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Diverse Routing in Networks with Probabilistic Failures

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Abstract—We develop diverse routing schemes for dealing with multiple, possibly correlated, failures. While disjoint path protection can effectively deal with isolated single link failures, recovering from multiple failures is not guaranteed. In particular, events such as natural disasters or intentional attacks can lead to multiple correlated failures, for which recovery mechanisms are not well understood. We take a probabilistic view of network failures where multiple failure events can occur simultaneously, and develop algorithms for finding diverse routes with minimum joint failure probability. Moreover, we develop a novel *Probabilistic Shared Risk Link Group* (PSRLG) framework for modeling correlated failures. In this context, we formulate the problem of finding two paths with minimum joint failure probability as an Integer Non-Linear Program (INLP), and develop approximations and linear relaxations that can find nearly optimal solutions in most cases.

I. INTRODUCTION

This paper deals with protection in communication networks with correlated probabilistic link failures. The objective of protection is to provide reliable communication in the event of failure of network components such as nodes or links. Such protection mechanisms are classified as *link protection* and *path protection*. Link protection precomputes an alternate detour for each link, and recovers from a link failure by rerouting the traffic along its predetermined detour. In contrast, path protection assigns two paths, a primary and a backup, to each connection, and the traffic is switched onto the backup path in case of a primary path failure. Therefore, the primary and backup paths need to be *disjoint*, since otherwise the two paths will fail simultaneously if a link or node shared by the two paths fails. In this paper, we focus on path protection.

The disjoint-path based protection effectively addresses the case of a single point failure, but if more than one failure occur at the same time, protection is not guaranteed since both paths may fail simultaneously. There are several factors that can cause multiple failures. First, modern communication networks are deployed over an optical fiber network, and so multiple communication links can share the same fiber in the optical layer. Consequently, any fiber cut can lead to the failure of all the (upper-layer) communication links sharing that fiber. Second, multiple link failures can occur if the second link

fails before the first was repaired. Third, natural disasters or attacks can destroy several links (which do not necessarily share a fiber) in the vicinity of such events.

The concept of Shared Risk Link Group (SRLG) has been proposed in order to address multiple correlated link failures systematically [1]. An SRLG is a set of links sharing a common physical resource (cable, conduit, etc.) and thus a risk of failure. In this context, Bhandari first studied so called Physically Disjoint Paths (PDP) problem in [2], and proposed a shortest PDP algorithm for particular topologies. Since this pioneering work, there has been a large body of work [3]–[13] dealing with multiple failures in the context of SRLGs. Most recently, Hu showed the NP-completeness of the *SRLG-Disjoint Paths* Problem (SDPP) where SRLG-disjoint paths are two paths touching no common SRLG [14].

All of the previous SRLG works assume that once an SRLG failure event occurs, all of its associated links fail simultaneously. Here, we generalize the notion of an SRLG to account for probabilistic link failures. This generalized notion allows us to model correlated failures that may result from a natural or man-made disaster. For example, in the event of a natural disaster, some, but not necessarily all, of the links in the vicinity of the disaster may be affected. Such failures cannot be described using a *deterministic failure model*, and this raises the need for a systematic approach to dealing with *correlated probabilistic link failures*. We address this issue by modeling SRLG events probabilistically so that upon an SRLG failure event, links belonging to that SRLG fail with some probability (not necessarily one). Our probabilistic SRLG model is applicable to a number of real-world failure scenarios. Some examples include: (i) *WDM Networks* where the lightpaths traversing a fiber form an SRLG and fail (with probability 1) in the event of a fiber cut, (ii) *Satellite/wireless communication links* where links are subject to outage in the event of bad weather. In this case, the satellite links affected by the weather event form an SRLG, and may fail with some probability, (iii) *ElectroMagnetic Pulse (EMP) attack*: EMP is an intense energy field that can instantly overload or disrupt numerous electrical circuits at a distance [15]. In the event of an EMP attack, the fiber links in the vicinity of the attack may have a high probability of failure and those distant from the attack would fail with low probability due to signal attenuation, and (iv) *Natural/man-made disasters* such

as earthquakes or floods where communication links in the vicinity of the disaster may fail. For example, an undersea cable was cut during the Taiwan earthquake of 2006 [16], disrupting most communications out of Taiwan. Similarly, during the Baltimore tunnel fire in 2001 [17], the fire melted away the fiber along the tunnel, leading again to a large number of correlated failures.

There are a number of papers dealing with probabilistic link failures [18]–[20]. Typically, they consider the *availability* (i.e., probability) that a connection is in the operating state, and seek to find a path pair satisfying minimum availability requirement [18], [19] or a path pair with maximum availability [20]. In [21], a primary/backup path allocation problem is defined to find a pair of paths having minimum joint failure probability. They adopt a *correlated link failure probability model* where the correlation between the links is represented by their joint failure probability. This correlation model requires exponential number of conditional probabilities in general, prohibiting a simple formulation. Due to this difficulty, they take into account only the first order correlation, i.e., the correlation between pairs of links.

In this paper, we consider finding a pair of paths with minimum joint failure probability in a network where the link failures occur randomly and are possibly correlated. We propose an alternative model that enables a simple formulation and captures the essence of correlated link failures. Our model assumes that once an SRLG failure event occurs, its associated links fail with some probabilities. Thus, the correlation exists among the links only when they belong to the same SRLG. Clearly, this model can be viewed as a generalization of the traditional (deterministic) SRLG model.

Our contributions can be summarized as follows:

- We generalize the SRLG framework to a probabilistic SRLG (PSRLG). This new framework enables us to effectively model correlated link failures, and develop efficient formulations to otherwise intractable problems involving correlated link failures.
- We develop mathematical formulations for the problem of finding a pair of paths with minimum joint failure probability. This new approach enables the generalization of disjoint path protection schemes to the case of multiple (probabilistic) failures.
- We develop heuristic algorithms for finding a pair of paths with minimum joint failure probability. Our algorithms are based on linear approximations and Lagrangian relaxations, and are shown to find nearly optimal solutions.

The rest of the paper is organized as follows. In Section II, we present our new probabilistic SRLG model and describe the generalized path protection problems. In Section III, we study the case of independent link failures, which provides fundamental insights to the study of correlated failures. In Section IV, we formulate the path protection problems using the PSRLG model and develop algorithms for finding paths with minimum joint failure probability. Finally, in Section V, we analyze the performance of our algorithms via simulations.

II. MODEL AND PROBLEM DESCRIPTION

Consider a directed network graph $G = (V, E)$ where V is a set of nodes and E is a set of links. Any link in E will be denoted by (i, j) for $i, j \in V$, meaning that the link starts from node i and ends at node j. There is a set R of SRLG events that can incur link failures. Each SRLG event $r \in R$ occurs with probability π_r, and once an SRLG event r occurs, link (i, j) will fail with probability $p_{ij}^r \in [0, 1]$. For example, if link (i, j) is never affected by event r, then we define $p_{ij}^r = 0$. On the other hand, if the event r is a cable cut and link (i, j) traverses that cable, then we will have $p_{ij}^r = 1$. In the following we generalize the traditional notion of an SRLG to include probabilistic correlated failures.

Definition 1: A *probabilistic SRLG* (PSRLG) is a set of links with positive failure probability in the event of an SRLG failure. Namely, link (i, j) belongs to SRLG r if $p_{ij}^r > 0$, and SRLG $r = \{(i, j) \in E : p_{ij}^r > 0\}$.

We say that links (i, j) and (k, l) are *correlated* if there exists an SRLG r such that $p_{ij}^r, p_{kl}^r > 0$. Clearly, this model is a generalization of the traditional SRLG model, and enables us to deal with correlated probabilistic link failures.

We consider a single source-destination pair. Let $s \in V$ and $t \in V$ be source and destination nodes respectively. Our objective is to find a pair of primary and backup paths from s to t with minimum joint failure probability. This problem will be considered using two different models: (i) independent link failure and (ii) SRLG-based correlated link failure. In each case, we seek to find a pair of paths with minimum joint failure probability.

Let $x_{ij} = 1$ if the primary path traverses link (i, j), and 0 otherwise. Similarly, define another binary variable y_{ij} for the backup path. We will just drop the index of a variable to denote its vector version, e.g., x represents the vector $[x_{ij}, (i, j) \in E]$. The set of n-dimensional binary vectors will be defined as B_n, i.e., $B_n = \{0, 1\}^n$. Our problems will be formulated as Integer Programs (IPs).

III. INDEPENDENT LINK FAILURE MODEL

In order to gain insights into the problem, we start by first considering the independent link failure model. Moreover, we begin by considering the simple case of finding a single path with minimum failure probability. We then use the insights gained in order to formulate the problem of finding a pair of paths with minimum joint failure probability. In Section IV, we will further generalize our formulations to deal with correlated (SRLG) failures.

A. Single Path Problem

First, consider the problem of finding a single path having minimum failure probability. Let p_{ij} be the probability that link (i, j) fails, then link (i, j) will survive with probability $1 - p_{ij}$. Consequently, the survivability probability of path x is given in a product form by $\prod_{(i,j)} (1 - p_{ij} x_{ij})$. The problem

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is formulated as:

$$\textbf{(P1.1)}: \quad \min_{x \in B_{|E|}} 1 - \prod_{(i,j)} (1 - p_{ij} x_{ij})$$

subject to

$$\sum_{j:(i,j) \in E} x_{ij} - \sum_{j:(j,i) \in E} x_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t, \forall i \in V, \\ 0, & \text{o.w.} \end{cases} \quad (1)$$

where $|E|$ is the cardinality of E. The constraints in (1) require that the set of links selected by x forms a path from node s to node t. For simplicity, we will denote this constraint by $CC(x)$, representing the connectivity constraint on x for a path from s to t. As all the variables in this work are binary, we will sometimes omit the binary constraint for convenience. The problem (P1.1) is an Integer NonLinear Program (INLP), which generally is very difficult to solve. However, using the following theorem, we are able to reformulate (P1.1) as an Integer Linear Program (ILP).

Theorem 1: Assume $p_{ij} \in [0,1), \forall(i,j)$, then the problem (P1.1) is equivalent to the following ILP:

$$\textbf{(P1.1L)}: \quad \min_{x \in B_{|E|}} - \sum_{(i,j)} x_{ij} \log(1 - p_{ij})$$

$$\text{subject to} \quad CC(x),$$

where $CC(x)$ is the connectivity constraint as given in equation (1).

Proof: First, the objective in (P1.1) can be equivalently written as $\max \prod_{(i,j)} (1 - p_{ij} x_{ij})$. Taking logarithm over the entire function gives $\max \sum_{(i,j)} \log(1 - p_{ij} x_{ij})$ without affecting the optimal solution. The proof is completed by applying the identity $\log(1 - p_{ij} x_{ij}) = x_{ij} \log(1 - p_{ij})$ for binary variable x_{ij}, and noting that $\max f(x)$ is the same as $\min -f(x)$. ∎

Observation 1: Theorem 1 shows that the path with minimum failure probability is the *shortest path* under link weights $-\log(1 - p_{ij}), \forall(i,j)$. It immediately follows that if the failure probability is sufficiently small (i.e., $p_{ij} \ll 1, \forall(i,j)$), then the probability-wise shortest path has the minimum failure probability because $-\log(1 - p_{ij}) \approx p_{ij}$ for small p_{ij}. Further, with uniform failure probability, i.e., $p_{ij} = q, \forall(i,j) \in E$, the shortest-hop path has the lowest failure probability. This result will be used in developing heuristic algorithms.

B. Path Pair Problem with Disjointness Constraint

Let $F_1(p,x)$ be the objective function of problem (P1.1), i.e., $F_1(p,x)$ is the failure probability of path x for given link failure probability vector p. Suppose that the two paths x and y are link-disjoint, then their failures are mutually independent because the link failures are independent and further the paths do not share any link. Then, the joint failure probability of two disjoint paths x and y is given by $F_1(p,x) \cdot F_1(p,y)$. The path pair problem with disjointness constraint (DC) is thus

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Algorithm 1 Heuristic: IND w/ DC
- 1: Set link weight $w_{ij} = p_{ij}, \forall(i,j) \in E$
- 2: Find shortest path x
- 3: Remove all the links used by x
- 4: Find shortest path y

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formulated as:

$$\textbf{(P1.2)}: \quad \min_{x,y} F_1(p,x) F_1(p,y)$$

$$\text{subject to} \quad \sum_{j:(i,j) \in E} x_{ij} - \sum_{j:(j,i) \in E} x_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t, \forall i \in V \\ 0, & \text{o.w.} \end{cases}$$

$$\sum_{j:(i,j) \in E} y_{ij} - \sum_{j:(j,i) \in E} y_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t, \forall i \in V \\ 0, & \text{o.w.} \end{cases}$$

$$x_{ij} + y_{ij} \leq 1, \forall(i,j) \in E.$$

Again, the first and second constraints are the connectivity constraints requiring that x and y are path from s to t. The last constraints require that x and y cannot share any link, i.e., they are link-disjoint. Hence, x and y satisfying all the constraints in (P1.2) will form a pair of disjoint paths from s to t. For brevity, throughout this paper, we will denote by $CC(x)$ the connectivity constraint on any binary link selection vector x, and $DC(x,y)$ the disjointness constraint (DC) on paths x and y.

Using the above formulation, we can make the following important observation.

Observation 2: Consider two paths with total length c, i.e., $\sum_{ij}(x_{ij} + y_{ij}) = c$, and uniform failure probability, i.e., $p_{ij} = 1 - q, \forall(i,j)$. Then, the objective function can be rewritten as $1 - q^X - q^{c-X} + q^c$ where $X = \sum_{ij} x_{ij}$. This new objective function is strictly concave with respect to X, and symmetric around $X = c/2$. Consequently, it is minimized when X takes minimum or maximum possible value. This implies that it is always better to choose an unbalanced pair of paths (given that the total length is fixed). Of course, this is a very limited case, but it gives us the intuition that good-bad path pair will be better than medium-medium pair. As an extreme example, we would prefer the pair having path failure probabilities $(0,1)$ to the one having $(0.01, 0.01)$; because in the former case, the probability of joint failure is 0.

According to this observation, it is important to include the best path (i.e., path having minimum failure probability) in the pair. This leads to a heuristic algorithm which selects the best path first and then selects the next best disjoint path (See Algorithm 1). Moreover, using Observation 1, the best path is obtained by the probability-wise shortest path. Note that Algorithm 1 only needs to run a shortest path algorithm twice whose complexity is $O(|V|^2)$.

1) ILP Approximation of (P1.2) and Its Lagrangian Relaxation: The problem (P1.2) is an Integer NonLinear Program (INLP) which is generally difficult to solve. So, we will instead approximate the problem by an ILP. First, the objective

function in (P1.2) can be expanded as

$$F_1(p,x)F_1(p,y) = 1 - \prod_{(i,j)}(1-p_{ij}x_{ij}) - \prod_{(i,j)}(1-p_{ij}y_{ij})$$
$$+ \prod_{(i,j)}(1-p_{ij}x_{ij})\prod_{(i,j)}(1-p_{ij}y_{ij}).$$
(2)

Further expanding the product terms and canceling out common terms yields

$$F_1(p,x)F_1(p,y) = \sum_{(i,j)}\sum_{(k,l)} p_{ij}p_{kl}x_{ij}y_{kl} + HOT, \quad (3)$$

where HOT stands for high order terms; namely, terms involving the product of 3 or more failure probabilities. In the low failure probability regime, i.e., $p_{ij} \ll 1, \forall(i,j)$, the HOTs can be neglected and the ILP can be formulated as follows:

$$\textbf{(P1.2L)}: \quad \min_{x,y,z} \quad \sum_{(i,j)}\sum_{(k,l)} p_{ij}p_{kl}z_{ij}^{kl}$$
$$\text{subject to} \quad CC(x), CC(y), DC(x,y)$$
$$(C1) \quad z_{ij}^{kl} \geq x_{ij} + y_{kl} - 1, \forall(i,j),(k,l) \in E,$$

where we have introduced the binary variables $z_{ij}^{kl}, \forall(i,j),(k,l)$ such that $z_{ij}^{kl} = 1$ only if both of x_{ij} and y_{kl} are 1. That is, link (i,j) is used by the primary and (k,l) by the backup path. This enables us to use z_{ij}^{kl} instead of $x_{ij}y_{kl}$ in the objective, hence resulting in a linear formulation. Consequently, the objective function represents the joint failure probability based on the pair-wise (one from x and one from y) joint link failure.

While generally ILPs are difficult to solve, in this case we also observe that the constraints $CC(x)$ and $CC(y)$ are totally unimodular[1] (TU) [22], hence the linear program (LP) relaxation has an integral optimal solution [22]. Further, we can use Lagrangian relaxation on the constraints $DC(x,y)$ and (C_1) to further simplify the problem. In particular, define the Lagrangian function as $L(x,y,z,\mu,\nu) = \sum_{(i,j)}\left(\mu_{ij} + \sum_{(k,l)}\nu_{ij}^{kl}\right)x_{ij} + \sum_{(i,j)}\left(\mu_{ij} + \sum_{(k,l)}\nu_{kl}^{ij}\right)y_{ij} + \sum_{(i,j),(k,l)}\left(p_{ij}p_{kl} - \nu_{ij}^{kl}\right)z_{ij}^{kl}$ where μ and ν are Lagrangian multiplier vectors associated with $DC(x,y)$ and $(C1)$, respectively. The (Lagrangian) relaxed problem is given by

$$\textbf{(P1.2LR)}: \quad \min_{x,y,z} \quad L(x,y,z,\mu,\nu)$$
$$\text{subject to} \quad CC(x), CC(y).$$

The above problem is TU, and so it can be solved by LP relaxation which is polynomial time solvable. Moreover, for given μ and ν, the problem (P1.2LR) is completely separable with respect to x, y and z. Namely, the optimal x and y are shortest paths respectively, and optimal z_{ij}^{kl} is obtained as: $z_{ij}^{kl} = 1$ if $\nu_{ij}^{kl} > p_{ij}p_{kl}$, and 0 otherwise. Now, the above optimization can be solved using a simple primal-dual method as described

[1]A matrix A is said to be *totally unimodular* if the determinant of each square submatrix of A is 0, 1, or -1 (The network flow conservation matrices are TU). If the constraint matrix of an ILP is TU, then its linear program relaxation has an integral optimal solution, that is equivalent to the optimal solution of the ILP. Hence, the optimal solution of such an ILP can be obtained by solving its linear program relaxation, which is polynomial time solvable.

Algorithm 2 Lagrangian Relaxation: IND w/ DC

1: Initialization: $m = 0$, $\mu_{ij}(0) = p_{ij}$ and $\nu_{ij}^{kl}(0) = p_{ij}p_{kl}, \forall(i,j),(k,l) \in E$, and $p_{\text{best}} = \infty$
2: While $m < M$
  2.1 Set $w_{ij} = \mu_{ij}(m) + \sum_{(k,l)}\nu_{ij}^{kl}(m), \forall(i,j) \in E$;
    Find shortest path $x(m)$
  2.2 Set $w_{ij} = \mu_{ij}(m) + \sum_{(k,l)}\nu_{kl}^{ij}(m), \forall(i,j) \in E$;
    Find shortest path $y(m)$
  2.3 $z_{ij}^{kl}(m) = \begin{cases} 1, & \text{if } \nu_{ij}^{kl}(m) > p_{ij}p_{kl} \\ 0, & \text{otherwise} \end{cases}, \forall(i,j),(k,l)$
  2.4 $\mu_{ij}(m+1) = [\mu_{ij}(m) + \gamma_m(x_{ji}(m) + y_{ij}(m) - 1)]^+$
  2.5 $\nu_{ij}^{kl}(m+1) = [\nu_{ij}^{kl}(m) + \gamma_m(x_{ji}(m) + y_{kl}(m) - z_{ij}^{kl}(m) - 1)]^+, \forall(i,j),(k,l) \in E$
  2.6 Set $(x^{\text{best}}, y^{\text{best}}) = (x(m), y(m))$ if $p(m) < p_{\text{best}}$ where $p(m)$ is the joint failure prob. of $(x(m), y(m))$
  2.7 $m = m + 1$

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in Algorithm 2, where γ_m is a positive diminishing step size, M is the maximum number of iterations, and w_{ij} is the weight of link (i,j). Note that steps 2.1-2.3 solve the relaxed problem (P1.2LR), and steps 2.4-2.5 are the subgradient-based update of Lagrangian multipliers. The algorithm keeps the best path pair all over the iterations (step 2.6), and takes it as the final solution. Such a Lagrangian relaxation method for IP does not guarantee an optimal solution due to the duality gap, but it has been very successful in solving many IPs [23].

C. Path Pair Problem without Disjointness Constraint

The disjointness constraint is a necessary condition for surviving a single link failure in traditional deterministic failure model. However, in a probabilistic model, a link may be shared if it is known that the link is very reliable. If link (i,j) is shared by x and y, then (i,j)'s failure leads to the simultaneous failure of both x and y. Hence, the probability that both x and y fail can be written as

$$F_S(p,x,y) + (1 - F_S(p,x,y))F_{NS}(p,x,y), \quad (4)$$

where $F_S(p,x,y)$ is the probability that both x and y fail due to a *shared link failure*, and $F_{NS}(p,x,y)$ is the probability that both x and y fail due to the *failure of non-shared links*. For path pair (x,y), let E_{xy} denote the set of links shared by x and y, i.e., $E_{xy} = \{(i,j) \in E : x_{ij} = 1, y_{ij} = 1\}$. Then, the probability $F_S(p,x,y)$ can be written as

$$\begin{aligned} F_S(p,x,y) &= 1 - \prod_{(i,j)\in E_{xy}}(1-p_{ij}) \\ &= 1 - \prod_{(i,j)\in E}(1-p_{ij}x_{ij}y_{ij}). \end{aligned} \quad (5)$$

For a binary vector v, define its complement as $\bar{v} = \vec{1} - v$ where $\vec{1}$ is a vector of 1's with appropriate dimension. Then, the vector $\bar{y}$ only includes the links which are not selected by y. Hence, the probability that x fails due to the failure of non-shared links is equivalent to the probability that both x and $\bar{y}$ fail due to the failure of the links shared by x and $\bar{y}$. This probability can be subsequently expressed as $F_S(p,x,\bar{y})$ following to the definition $F_S(p,x,y)$. Similarly, $F_S(p,\bar{x},y)$ denotes the probability that y fails due to the failure of non-shared links. The probability $F_{NS}(p,x,y)$ is

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Algorithm 3 Heuristic: IND w/o DC

1: Set $w_{ij} = p_{ij}, \forall (i,j) \in E$; Find shortest path x

2: $w_{ij} = \begin{cases} p_{ij} \sum\limits_{(k,l)} p_{kl} x_{kl}, & \text{if } x_{ij} = 0 \\ p_{ij}, & \text{if } x_{ij} = 1 \end{cases}, \forall (i,j) \in E;$

Find shortest path y

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then given by $F_{NS}(p,x,y) = F_S(p,x,\bar{y})F_S(p,\bar{x},y)$, leading to the following formulation:

$$\textbf{(P1.3)}: \quad \min_{x,y} \quad F_S(p,x,y) + (1 - F_S(p,x,y))F_{NS}(p,x,y)$$
$$\text{subject to} \quad CC(x), CC(y).$$

The problem (P1.3) has an equivalent (if disjoint paths are optimal in (P1.3)) or better optimal solution compared to the problem (P1.2).

Under the low failure probability regime, we can approximate the problem (P1.3) by an ILP as follows:

$$\textbf{(P1.3L)}: \quad \min_{x,y,z,\tilde{z}} \quad \sum_{(i,j)} p_{ij}\tilde{z}_{ij} + \sum_{(i,j)} \sum_{(k,l)} p_{ij}p_{kl}z_{ij}^{kl}$$
$$\text{subject to} \quad CC(x), CC(y)$$
$$(C1) \quad \tilde{z}_{ij} \geq x_{ij} + y_{ij} - 1, \forall (i,j) \in E$$
$$(C2) \quad z_{ij}^{kl} \geq x_{ij} - y_{ij} + y_{kl} - x_{kl} - 1, \forall (i,j),(k,l).$$

In constraint $(C1)$, $\tilde{z}_{ij} = 1$ only if $x_{ij} = y_{ij} = 1$ which means that link (i,j) is shared. Hence, the first term in the objective function is the joint failure probability due to the failure of shared links. In constraint $(C2)$, $z_{ij}^{kl} = 1$ only if $x_{ij} = y_{kl} = 1$ and $y_{ij} = x_{kl} = 0$ which means links (i,j) and (k,l) are respectively used by x and y, but neither of them are shared. Hence, the second term is the joint failure probability due to the failure of non-shared links.

The formulation (P1.3L) is a standard ILP that can be solved using an ILP solver such as CPLEX. However, it can take unacceptably long to run because it is NP-complete in general. Inspired by the approximation (P1.3L), we propose a simple heuristic in Algorithm 3. Similar to Algorithm 1, Step 1 finds a shortest path for x using link weights $w_{ij} = p_{ij}, \forall (i,j)$ and this gives a primary path x with minimum failure probability. For the backup path y, the weight of each link (i,j) is set to the joint path failure probability due to the failure of link (i,j) and the links in x. Hence, two different cases have to be considered. First, if link (i,j) has not been selected by the primary path x, then its weight is set to the product of (i,j)'s failure probability p_{ij} and the (approximated) failure probability $\sum_{(k,l)} p_{kl} x_{kl}$ of path x. If (i,j) was selected by x, then (i,j)'s failure leads to joint path failure (provided that y also selects (i,j)), and so its weight is set to p_{ij}. The shortest path under these link weights will minimize the joint path failure probability and will be used as the backup path y. Note that if link (i,j) is to be shared, its weight is set to a first-order value, i.e., p_{ij} which is obviously larger than the second-order weight in the non-shared case. Hence, the links with relatively low failure probability will be more likely to be shared.

D. Extension to Correlated Failures

As discussed in the introduction, many failure scenarios involve multiple links. Hence, link failure events may be corre-lated. In order to account for such correlation, the path failure probability expressions must include conditional probabilities for joint link failures. These conditional probability expressions involve an exponential number of terms accounting for the joint failure probability of multiple links. Hence, formulating the above problems under correlated failures seems to be intractable.

In order to better account for correlated failures, we propose a new model using *probabilistic SRLGs*. In our model, once an SRLG failure event occurs, its associated links fail with some probabilities. So, the link failures are correlated only if the links belong to the same SRLG (while in the link-wise model, the correlation is considered between every two links). Moreover, under the condition that an SRLG event occurs, its associated link failures are mutually independent, and thus the formulations developed in the independent model can be used. This enables a simple formulation for the path protection problems with correlated failures. More importantly, it can be used to model most correlated failure scenarios; as events leading to failures can be modeled as a probabilistic SRLG (PSRLG).

IV. PSRLG-BASED CORRELATED FAILURE MODEL

We consider a single SRLG model where only one SRLG failure event can take place at a time. Let π_r be the probability that the failed SRLG is $r \in R$, then we will have $\sum_{r \in R} \pi_r = 1$. We refer to this model as the *mutually exclusive PSRLGs*. Note that the traditional deterministic SRLG model also assumes a single SRLG failure, and so our model of mutually exclusive PSRLGs is a probability-wise generalization of the traditional model.

A. Single Path Problem

Again, we start by considering a single path problem. Given that SRLG event r happens, each link (i,j) will fail with probability p_{ij}^r as if they are independent. Hence, the failure probability of path x is given by $1 - \prod_{(i,j)}(1 - p_{ij}^r x_{ij})$, and according to the definition of F_1 in Section III, this probability can be denoted by $F_1(p^r, x)$ where $p^r = [p_{ij}^r, \forall (i,j) \in E]$. The single path problem can be simply written as

$$\textbf{(P2.1)}: \quad \min_{x \in B_{|E|}} \quad \sum_{r \in R} \pi_r F_1(p^r, x)$$
$$\text{subject to} \quad CC(x).$$

Note that the path failure probability is averaged over all SRLGs because they are mutually exclusive. As shown in Section III-A, the single path problem in independent failures is an easy shortest path problem. However, the same problem in the correlated failures case (i.e., (P2.1)) has a special structure which is difficult to solve in general.

Theorem 2: The problem (P2.1) is a convex maximization.

Proof: Using the identity $\min f(x) \equiv \max -f(x)$, the objective in (P2.1) can be rewritten as

$$\max_x \sum_r \pi_r \prod_{(i,j)}(1 - p_{ij}^r x_{ij}). \quad (6)$$

Note that for binary variable x_{ij}, the following identity holds: $1 - p_{ij}x_{ij} = e^{x_{ij}\log(1-p_{ij})}$. Applying this identity to (6) yields

$$\max_x \sum_r \pi_r e^{\sum_{(i,j)} x_{ij}\log(1-p_{ij})}, \qquad (7)$$

where each term in the summation is convex with respect to x. Because the nonnegative-weighted sum of convex functions is also convex, the problem (7) is a convex maximization problem. ∎

Generally, the convex maximization (or concave minimization) problem is *NP-hard* [24], and thus Theorem 2 implies that the problem (P2.1) may not be easy to solve.

Due to the difficulty, its approximation is again considered. Under the low failure probability assumption (i.e., $p_{ij}^r \ll 1, \forall r, \forall (i,j)$), the objective function can be written as

$$\sum_{(i,j)} \left(\sum_{r \in R} \pi_r p_{ij}^r \right) x_{ij}. \qquad (8)$$

We first begin by considering the simple case of uniform SRLG failure probability. Namely, let $\pi_r = 1/|R|, \forall r \in R$ and $p_{ij}^r = q, \forall r \in R(i,j), \forall (i,j) \in E$ where $R(i,j) = \{r \in R : p_{ij}^r > 0\}$.

Observation 3: Under the uniform failure probability, minimizing the objective function (8) is equivalent to minimizing $\sum_{(i,j)} |R(i,j)|x_{ij}$ where each term in the summation is the number of SRLGs to which link (i,j) belongs. This shows that the path touching the minimum number of SRLGs has the lowest failure probability.

The traditional SRLG model falls into a special case of the uniform failure probability with $q = 1$, and hence, Observation 3 implies that in the traditional model, the number of SRLGs touched by a link is an important link-weight metric when finding a reliable path. In fact, some previous works have used this metric [11], and our result provides a theoretical basis for those works in traditional SRLG model.

B. Path Pair Problem with Disjointness Constraint

The path pair problem can also be formulated in a simple form as in the case of single path. Once an SRLG event r occurs, paths x and y will fail with probabilities $F_1(p^r, x)$ and $F_1(p^r, y)$, respectively. In this case, their joint failure probability is given by the product $F_1(p^r, x)F_1(p^r, y)$ because the link failures are independent, under the condition that SRLG event r has occurred. The problem can be formulated as follows:

$$\textbf{(P2.2)}: \quad \min_{x,y} \quad \sum_{r \in R} \pi_r F_1(p^r, x)F_1(p^r, y)$$
$$\text{subject to} \quad CC(x), CC(y), DC(x,y).$$

Our probabilistic SRLG model has enabled us to express the joint failure probability in a product form leading to a simple formulation. Namely, the objective function in (P2.2) is the combination of the objective functions in (P1.2) and (P2.1). That is, for given SRLG r, the joint path failure probability is equivalent to the joint path failure probability with link failure probability vector p^r in the independent model, and those joint failure probabilities are averaged over all SRLGs,

as done in (P2.1). This is in sharp contrast with the link-wise correlated failure model where the path failure probability would include terms of the conditional probabilities involving all the combinations of link failures.

It is obvious that the path pair problem is harder than the single path problem, and thus, we can infer from Theorem 2 that the problem (P2.2) will be difficult. Further, we can show its NP-completeness as below.

Theorem 3: The problem (P2.2) is NP-complete.

Proof: First, note that the objective value of (P2.2) is nonnegative, and so if any path pair results in zero objective value, then it is optimal. The probability $F_1(p^r, x)$ in (P2.2) can be written as

$$\begin{aligned} F_1(p^r, x) &= 1 - \prod_{(i,j)}(1 - p_{ij}^r x_{ij}) \\ &= 1 - \prod_{(i,j) \in r}(1 - p_{ij}^r x_{ij}), \end{aligned} \qquad (9)$$

because $p_{ij}^r = 0$ if link (i,j) does not belong to SRLG r, i.e., $(i,j) \notin r$. Consequently, the function $F_1(p^r, x)$ becomes zero for the path x which does not touch SRLG r, i.e., $x_{ij} = 0, \forall (i,j) \in r$. Hence if x and y do not share any SRLG, then the product $F_1(p^r, x)F_1(p^r, y)$ will be zero for every $r \in R$, thereby leading to zero objective value. This implies that any pair of SRLG-disjoint paths is an optimal solution to the problem (P2.2). Subsequently, the problem (P2.2) becomes an SRLG-disjoint paths problem if one exists. Therefore, the problem (P2.2) is NP-complete because it includes (as a special case) the SRLG-disjoint paths problem which is NP-complete [14]. ∎

Again, it is easy to show that when the link failure probabilities are low, the objective function of (P2.2) can be expressed as

$$\sum_{(i,j)} \sum_{(k,l)} \left(\sum_{r \in R} \pi_r p_{ij}^r p_{kl}^r \right) x_{ij} y_{kl}. \qquad (10)$$

Next, we observe that the problem is still NP-complete even after the approximation.

Observation 4: Under the uniform failure probability (i.e., $\pi_r = 1/|R|, \forall r \in R$ and $p_{ij}^r = q, \forall r \in R(i,j), \forall (i,j) \in E$), the objective function (10) becomes $\sum_{(i,j)} \sum_{(k,l)} |R(i,j) \bigcap R(k,l)|x_{ij}y_{kl}$ where each term in the summation represents the number of SRLGs shared by corresponding link pair in x and y. This obviously contains SRLG-disjoint paths problem as a special case (if there exist SRLG-disjoint paths), and so, it is NP-complete (following to the proof of Theorem 3). Subsequently, the approximated problem (10) is also NP-complete.

As this approximation is still difficult to solve, we propose a heuristic in Algorithm 4 using the approximations (8) and (10). For primary path x, we set the link weights and find the shortest path according to (8). This will give a path having minimum failure probability. Then, all the links selected by x are removed for disjointness. Finally, the obtained primary x is substituted into (10), and the backup path y is computed by minimizing (10) for fixed x. One can also develop a Lagrangian relaxation based algorithm by linearizing the problem. However, this development is nearly identical to that in Section III-B1 and omitted for brevity.

Algorithm 4 Heuristic: MES w/ DC

1: Set $w_{ij} = \sum_{r \in R} \pi_r p_{ij}^r, \forall (i,j) \in E$
2: Find shortest path x
3: Remove all the links used by x
4: Set $w_{ij} = \sum_{(k,l)} x_{kl} \sum_{r \in R} \pi_r p_{ij}^r p_{kl}^r, \forall (i,j) \in E$
5: Find shortest path y

The path pair problem without disjointness constraint was also studied, but due to the limited space, only the results will be shown in the next section.

V. PERFORMANCE EVALUATION

In this section, we evaluate and compare the performance of the algorithms developed in this paper. In particular, we consider the following four algorithms:

- The brute-force solution to the ILP formulations using the CPLEX solver (denoted by CPLEX).
- The Lagrangian relaxation for the ILP (Algorithm 2; denoted by LR).
- The heuristic algorithms that select the first path with minimum failure probability and the second path with adjusted link weights to reduce the joint path failure probability (i.e., Algorithms 1, 3 and 4; denoted by Heu).
- The shortest disjoint paths algorithm that finds a pair of disjoint paths with minimum total weight, where the weight of a link is its failure probability (i.e., for each link (i,j), $w_{ij} = p_{ij}$ in the independent model or $w_{ij} = \sum_r \pi_r p_{ij}^r$ in the PSRLG model). This algorithm is a straightforward approach simply selecting a shortest path pair, and as mentioned in Observation 2, such a pair does not necessarily have minimum joint failure probability. Note that this joint shortest path approach is in contrast with our heuristic which is one-by-one shortest path approach. (denoted by SDP)

The protection quality (i.e., joint path failure probability) and run-time of the above algorithms will be compared.

First, we compare the ILP (P1.1L) and probability-wise shortest path (PSP) algorithm that finds a shortest path under link weights $w_{ij} = p_{ij}, \forall (i,j)$. As discussed in Theorem 1 and Observation 1, the ILP finds a path with minimum failure probability while the PSP algorithm approximates the optimal path in the low failure probability regime. Because the PSP algorithm is used in our heuristics (Algorithms 1, 3 and 4) to find a path with minimum failure probability, this comparison will demonstrate the suitability of the PSP algorithm in our heuristics. We generated 100 random graphs, each of which has 10 nodes and maximum node degree of 5, and is 3-connected[2] from s to t. In each graph, the failure probability of each link (i,j) is assigned as follows:

$$p_{ij} = \alpha(\beta + (1 - \beta)u),$$

[2]A graph is said to be k-connected if it remains connected after up to $k - 1$ link (or node) failures. Equivalently, if every source-destination pair has at least k disjoint paths, then it is k-connected. In our simulation, once a graph is generated, we examine the 3-connectedness of the graph, and if it is not 3-connected, it is discarded.
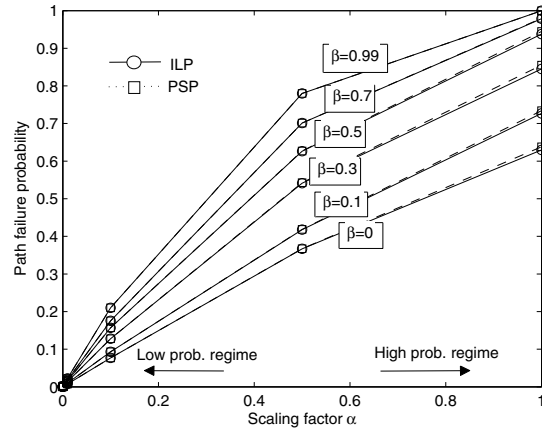

Fig. 1. Comparison of ILP and probability-wise shortest path (PSP)

where α and β are constants in $[0, 1]$, and u is a random number uniformly distributed on the interval $(0, 1)$. Note that as β increases, the network approaches to the uniform failure probability regime (i.e., $p_{ij} = q, \forall (i,j)$). For example, if $\beta = 1$, it will be $p_{ij} = \alpha, \forall (i,j)$, which implies uniform failure probability regime. In contrast, if $\beta = 0$, p_{ij} will be a random number from $(0, \alpha)$. On the other hand, small α corresponds to the low failure probability regime and large α to the high failure probability regime.

Fig. 1 plots the path failure probability for each combination of (α, β) where each point is the average of the results of 100 random graphs. As expected in Observation 1, the PSP algorithm finds an optimal path in the uniform or low failure probability regime (large β or small α, respectively). Furthermore, even in high probability regime (large α), the PSP approximates the ILP very well. When the network is nearly in the uniform probability regime (large β), shortest-hop path would be optimal, and the PSP obviously finds this shortest path. With small β, the network is in a mixed regime having high and low failure probabilities. In this case, both the ILP and PSP would select only the links with low failure probability whenever feasible. Then, it is highly likely that the same path is optimal after the links with high failure probability are removed. This is equivalent to being in the low probability regime, and therefore, the PSP performs comparably to the ILP. Overall, the PSP algorithm finds an optimal path in most cases, as desired.

Next, we consider the problem of finding the path pair with minimum joint failure probability. The proposed heuristic and LR-based algorithms are compared with CPLEX and the SDP algorithm using various topologies. The CPLEX solves the ILP version of every path pair problem. For LR-based iterative algorithms, the following parameters are used: maximum iteration number $M = 2 \times 10^4$ and step size $\gamma_m = 10^{-9}/\sqrt{m}$. The comparison is performed by changing the number of nodes, and 100 random graphs are generated for each case. As in the above, each graph is 3-connected (from s to t) with maximum node degree of 5. For the independent model, the failure probability of each link is set to a random number uniformly distributed on the interval

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$(0, 10^{-3})$, hence $p_{ij} \in (0, 10^{-3}), \forall (i, j) \in E$. For the PSRLG model, 20 SRLGs are generated for every graph and their failure event probabilities π_r's are set to uniformly distributed random numbers such that $\sum_r \pi_r = 1$. Each SRLG includes a randomly selected set of links, and once a link, say (i, j), is included in SRLG r, its failure probability p_{ij}^r is set to a uniformly distributed random number in $(0, 10^{-3})$.

Fig. 2 plots the joint path failure probability achieved by each algorithm in the independent model. With disjointness constraint, our heuristic (Heu) always finds a better path pair than CPLEX (See Fig. 2(a)). This is partly because the heuristic tries to find an optimal solution to the original INLP while CPLEX solves its ILP relaxation whose solution is not necessarily an optimal solution to the INLP. Observe further that the LR-based algorithm also performs better than CPLEX. This is more surprising because the CPLEX and LR solve the same ILP. This is because CPLEX solves an ILP using general approaches such as branch-and-bound and linear program relaxation. Hence, it does not take advantage of the special structure (i.e., total unimodularity of the connectivity constraint) in our problem. In contrast, our LR-based algorithm takes full advantage of this property; namely, it obtains an easy problem by relaxing the complicating constraints and applies the dual subgradient method. Hence, our result confirms that an algorithm developed by taking into account the structure of a problem can achieve better performance than a general solver.

The SDP algorithm finds a better pair than the heuristic, but the difference is almost negligible. Hence, the proposed heuristic and LR-based algorithms can effectively find a reliable pair of disjoint paths. Without disjointness constraint (Fig. 2(b)), the heuristic still finds a more reliable path pair than CPLEX, but the performance of LR algorithm is substantially degraded as the number of nodes increases.

The run-time of each algorithm is shown in Fig. 3. As the number of nodes increases, the run-time of CPLEX increases exponentially. This shows that CPLEX takes a brute-force approach having exponential run-time and hence, may be prohibitively complex. The LR algorithm also takes a long time, but its run-time increases much more slowly than CPLEX. On the other hand, the heuristic and SDP find a path pair in minimal time, almost independent of the problem size. Therefore, both the heuristic and the SDP algorithms find the most reliable pair of paths with short run-times.

The joint path failure probabilities in the PSRLG model are shown in Fig. 4. As in the independent model, our heuristics find the most reliable pair of paths in all cases. Fig. 4 also shows that our one-by-one heuristic provides better protection than the SDP. This is due to the fact that our heuristic adjusts the failure probability before selecting the second path in order to reduce the joint failure probability while the SDP algorithm fails to take correlation into account.

As mentioned in Section III-C, relaxing the disjointness constraint (DC) should improve the protection quality (if non-disjoint path pair is optimal). Fig. 5 shows that the joint failure probability is decreased by relaxing the DC. Observe
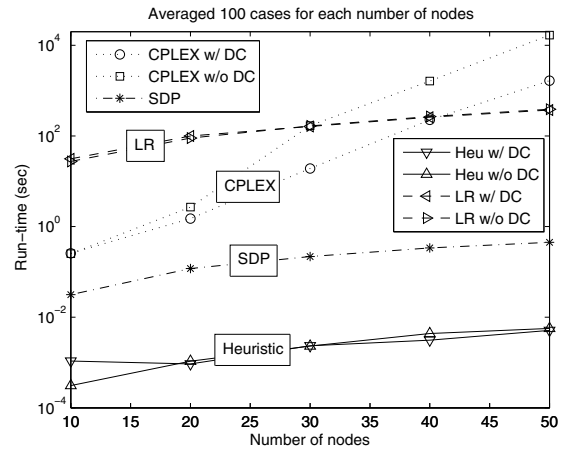

Fig. 3.   Run-time (on DELL Workstation T7400) in independent model
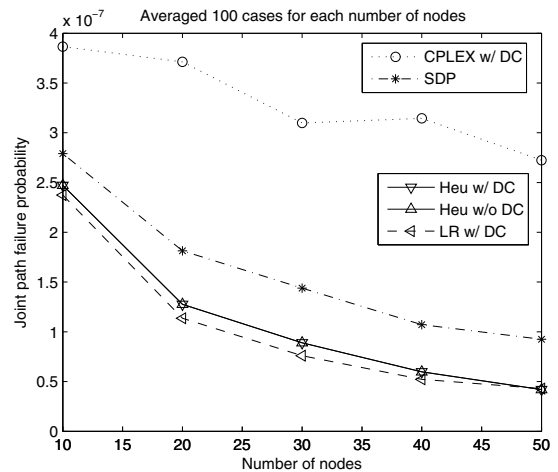

Fig. 4.   Joint path failure probability in PSRLG model

further that our one-by-one heuristic finds a more reliable path pair than SDP, verifying Observation 2 that it is important to include the best path in the pair.

VI. CONCLUSION

In this paper, we studied path protection problems in a network with multiple, possibly correlated, failures. In such a network, protection cannot be guaranteed by simply providing disjoint paths, and thus we sought to find diverse routes that maximize reliability, i.e., have the minimum joint failure probability. To that end, we first developed a probabilistic SRLG (PSRLG) framework by generalizing the traditional notion of SRLG. Under this model, given an SRLG failure, links belonging to that SRLG fail independently; significantly simplifying the computation of the joint failure probability between two paths. This enables a simple formulation to the path protection problem under correlated failures which would be otherwise intractable. Using this model, we formulated the path protection problem of finding a pair of paths with minimum joint failure probability as an Integer Non-Linear Program (INLP).

Further, using linear approximations, we transformed the INLP to an ILP and developed algorithms for finding a pair of paths with minimum joint failure probability. Finally, we

(a) With disjointness constraint (DC)

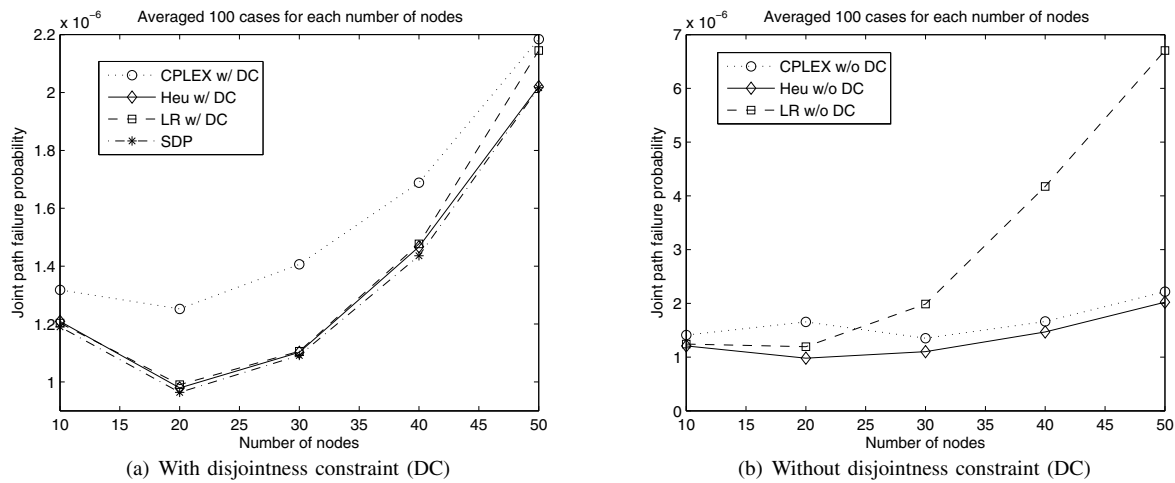(b) Without disjointness constraint (DC)

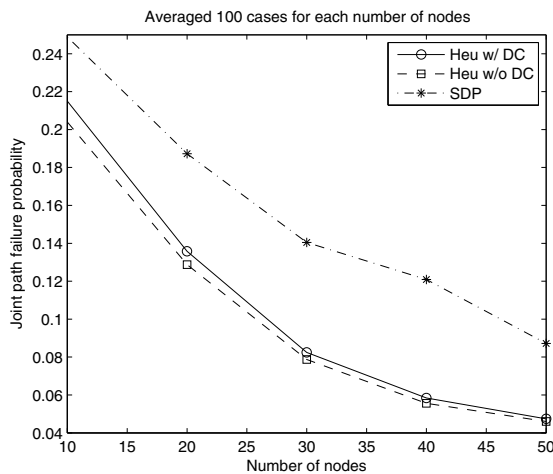Fig. 2.   Joint path failure probability in independent model

Fig. 5.   Improvement of protection quality by relaxing disjointness constraint: PSRLG model with 20 SRLGs and $p_{ij}^r \in (0,1), \forall (i,j) \in r, \forall r \in R$

showed through simulations that our heuristic algorithms often find a better path pair and require less run-time than a brute-force solution using an ILP solver such as CPLEX.

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