6
190	14.5	8.8
200	14.8	9.0

APPENDIX II $\frac{P-P_0}{k_0}$ vs. $\frac{2\rho_0 E}{k_0}$

$\frac{P-P_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$	$\frac{P-P_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$	$\frac{P-P_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$
0.0001	$1\text{x}10^{-8}$	0.14	$1.5\text{x}10^{-2}$	0.45	$1.05\text{x}10^{-1}$
0.0002	4	0.15	1.7	0.46	1.09
0.0003	9	0.16	1.9	0.47	1.12
0.0004	$1.6\text{x}10^{-7}$	0.17	2.1	0.48	1.16
0.0005	2.5	0.18	2.4	0.49	1.20
0.0006	3.6	0.19	2.6	0.50	1.23
0.0007	4.9	0.20	2.8	0.51	1.27
0.0008	6.4	0.21	3.1	0.52	1.31
0.0009	8.1	0.22	3.3.	0.53	1.35
0.001	$1\text{x}10^{-6}$	0.23	3.6	0.54	1.39
0.002	4	0.24	3.8	0.55	1.43
0.003	9	0.25	4.1	0.56	1.47
0.004	$1.6\text{x}10^{-5}$	0.26	4.3	0.57	1.51
0.005	2.5	0.27	4.6	0.58	1.55
0.006	3.6	0.28	4.9	0.59	1.59
0.007	4.9	0.29	5.2	0.60	1.63
0.008	6.4	0.30	5.5	0.61	1.68
0.009	8.1	0.31	5.8	0.62	1.72
0.01	$1\text{x}10^{-4}$	0.32	6.1	0.63	1.76
0.02	4	0.33	6.4	0.64	1.80
0.03	9	0.34	6.7	0.65	1.84
0.04	$1.5\text{x}10^{-3}$	0.35	7.0	0.66	1.89
0.05	2.3	0.36	7.4	0.67	1.93
0.06	3.2	0.37	7.7	0.68	1.97
0.07	4.3	0.38	8.0	0.69	2.02
0.08	5.4	0.39	8.4	0.70	2.06
0.09	6.6	0.40	8.7	0.71	2.11
0.10	7.9	0.41	9.1	0.72	2.15
0.11	9.5	0.42	9.4	0.73	2.20
0.12	$1.1\text{x}10^{-2}$	0.43	9.7	0.74	2.24
0.13	1.3	0.44	$1.01\text{x}10^{-1}$	0.75	2.29

$\frac{\rho-\rho_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$	$\frac{\rho-\rho_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$	$\frac{\rho-\rho_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$
0.76	2.33×10^{-1}	1.7	7.26×10^{-1}	4.8	2.76
0.77	2.37	1.8	7.84	4.9	2.83
0.78	2.42	1.9	8.41	5.0	2.90
0.79	2.47	2.0	9.02	5.1	2.97
0.80	2.51	2.1	9.62	5.2	3.05
0.81	2.56	2.2	1.02x1	5.3	3.12
0.82	2.60	2.3	1.09	5.4	3.19
0.83	2.65	2.4	1.15	5.5	3.26
0.84	2.70	2.5	1.21	5.6	3.33
0.85	2.74	2.6	1.27	5.7	3.41
0.86	2.79	2.7	1.34	5.8	3.48
0.87	2.83	2.8	1.40	5.9	3.55
0.88	2.88	2.9	1.47	6.0	3.63
0.89	2.93	3.0	1.53	6.1	3.70
0.90	2.98	3.1	1.60	6.2	3.77
0.91	3.03	3.2	1.66	6.3	3.85
0.92	3.07	3.3	1.73	6.4	3.92
0.93	3.12	3.4	1.80	6.5	3.99
0.94	3.17	3.5	1.86	6.6	4.07
0.95	3.22	3.6	1.93	6.7	4.14
0.96	3.27	3.7	2.00	6.8	4.21
0.97	3.32	3.8	2.07	6.9	4.29
0.98	3.37	3.9	2.13	7.0	4.36
0.99	3.42	4.0	2.20	7.1	4.44
1.00	3.47	4.1	2.27	7.2	4.51
1.1	3.97	4.2	2.34	7.3	4.59
1.2	4.49	4.3	2.41	7.4	4.66
1.3	5.01	4.4	2.48	7.5	4.74
1.4	5.56	4.5	2.55	7.6	4.81
1.5	6.12	4.6	2.62	7.7	4.89
1.6	6.69	4.7	2.69	7.8	4.96

$\frac{p-p_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$	$\frac{p-p_0}{k_0}$	$\frac{2\rho_0 E}{k_0}$
7.9	5.04	20	14.8
8.0	5.12	30	23
8.1	5.19	40	32
8.2	5.27	50	41
8.3	5.34	60	49
8.4	5.42	70	58
8.5	5.50	80	67
8.6	5.58	90	76
8.7	5.65	100	85
8.8	5.73	110	95
8.9	5.80	120	104
9.0	5.88	130	113
9.1	5.96	140	122
9.2	6.03	150	132
9.3	6.11	160	141
9.4	6.19	170	150
9.5	6.27	180	159
9.6	6.35	190	169
9.7	6.42	200	179
9.8	6.50		
9.9	6.58		
10	6.66		
11	7.4		
12	8.2		
13	9.0		
14	9.8		
15	10.7		
16	11.5		
17	12.2		
18	13.1		
19	13.9		

APPENDIX III m vs. $\left(\frac{a}{R}\right)^n$

$m = R/a$	$\frac{a}{R}$	$\left(\frac{a}{R}\right)^2$	$\left(\frac{a}{R}\right)^3$
1.00	1.0000	1.000	1.000
1.01	0.9900	0.980	0.970
1.02	0.9804	0.961	0.942
1.03	0.9709	0.943	0.915
1.04	0.9615	0.924	0.889
1.05	0.9524	0.907	0.864
1.06	0.9434	0.890	0.840
1.07	0.9346	0.873	0.816
1.08	0.9259	0.857	0.794
1.09	0.9174	0.842	0.772
1.10	0.9091	0.826	0.751
1.11	0.9009	0.812	0.731
1.12	0.8929	0.797	0.712
1.13	0.8850	0.783	0.693
1.14	0.8772	0.769	0.675
1.15	0.8696	0.756	0.658
1.16	0.8621	0.743	0.641
1.17	0.8547	0.731	0.624
1.18	0.8475	0.718	0.609
1.19	0.8403	0.706	0.593
1.2	0.8333	0.694	0.579
1.3	0.769	5.91×10^{-1}	4.55×10^{-1}
1.4	0.714	5.08	3.64
1.5	0.667	4.45	2.97
1.6	0.625	3.91	2.44
1.7	0.588	3.46	2.03
1.8	0.556	3.09	1.72
1.9	0.526	2.77	1.46
2.0	0.500	2.50	1.25
2.1	0.476	2.27	1.08
2.2	0.455	2.07	9.42×10^{-2}
2.3	0.435	1.89	8.23
2.4	0.417	1.74	7.25
2.5	0.400	1.60	6.40
2.6	0.385	1.48	5.71
2.7	0.370	1.37	5.07
2.8	0.357	1.27	4.55
2.9	0.345	1.19	4.11
3.0	0.333	1.11	3.69
3.1	0.323	1.04	3.37
3.2	0.313	9.80×10^{-2}	3.07
3.3	0.303	9.18	2.78
3.4	0.294	8.64	2.54
3.5	0.286	8.18	2.34

$m = R/a$	$\frac{a}{R}$	$\left(\frac{a}{R}\right)^2$	$\left(\frac{a}{R}\right)^3$
3.6	0.278	7.73	2.15
3.7	0.270	7.29	1.97
3.8	0.263	6.92	1.82
3.9	0.256	6.55	1.68
4.0	0.250	$6.25\text{x}10^{-2}$	$1.56\text{x}10^{-2}$
4.1	0.244	5.95	1.45
4.2	0.238	5.66	1.35
4.3	0.233	5.43	1.26
4.4	0.227	5.15	1.17
4.5	0.222	4.93	1.09
4.6	0.217	4.71	1.02
4.7	0.2128	4.53	$9.64\text{x}10^{-3}$
4.8	0.2083	4.34	9.04
4.9	0.2041	4.17	8.50
5.0	0.2000	4.00	8.00
5.1	0.1961	3.85	7.54
5.2	0.1923	3.70	7.11
5.3	0.1887	3.56	6.72
5.4	0.1852	3.43	6.35
5.5	0.1818	3.31	6.01
5.6	0.1786	3.19	5.70
5.7	0.1754	3.08	5.40
5.8	0.1724	2.97	5.12
5.9	0.1695	2.87	4.87
6.0	0.1667	2.78	4.63
6.1	0.1639	2.69	4.40
6.2	0.1613	2.60	4.20
6.3	0.1587	2.52	4.00
6.4	0.1563	2.44	3.82
6.5	0.1538	2.37	3.64
6.6	0.1515	2.30	3.48
6.7	0.1493	2.23	3.33
6.8	0.1471	2.16	3.18
6.9	0.1449	2.10	3.04
7.0	0.1429	$2.04\text{x}10^{-2}$	$2.92\text{x}10^{-3}$
7.1	0.1408	1.98	2.79
7.2	0.1389	1.93	2.68
7.3	0.1370	1.88	2.57
7.4	0.1351	1.69	2.47
7.5	0.1333	1.78	2.37
7.6	0.1316	1.73	2.28
7.7	0.1299	1.69	2.19
7.8	0.1282	1.64	2.11
7.9	0.1266	1.60	2.03

m = R/a	$\frac{a}{R}$	$\left(\frac{a}{R}\right)^2$	$\left(\frac{a}{R}\right)^3$
8.0	0.1250	1.56	1.95
8.1	0.1235	1.53	1.88
8.2	0.1220	1.49	1.82
8.3	0.1205	1.45	1.75
8.4	0.1190	1.42	1.69
8.5	0.1176	1.38	1.63
8.6	0.1163	1.35	1.57
8.7	0.1149	1.32	1.52
8.8	0.1136	1.29	1.47
8.9	0.1124	1.26	1.42
9.0	0.1111	1.23	1.37
9.1	0.1099	1.21	1.33
9.2	0.1087	1.18	1.28
9.3	0.1075	1.16	1.24
9.4	0.1064	1.13	1.20
9.5	0.1053	1.11	1.17
9.6	0.1042	1.09	1.13
9.7	0.1031	1.06	1.10
9.8	0.1020	1.04	1.06
9.9	0.1010	1.02	1.03
10	0.1000	1.00×10^{-2}	1.00×10^{-3}
11	9.09×10^{-2}	8.26×10^{-3}	7.51×10^{-4}
12	8.33	6.94	5.78
13	7.69	5.91	4.55
14	7.14	5.08	3.64
15	6.67	4.45	2.97
16	6.25	3.91	2.44
17	5.88	3.46	2.03
18	5.56	3.09	1.72
19	5.26	2.77	1.46
20	5.00	2.50	1.25
21	4.76	2.27	1.08
22	4.55	2.07	9.42×10^{-5}
23	4.35	1.89	8.23
24	4.17	1.74	7.25
25	4.00	1.60	6.40
26	3.85	1.48	5.71
27	3.70	1.37	5.07
28	3.57	1.27	4.55
29	3.45	1.19	4.11
30	3.33	1.11	3.69
31	3.23	1.04	3.37
32	3.13	9.80×10^{-4}	3.07
33	3.03	9.18	2.78

$m = R/a$	$\frac{a}{R}$	$\left(\frac{a}{R}\right)^2$	$\left(\frac{a}{R}\right)^3$
34	2.94	8.64	2.54
35	2.86	8.18	2.34
36	2.78	7.73	2.15
37	2.70	7.29	1.97
38	2.63	6.92	1.82
39	2.56	6.55	1.68
40	2.50×10^{-2}	6.25×10^{-4}	1.56×10^{-5}
41	2.43	5.95	1.45
42	2.38	5.66	1.35
43	2.33	5.43	1.26
44	2.27	5.15	1.17
45	2.22	4.93	1.09
46	2.17	4.71	1.02
47	2.13	4.54	9.64×10^{-6}
48	2.08	4.33	9.04
49	2.04	4.16	8.50
50	2.00	4.00	8.00
55	1.82	3.31	6.01
60	1.67	2.78	4.63
65	1.54	2.37	3.64
70	1.43	2.04	2.92
75	1.33	1.78	2.37
80	1.25	1.56	1.95
85	1.18	1.38	1.63
90	1.11	1.23	1.37
95	1.05	1.11	1.17
100	1.00	1.00	1.00
110	9.09×10^{-3}	8.26×10^{-5}	7.51×10^{-7}
120	8.33	6.94	5.78
130	7.69	5.91	4.55
140	7.14	5.08	3.64
150	6.67	4.45	2.97
160	6.25	3.91	2.44
170	5.88	3.46	2.03
180	5.56	3.09	1.72
190	5.26	2.77	1.46
200	5.00	2.50	1.25
300	3.33×10^{-3}	1.11×10^{-5}	3.69×10^{-8}
400	2.50	6.25×10^{-6}	1.56
500	2.00	4.00	8.00×10^{-9}
600	1.67	2.78	4.63
700	1.43	2.04	2.92
800	1.25	1.56	1.95
900	1.11	1.23	1.37
1000	1.00	1.00	1.00

B I O G R A P H Y

The author, Sven Treitel, was born on March 5, 1929, in Freiburg i/Breisgau, Germany. He attended an American high school in Buenos Aires, Argentina, and entered M.I.T. as a freshman in the fall of 1949. In June 1953 he received his B.S. degree in Geophysics, and in September of that year began his graduate work in the Department of Geology and Geophysics. He received his M.S. degree in Geophysics in June, 1955.

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