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Geometric Properties of Gradient Projection
Anti-windup Compensated Systems

Justin Teo and Jonathan P. How

Abstract—The gradient projection anti-windup (GPAW) scheme was recently proposed as an anti-windup method for nonlinear multi-input-multi-output systems/controllers, which was recognized as a largely open problem in a recent survey paper. Here, we show that for controllers whose output equation depends only on its state, the GPAW compensated controller achieves exact state-output consistency when appropriately initialized. In a related paper analyzing the GPAW scheme on a simple constrained system, this property was crucial in proving that the GPAW scheme can only maintain/enlarge the exact region of attraction of the uncompensated system. When the nominal controller does not have the required structure, an arbitrarily close approximating controller can be constructed. Further geometric properties of GPAW compensated systems are then presented, which illuminates the role of the GPAW tuning parameter.

I. INTRODUCTION


In a related paper [7], we analyzed the GPAW scheme when applied to a constrained first order linear time invariant (LTI) system driven by a first order LTI controller, where the objective is to regulate the system state about the origin. The main result of [7] shows that the GPAW scheme can only maintain/enlarge the exact region of attraction of the uncompensated system. This shows the GPAW scheme to be a valid anti-windup method for this simple system. A crucial part of these results relies on the fact that the GPAW compensated controller maintains exact state-output consistency, i.e. sat(u) ≡ u, when the controller state is appropriately initialized. While this fact is easily seen for the simple system considered in [7], it is not immediately clear for a more general GPAW compensated controller. We present this result for general GPAW compensated controllers as described below.

We first describe the nominal tracking system and the derived GPAW compensated controller in Sections II and III respectively. It is shown in Theorem 1 that when appropriately initialized, the GPAW compensated controller achieves exact state-output consistency, i.e. sat(u) ≡ u for all times, provided that the output equation of the nominal controller depends only on its state, and specifically, not on measurements or exogenous signals. When the nominal controller does not possess this property, we show in Section IV how an arbitrarily close approximating controller can be constructed. In Section V, we present further geometric properties of GPAW compensated systems, the main result of which, Theorem 2, illuminates the role of the GPAW tuning parameter.

We will adopt the following conventions in the sequel. For inequalities involving vectors, the inequality is to be interpreted element-wise. Let I and J be two sets. The cardinality of I will be denoted by |I|, and I \ J is the relative complement of J in I. For any set X ∈ R^n, the boundary of X is denoted by ∂X. The dot product of two vectors x, y ∈ R^n is denoted by (x, y) ∈ R.

II. CONSTRAINED NOMINAL SYSTEM

Consider the input constrained nonlinear system

\[ \dot{x} = f(x, \text{sat}(u)), \quad x(0) = x_0, \]
\[ y = g(x, \text{sat}(u)), \]

where x, x_0 ∈ R^n are the state and initial state, u ∈ R^m is the plant input, y ∈ R^p is the measurement, and sat: R^m → R^m is the familiar saturation function. Let C^k([0, ∞), R^n) be the vector space of k times continuously differentiable functions [0, ∞) → R^n, and let R ∈ C^1([0, ∞), R^n) be a class of admissible reference signals evolving in R^n that is at least continuously differentiable. Assume that the control objective is to have x track a reference signal r ∈ R, so that the instantaneous tracking error at time t is e = x − r(t). The time-varying tracking error dynamics are then given by

\[ \dot{e} = f(e + r(t), \text{sat}(u)) - \dot{r}(t), \quad e(0) = x_0 - r(0), \]
\[ y = g(e + r(t), \text{sat}(u)). \]

For control designs that require smoother than C^1 reference signals, we can always restrict R appropriately, e.g. by requiring that R ∈ C^k([0, ∞), R^n), so that we can define the instantaneous controller reference \( \dot{r}(t) = \]

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\((r(t), \dot{r}(t), \ldots, r^{(k)}(t)) \in \mathbb{R}^{(k+1)n}\), where \(r^{(i)}(t)\) denotes the \(i\)-th time derivative of \(r\) at time \(t\). Let the nominal controller take the form

\[
\dot{x}_c = f_c(x_c, y, \hat{r}(t)), \quad x_c(0) = x_{c0},
\]

\[
u = g_c(x_c),
\]

where \(x_c, x_{c0} \in \mathbb{R}^q\) are the state and initial state, \(u \in \mathbb{R}^m\) is the output, and \(g_c\) depends only on the controller state \(x_c\).

**Remark 1:** Memoryless static feedback controllers of the form \(u = g_c(y, \hat{r}(t))\) can be approximated to have the form of (2) (see Section IV).

**Remark 2:** It is assumed in (2) that \(r(t)\) and all its time derivatives up to the \(k\)-th order are available to the controller. See [8, pp. 194–195] for a justification.

The closed loop system defined by (1) and (2) is called the nominal system, which can be written as

\[
\dot{e} = f(e + r(t), \text{sat}(g_c(x_c))) - \hat{r}(t),
\]

\[
\dot{x}_g = f_g(x_g, g(e + r(t), \text{sat}(g_c(x_c))), \hat{r}(t)),
\]

with initial state \((e(0), x_g(0)) = (x_0 - r(0), x_{c0})\).

### III. Gradient Projection Anti-windup Compensated System

Here, we apply the GPAW scheme [1] on (2) and show that the GPAW compensated controller achieves exact state-output consistency, i.e., \(\text{sat}(u) \equiv u\), for “almost all” times (stated more precisely as Theorem 1) when \(g_c\) depends only on the controller state. The GPAW compensated controller is derived from (2) and takes the form

\[
\dot{x}_g = f_g(x_g, y, \hat{r}(t)), \quad x_g(0) = x_{c0},
\]

\[
u = g_c(x_g),
\]

in which the only difference with (2) is the definition of an independent state \(x_g \in \mathbb{R}^q\) and the controller state update law \(f_g\). The following shows how \(f_g\) is constructed [1], leading to its definition in (9).

**Remark 3:** Even though the GPAW controller (4) may not appear to conform to the conventional anti-windup paradigm where the nominal controller is to remain unaltered, it can always be transformed in a way such that the nominal controller need not be modified. For example, if the anti-windup compensator’s output is to be combined additively with that of the nominal controller, and the output of the nominal controller can be measured, then by subtracting the nominal controller’s output from that of the GPAW controller’s, we obtain the desired anti-windup signal. Alternatively, one can build a model of the nominal controller, and with knowledge of the initial controller state, the same can be achieved. We avoid the difficulties associated with the individual robustness issues of each realization by focusing only on the effective composite controller (4).

In the following, we consider a fixed point in time, so that \((x_g, y, \hat{r}(t)) \in \mathbb{R}^{q+p+(k+1)n}\) are fixed. Let \(I_j = \{1, 2, \ldots, j\}\) where \(j\) is some positive integer. First, observe that the saturation function \(\text{sat} : \mathbb{R}^m \rightarrow \mathbb{R}^m\) in (1) is defined with \(m\) lower and upper saturation limits \(u_{IL}, u_{IU} \in \mathbb{R}\) satisfying \(u_{IL} < u_{IU}\) for \(i \in I_m\). Let \(g_c\) in (4) be decomposed as

\[
g_c(x_g) = [g_{c1}(x_g), g_{c2}(x_g), \ldots, g_{cm}(x_g)]^T.
\]

The GPAW scheme constructs \(f_g\) in a way to maintain the feasibility of the \(2m\) saturation constraints

\[
h_i(x_g) = g_{ci}(x_g) - u_{IU} \leq 0,
\]

\[
h_{i+m}(x_g) = -g_{ci}(x_g) + u_{IL} \leq 0,
\]

with associated gradient vectors

\[
\nabla h_i(x_g) = -\nabla h_{i+m}(x_g) = \left(\frac{\partial g_{ci}(x_g)}{\partial x_c}\right)^T \in \mathbb{R}^q,
\]

for \(i \in I_m\). For any non-empty set of indices \(I \subset \mathbb{I}_{2m}\), \(|I| = s > 0\), define the \(q \times s\) matrix

\[
N_I(x_g) = [\nabla h_{s+1}(x_g), \nabla h_{s+2}(x_g), \ldots, \nabla h_{s+|I|}(x_g)],
\]

where \(s: \mathbb{I}_s \rightarrow I\) is any chosen bijection that assigns an integer in \(I\) to an integer in \(I_s = \{1, 2, \ldots, s\}\). For \(I = \emptyset\), define \(N_I(x_g) = 0 \in \mathbb{R}^q\).

**Remark 4:** Any bijection \(s: \mathbb{I}_s \rightarrow I\) suffices. For example, we can take the ascending order map defined recursively by \(s(1) = \min(I \cup \{j^{-1} s(j) \})\) for all \(1 \in I_s\). The final result will be independent of the choice of \(s\).

In contrast to numerous anti-windup schemes, the GPAW scheme has only a single tuning parameter, a chosen symmetric positive definite matrix \(\Gamma \in \mathbb{R}^{q \times q}\). For any \(I \subset \mathbb{I}_{2m}\) such that \(|I| = 0\), or \(0 < |I| < q\) and \(N_I(x_g)\) is full rank, define

\[
f_I(x_g, y, \hat{r}(t)) = R_I(x_g)f_g(x_g, y, \hat{r}(t)),
\]

where

\[
R_I(x_g) = \begin{cases}
I - \Gamma N_I^T (N_I^T \Gamma N_I)^{-1} N_I^T(x_g), & \text{if } |I| > 0, \\
I, & \text{otherwise}.
\end{cases}
\]

Define the set of indices corresponding to active saturation constraints as

\(I_{sat} = \{i \in I_{2m} | h_i(x_g) \geq 0\}\).

Let \(J\) be the set of all subsets of \(I_{sat}\) with cardinality less than or equal to \(q\). Define the following combinatorial optimization subproblem

\[
\max_{I \in J} f_I^T(x_g, y, \hat{r}(t)) \Gamma^{-1} f_I(x_g, y, \hat{r}(t)),
\]

subject to

\[
\text{rank}(N_I(x_g)) = |I|,
\]

\[
N_I^T(x_g) f_I(x_g, y, \hat{r}(t)) \leq 0.
\]

The following result asserts the existence of solutions to subproblem (7).

**Proposition 1:** For any fixed \((x_g, y, \hat{r}(t)) \in \mathbb{R}^{q+p+(k+1)n}\), there exists a solution to subproblem (7).

**Proof:** To simplify the notation, we will omit all function arguments. If \(I_{sat} = \emptyset\), then \(J = \{\emptyset\}\), and it can be verified that \(I^* = \emptyset\) is the unique optimal solution. If \(\text{rank}(N_{I_{sat}}) = \emptyset\) and \(I \subset I_{sat}\), be any set of indices of \(v\) linearly independent gradient vectors, \(\nabla h_i\) for \(i \in I_{sat}\), so that \(\text{rank}(N_I) = v = |I|\). Since \(\text{rank}(N_{I_{sat}}) = v\), the columns of \(N_{I_{sat}}\) are linearly
dependent if \( s := |I_{\text{sat}}| > v \). Then, \( N_{I_{\text{sat}}} I \in \mathbb{R}^{q \times (s-v)} \) can be written as \( N_{I_{\text{sat}}} I = N_I \Psi \) for some \( \Psi \in \mathbb{R}^{v \times (s-v)} \). It can be verified from (6) that \( N_{I_{\text{sat}}} I f_I = \Psi^T f_I = 0 \in \mathbb{R}^{s-v} \), and hence \( I \) is a feasible (not necessarily optimal) solution to subproblem (7). Since there can only be a finite number of active saturation constraints, \( |I_{\text{sat}}| = s < \infty \), the number of candidate solutions \( \sum_{t=0}^{\min(q,s)} (s_t) \) is also finite. It follows that optimal solutions always exist that can be found by an exhaustive search algorithm.

The following lemma asserts a property of a solution to subproblem (7) when the gradient vectors of the active constraint functions are \textit{linearly independent}. This property is similar to one of the necessary and sufficient optimality conditions of the gradient projection method of nonlinear programming [4, Theorem 4].

\textbf{Lemma 1:} If the columns of \( N_{I_{\text{sat}}} I(x_g) \) are \textit{linearly independent}, i.e. \( \text{rank}(N_{I_{\text{sat}}} I(x_g)) = |I_{\text{sat}}| \), then any solution \( I^* \) to subproblem (7) satisfy
\[
(N_T I^T I_{I_{\text{sat}}})^{-1} N_T I^T f_I (x_g, y, \tilde{r}(t)) \geq 0.
\]
\textit{Proof:} See Appendix.

When the gradient vectors of the active constraint functions are \textit{linearly dependent}, we have the following result, which is needed in the proof of Theorem 2 in Section V.

\textbf{Lemma 2:} There exists a solution \( I^* \) of subproblem (7) such that (8) holds.

\textit{Proof:} See Appendix.

By Proposition 1 and Lemma 2, we can choose at each fixed time (so that \( (x_g, y, \tilde{r}(t)) \) is fixed), a solution \( I^* \) to subproblem (7) such that (8) holds. The GPAW compensated controller derived from (2) is then given by (4) with
\[
f_I(x_g, y, \tilde{r}(t)) = f_{I^*}(x_g, y, \tilde{r}(t)).
\]

The following is a key property of general GPAW compensated controllers that was crucial in obtaining the results of [7]. For the particular first order controller in [7], this property is readily seen by inspecting the defining equations of the GPAW compensated controller. However, this property is not immediately clear for more general GPAW compensated controllers, and it is shown here.

\textbf{Theorem 1 (Controller State-Output Consistency):} Consider the GPAW compensated controller defined by (4) and (9). If there exists a \( T \in \mathbb{R} \) such that \( \text{sat}(u(T)) = u(T) \), then \( \text{sat}(u(t)) = u(t) \) holds for all \( t \geq T \).

\textit{Proof:} Observe that \( \text{sat}(u(t)) = u(t) \) if and only if \( h_i(x_g(t)) \leq 0 \) for all \( i \in \mathbb{I}_{m_T} \) (see (5)). By assumption, we have \( h_i(x_g(T)) \leq 0 \), for all \( i \in \mathbb{I}_{m_T} \). Hence it is sufficient to show that for all \( i \in \mathbb{I}_{m_T} \), whenever \( h_i(x_g(t)) = 0 \), then \( h_i(x_g(t)) \leq 0 \) holds. Taking the time derivative, we have
\[
h_i(x_g(t)) = \nabla h_i^T(x_g(t)) f_{I^*}(x_g(t), y(t), \tilde{r}(t)).
\]
If \( h_i(x_g(t)) = 0 \), then \( i \in I_{\text{sat}} \). We need to show that
\[
N_{I_{\text{sat}}}^T I^T f_{I^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0,
\]
or equivalently, since \( I^* \subset I_{\text{sat}} \),
\[
N_{I_{\text{sat}}}^T I^T f_{I^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0,
\]
\[
N_{I^*}^T f_{I^*}(x_g(t), y(t), \tilde{r}(t)) \leq 0 \quad \text{if} \quad I^* \neq \emptyset.
\]

Observe that the second inequality above needs to be satisfied only when \( I^* \neq \emptyset \) since for \( I^* = \emptyset \), the first inequality is equivalent to (10). Since \( I^* \) is a solution to subproblem (7), the first inequality holds due to the second constraint of subproblem (7). For \( I^* \neq \emptyset \), the definition of \( f_{I^*} \) (6) yields
\[
N_{I^*}^T f_{I^*}(x_g(t), y(t), \tilde{r}(t)) = 0 \in \mathbb{R}^{I^*-1},
\]
so that the second inequality holds. Since these two inequalities hold for all \( t \in \mathbb{R} \), the conclusion follows.

\textbf{Remark 5:} Note that Theorem 1 depends critically on \( h_i \) (and hence \( g_i \)) being dependent only on the controller state. See [7, Fig. 3] for a numerical illustration of this result.

The closed loop system defined by (1), (4) and (9) is called the \textit{GPAW compensated system}, rewritten as
\[
\begin{align*}
\dot{e} &= f(e + r(t), \text{sat}(g_c(x_g))) - \dot{r}(t), \\
\dot{x}_g &= f_{I^*}(x_g, g(e + r(t), \text{sat}(g_c(x_g))), \tilde{r}(t)),
\end{align*}
\]
with initial state \( (e(0), x_g(0)) = (e_0 - r(0), x_{c0}) \). Observe that if there exists a \( T \in \mathbb{R} \) (possibly \( T = 0 \)) such that \( \text{sat}(g_c(x_g(T))) = g_c(x_g(T)) \), then for all \( t \geq T \), Theorem 1 allows (11) to be simplified to
\[
\begin{align*}
\dot{e} &= f(e + r(t), g_c(x_g)) - \dot{r}(t), \\
\dot{x}_g &= f_{I^*}(x_g, g(e + r(t), g_c(x_g)), \tilde{r}(t)).
\end{align*}
\]

\textbf{IV. APPROXIMATING CONTROLLER}

Theorem 1 requires that the output equation of the nominal controller (2) depends only on the controller state. If this is not true, we show here that an arbitrarily close approximation of the nominal controller can be constructed that has the required structure. Note that this construction is not unique. The main idea is to replace the signal components in the controller output equation that are not part of the controller state by its low-pass filtered signal, and design the low-pass filter such that its bandwidth is much larger than the effective bandwidth of the closed loop system. It is clear that the approximation will be enhanced as the bandwidth of the low-pass filter is increased. Importantly, the main purpose of this low-pass filter is \textit{not} for noise rejection or performance/robustness enhancements.

Consider the nominal controller
\[
\begin{align*}
\dot{x}_c &= f_c(x_c, y, \tilde{r}(t)), \quad x_c(0) = x_{c0}, \\
u &= g_c(x_c, y),
\end{align*}
\]
whose output equation depends not only on the state, but on measurement \( y \) as well. For simplicity, we have assumed that the output equation is not dependent on the controller reference \( \tilde{r}(t) \). If it indeed does, the treatment is similar, and also simpler due to the structure of \( \tilde{r}(t) \).

\textbf{Remark 6:} When \( g_c \) depends on the measurement \( y \) as in (12), the closed loop system (1), (12) will contain an \textit{algebraic loop} whenever \( \frac{\partial g_c}{\partial x_c} \frac{\partial x_c}{\partial y} \neq 0 \).

Consider augmenting the controller state to be \( \hat{x}_c = (x_c, \hat{y}) \), with \( \hat{y} = y \). Then the controller output equation \( u = g_c(x_c, \hat{y}) = g_c(\hat{x}_c) \) will be of the desired form (2).
The state equation of the augmented controller with state \( \hat{x}_c \) needs to satisfy
\[
\dot{\hat{x}}_c = f_c(x_c, y, \hat{r}(t)), \quad \dot{\hat{y}} = \hat{y}.
\] (13)
Clearly, if the functions \( f \) and \( g \) in (1) are known exactly, realization of (13) is straightforward\(^1\) by taking the time derivative of \( y \) in (1) and using the knowledge of \( f \) and \( g \).

We avoid making such a conservative assumption by using an approximation. Consider \( \hat{y} \) obtained as the output of an exponentially stable, unity DC gain low-pass filter with input \( y \), parameterized by \( a \in (0, \infty) \)
\[
\dot{\hat{y}} = a(y - \hat{y}), \quad \hat{y}(0) = y(0).
\]
It can be seen that \( \hat{y}(t) \to y(t) \) for all \( t \geq 0 \) as \( a \to \infty \), so that the solution of the approximating controller can be made arbitrarily close to the nominal controller. While this can be shown formally for any fixed \( y(t), t \in [0, \infty) \) and \( r \in \mathcal{R} \) using singular perturbation theory [9, Chapter 11, pp. 423–468], the larger question is the effect of the approximation on the closed loop system, which we discuss next.

The approximate controller by the above considerations is
\[
\begin{align*}
\dot{x}_c &= f_c(x_c, y, \hat{r}(t)), \quad x_c(0) = x_{c0}, \\
\dot{\hat{y}} &= a(y - \hat{y}), \quad \hat{y}(0) = y(0), \\
u &= g_c(x_c, \hat{y}),
\end{align*}
\]
which, together with (1), gives the closed loop dynamics
\[
\begin{align*}
\dot{e} &= f(e + r(t), \text{sat}(g_c(x_c, \hat{y}))) - \hat{r}(t), \\
\dot{x}_c &= f_c(x_c, g(e + r(t), \text{sat}(g_c(x_c, \hat{y}))), \hat{r}(t)), \\
\dot{\hat{y}} &= g(e + r(t), \text{sat}(g_c(x_c, \hat{y}))) - \hat{y},
\end{align*}
\] (14)
where \( \epsilon = \frac{1}{2} a \). Observe that when \( \epsilon = 0 \), we recover the exact closed loop system obtained with controller (12), which corresponds to the reduced system in the singular perturbation framework. Here, we refer to (14) as the approximate system, and (14) with \( \epsilon = 0 \) as the exact system. When we assume existence and uniqueness of solutions to the exact system, then (14) is a standard singular perturbation model [9, pp. 424]. It can be shown that if \( g \) and \( g_c \) are such that the eigenvalue condition [9, pp. 433]
\[
\text{Re} \left( \lambda \left( \frac{\partial g_c}{\partial u} \frac{\partial g}{\partial y} - I \right) \right) < 0,
\]
holds uniformly for all \((t, e, x_c)\) in some domain, then the origin of the associated boundary layer model for the singular perturbation model (14) is exponentially stable. This, and assuming existence and uniqueness of solutions of the exact system, [9, Theorem 11.1, pp. 434] shows that on any finite time interval, the solution of the approximate system can be made arbitrarily close to the solution of the exact system when \( \epsilon \) is sufficiently small (\( a \) is sufficiently large).

When the origin is an exponentially stable equilibrium of the exact system, [9, Theorem 11.2, pp. 439–440] shows that the result extends to infinite intervals.

Observe that for constrained LTI systems driven by LTI controllers, local exponential stability is usually guaranteed so that the infinite time approximation result holds. If the exact system is not exponentially stable and the finite time approximation result indicated above is not sufficient, redoing the analysis with the approximating controller may be required. Because the approximation can be made arbitrarily well, it is likely that the approximate controller will be able to achieve the control objectives as well.

V. FURTHER GEOMETRIC PROPERTIES OF GPAW COMPENSATED SYSTEMS

This section presents further geometric properties of GPAW compensated systems, the main result of which (Theorem 2) illuminates the role of the GPAW tuning parameter, \( \Gamma \). These geometric properties are foreseen to be needed to extend the results of [7] and to prove general desirable properties of GPAW compensated systems.

First, we describe star domains, which will be needed to describe the unsaturated regions for GPAW compensated systems. For any two points \( x_1, x_2 \in \mathbb{R}^n \), let the line segment connecting them be
\[
\eta(x_1, x_2) = \{ x \in \mathbb{R}^n \mid x = \theta x_1 + (1 - \theta) x_2, \forall \theta \in [0, 1] \}.
\]

**Definition 1:** [10, Definition 1.4, pp. 5] Let \( X \subset \mathbb{R}^n \) be a nonempty set. The kernel of \( X \), denoted by \( \ker(X) \), is
\[
\ker(X) = \{ x \in \mathbb{R}^n \mid \eta(x, y) \subset X, \forall y \in X \} \subset X.
\]

**Definition 2:** [10, Definition 1.2, pp. 4] A nonempty set \( X \subset \mathbb{R}^n \) is a star domain, or star-shaped, if \( \ker(X) \neq \emptyset \). In other words, a nonempty set \( X \) is a star domain if there exists at least one point \( x \in X \) such that for every \( y \in X \), the line segment connecting \( x \) and \( y \) is contained within \( X \).

**Remark 7:** Clearly, any convex set \( X \) is also a star domain with \( \ker(X) = X \). For any non-convex star domain, \( \ker(X) \) is a strict subset of \( X \).

**Remark 8:** If \( X \subset \mathbb{R}^n \) is a star domain, then \( X \times \mathbb{R}^m \) is also a star domain in \( \mathbb{R}^{n+m} \) with kernel \( \ker(X) \times \mathbb{R}^m \).

When a star domain is defined by a set of constraint functions, the following gives a characterization of the gradient vectors of the constraint functions on the boundary of the star domain.

**Lemma 3:** Let \( X \) be defined by a set of \( m \) constraints
\[
X = \{ x \in \mathbb{R}^n \mid \hat{h}_i(x) \leq 0, \forall i \in I_m \} \subset \mathbb{R}^n.
\]
For any boundary point \( x \in \partial X \), define
\[
I_{lim}(x) = \{ i \in I_m \mid \hat{h}_i(x) = 0 \}.
\]
If \( X \) is a star domain, then for any \( x_{ker} \in \ker(X) \) and any boundary point \( x \in \partial X \), we have
\[
(x - x_{ker}, \nabla \hat{h}_i(x)) \geq 0, \quad \forall i \in I_{lim}(x).
\]
**Proof:** By the definition of \( \ker(X) \) and the star domain, we have \( y(\theta) := \theta x + (1 - \theta) x_{ker} \in X \) for all \( \theta \in [0, 1] \).

Hence \( \hat{h}_i(y(\theta)) \leq 0 \) for all \( i \in I_m \). View \( h_i(y(\theta)) \) as a...
function of \( \theta \) with \( x, x_{ker} \) fixed. Since \( x \in \partial X \), we must have \( \tilde{h}_i(y(1)) = \tilde{h}_i(x) = 0 \) for all \( i \in I_{lim}(x) \). Since \( \tilde{h}_i(y(\theta)) \leq 0 \) for all \( \theta \in [0, 1] \), \( \tilde{h}_i(y(\theta)) \) must be non-decreasing at \( \theta = 1 \). Hence by the chain rule,

\[
\frac{d\tilde{h}_i}{d\theta}(y(\theta)) = \frac{d\tilde{h}_i}{dx}(y(\theta))(x - x_{ker}) \geq 0, \quad \forall i \in I_{lim}(x),
\]

at \( \theta = 1 \). This gives \( \frac{d\tilde{h}_i}{dx}(x)(x - x_{ker}) \geq 0 \) for all \( i \in I_{lim}(x) \), which can be written in dot product form with the gradient vector as stated.

Let the nominal system (3) and GPAW compensated system (11) be represented as \( \dot{z} = f_n(t, z) \) and \( \dot{z} = f_p(t, z) \) respectively. Define the unsaturated region

\[
K = \{ x \in \mathbb{R}^q \mid \tilde{h}_i(x) \leq 0, \forall i \in I_{2m} \} \subset \mathbb{R}^q,
\]

where \( \tilde{h}_i \) are the saturation constraint functions in (5).

**Remark 9:** If \( g_e \) in (2) is a linear function, i.e. \( g_e(x) = C_e x \) where \( C_e \in \mathbb{R}^{m \times q} \), then \( K \) is convex (in fact, a convex polyhedron), and hence is also a star domain in \( \mathbb{R}^q \). □

The following is the main result of this section which illuminates a geometric property of all GPAW compensated systems.

**Theorem 2:** If \( K \subset \mathbb{R}^q \) is a star domain, then for all \( z \in (\mathbb{R}^n \times K) \) and any \( z_{ker} \in (\mathbb{R}^n \times \ker(K)) \),

\[
(z - z_{ker}, \Gamma^{-1}f_p(t, z)) \leq (z - z_{ker}, \Gamma^{-1}f_n(t, z)),
\]

holds for all \( t \in \mathbb{R} \), where \( \Gamma = [I \ 0] \in \mathbb{R}^{(n+q) \times (n+q)} \).

**Proof:** Let \( z = (e, x) \in (\mathbb{R}^n \times K) \) and \( z_{ker} = (e_{\infty}, x_\infty) \in (\mathbb{R}^n \times \ker(K)) \), \( \tilde{e} = e - e_\infty \in \mathbb{R}^n \) and \( \tilde{x} = x - x_\infty \in K \). With reference to (3) and (11), we need to show that

\[
\tilde{e}^T(f - \dot{r}(t)) + \tilde{x}^T\Gamma^{-1}f_{2r} \leq \tilde{e}^T(f - \dot{r}(t)) + \tilde{x}^T\Gamma^{-1}f_c,
\]

or equivalently,

\[
\tilde{x}^T\Gamma^{-1}f_{2r} \leq \tilde{x}^T\Gamma^{-1}f_c,
\]

where the function arguments have been dropped. If \( x \) is in the interior of \( K \), then \( h_i(x) < 0 \) for all \( i \in I_{2m} \) and \( I_{sat} = \emptyset \). Since \( I^* \subset I_{sat} \), we have \( I^* = \emptyset \). From the definition of \( f_{2r} \) (6), we have \( f_{2r} = f_c \) and (15) holds with equality. It suffices to consider boundary points \( x \in \partial K \). When \( x \in \partial K \) and \( I^* = \emptyset \), the same reasoning shows that (15) holds. Hence we need to show that (15) holds for \( x \in \partial K \) and \( I^* \neq \emptyset \).

Using the definition of \( f_{2r} \) (6), this reduces to showing that

\[
\tilde{x}^T N_{2r}, (N_{2r}^T \Gamma N_{2r})^{-1} N_{2r}^T f_c \geq 0, \quad \text{if } 1 \leq |I^*| \leq q.
\]

By assumption, \( K \) is a star domain defined by the functions \( h_i, i \in I_{2m} \). It follows from Lemma 3 that \( \tilde{x}^T N_{2r} \tilde{x} \geq 0 \). Since \( I^* \subset I_{sat} \), we have \( \tilde{x}^T N_{2r} \geq 0 \), which shows that \( \tilde{x}^T N_{2r} \) is a row vector of non-negative numbers. By Lemma 2 and our choice of \( I^* \), we have \( (N_{2r}^T \Gamma N_{2r})^{-1} N_{2r}^T f_c \) as a column vector of non-negative numbers. The sums and products of non-negative numbers must be non-negative. The conclusion follows. □

The geometric condition stated in Theorem 2 is illustrated in Fig. 1 with an example non-convex star domain for the unsaturated region. If the control objective is to regulate the system state about the origin, we can set \( z_{ker} = 0 \), provided \( \ker(K) \) contains the origin of \( \mathbb{R}^q \).

**VI. CONCLUSIONS**

When the output equation of the nominal controller depends only on the controller state, then exact controller state-output consistency is achieved for the GPAW compensated controller when appropriately initialized. When the nominal controller does not possess this structure, an arbitrarily close approximating controller can be constructed that has the required properties. Further geometric properties of GPAW compensated systems are established, which illuminates the role of the GPAW tuning parameter.

**APPENDIX**

**Proof of Lemma 1:** From (8), we only need to consider when \( (N_{2r}^T \Gamma N_{2r})^{-1} N_{2r}^T f_c \neq 0 \), which implies \( f_c \neq 0 \) and \( I^* \neq \emptyset \). Let \( I^* (\neq \emptyset) \) be a solution to subproblem (7), assume \( f_c \neq 0 \), and let \( s := |I^*| \) so that \( 0 < s \leq q \). The symmetric positive definite GPAW tuning parameter \( \Gamma \) can be decomposed as \( \Gamma = \Phi^T \Phi \), where \( \Phi \in \mathbb{R}^{n \times q} \) is nonsingular.

Since \( I^* \) is a solution of subproblem (7), \( N_{I^*} \) is full rank and its columns are linearly independent. Let \( \tilde{N} := \Phi N_{I^*} = [n_1, n_2, \ldots, n_s] \) where \( n_i = \Phi \nabla h(x_i, \cdot) \) for all \( i \in I_{sat} \). Let \( M := [m_1, m_2, \ldots, m_{q-s}] \), where \( m_i \in \mathbb{R}^n \) (for \( i \in I_{q-s} \)) together with \( n_j \) (for \( j \in I_{sat} \)) form a basis for \( \mathbb{R}^n \), and such that each \( m_i \) is orthogonal to each \( n_j \) for all \( i \in I_{q-s}, j \in I_{sat} \). Then \( \tilde{N}^T M = 0 \in \mathbb{R}^{s \times (q-s)} \), and any \( x \in \mathbb{R}^q \) can be expressed as \( x = \tilde{N} x_1 + M x_2 \) for some \( x_1 \in \mathbb{R}^s, x_2 \in \mathbb{R}^{q-s} \). Observe that when \( s := |I^*| = q \), the matrix \( M \) is not needed, and any \( x \in \mathbb{R}^q \) can be written as \( x = \tilde{N} x_1 \) for some \( x_1 \in \mathbb{R}^s \). In this case, the reasoning applies similarly.

Let \( \Phi^T f_c := (\Phi^T)^{-1} f_c = \tilde{N} v + M w \) for some \( v \in \mathbb{R}^s, w \in \mathbb{R}^{q-s} \). We need to show that

\[
v = (\tilde{N}^T \tilde{N})^{-1} \tilde{N}^T \Phi^T f_c = (N_{2r}^T \Gamma N_{2r})^{-1} N_{2r}^T f_c \geq 0.
\]

We will show that if \( v \geq 0 \) does not hold, then \( I^* \) cannot be an optimal solution to subproblem (7).

Let \( J(I) := f_{I^*}^T \Gamma^{-1} f_I \) be the objective function of subproblem (7), \( v = [v_1, v_2, \ldots, v_s]^T \) with \( v_1 < 0, v_i \in \mathbb{R} \)

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for all $i \in \{2, 3, \ldots, s\}$, so that $v \not\geq 0$, and define $I^o = I^* \setminus \{\sigma^*, 1\}$. We will show that
\[
J(I^o) > J(I^*), \quad \text{rank}(N_{I^o}^T \chi f_{I^o}) \leq 0,
\]
where the objective value is strictly increased, so that $I^*$ cannot be an optimal solution to subproblem (7). Since $N_{I^*} = [\nabla h_{\sigma^*}(1), N_{I^*}]$ is full rank, $N_{I^*}$ must also be full rank. It remains to show that the first and third conditions above hold.

Define $\tilde{N}_o := \Phi N_{I^o}$, so that $\tilde{N} = [n_1, \tilde{N}_o]$. We first show that $J(I^o) > J(I^*)$ holds, which can be verified by computation to be equivalent to
\[
v^T \tilde{N}^T (I - \tilde{N}_o (\tilde{N}^T \tilde{N}_o)^{-1} \tilde{N}^T) \tilde{N} v > 0,
\]
which in turn can be simplified to
\[
v_0^T n_1^T (I - \tilde{N}_o (\tilde{N}^T \tilde{N}_o)^{-1} \tilde{N}^T) n_1 > 0.
\]

By defining the projection matrix $P_o := I - \tilde{N}_o (\tilde{N}^T \tilde{N}_o)^{-1} \tilde{N}^T$ [4, Equation 2.15], it can be verified that $P_o P_o = P_o$ and $P_o = \tilde{P}_o^T = \tilde{P}_o$, so that $J(I^o) > J(I^*)$ is equivalent to $v_0^T \|P_o n_1\|^2 = v_0^T n_1^T P_o n_1 > 0$. Observing that $n_1$ must be linearly independent of the columns of $N_o$, [4, Theorem 2] gives $P_o n_1 \neq 0$. With $v_1 < 0$ (specifically, $v_1 \neq 0$), we have the strict inequality $v_0^T \|P_o n_1\|^2 > 0$, which shows that $J(I^o) > J(I^*)$.

It remains to show that $N_{I^o}^T \chi f_{I^o} \leq 0$. Since $I^o = I^* \setminus (\sigma^* \setminus \{1\})$, we have $I_{sat} \setminus I^o = (I_{sat} \setminus I^*) \cup (\sigma^* \setminus \{1\})$. Then $N_{I_{sat} \setminus I^*} f_{I^o} \leq 0$ is equivalent to satisfying the two conditions
\[
N_{I_{sat} \setminus I^*} f_{I^o} \leq 0, \quad \nabla h_{\sigma^*}(1) f_{I^o} \leq 0.
\]

It can be verified that $\Phi^{-T} f_{I^o} = M u + P_o n_1 v_1$, so that
\[
\nabla h_{\sigma^*}(1) f_{I^o} = n_1^T (M u + P_o n_1 v_1) = v_1^T P_o n_1^2.
\]

Since $v_1 < 0$, the second condition in (16) holds.

To complete the proof, we need to show that the first condition $N_{I_{sat} \setminus I^*} f_{I^o} \leq 0$ holds. Since by assumption, the columns of $N_{I_{sat} \setminus I^*}$ are linearly independent, no columns of $N_{I_{sat} \setminus I^*}$ can be in the span of the columns of $N_{I^o}$. This implies that all columns of $\Phi N_{I_{sat} \setminus I^*}$ must lie in the span of the columns of $M$, so that $\Phi N_{I_{sat} \setminus I^*} = M \Psi$ for some $\Psi \in \mathbb{R}^{(q-s) \times n}$ where $\alpha := |I_{sat}| - s$. Then we have
\[
N_{I_{sat} \setminus I^*} f_{I^o} = (N_{I_{sat} \setminus I^*}^T \Phi^T) (\Phi^{-T} f_{I^o}),
\]
\[
= \Psi^T M (M u + P_o n_1 v_1) = \Psi^T M T M u,
\]
where the last equality follows from $\Psi^T M T P_o n_1 v_1 = 0$ by direct computation. Because $I^*$ is a solution to subproblem (7), we have $N_{I_{sat} \setminus I^*}^T f_{I^o} = \Psi^T M T M u \leq 0$, which shows that $N_{I_{sat} \setminus I^*} f_{I^o} \leq 0$, as desired.

**Proof of Lemma 2:** We will show that when the columns of $N_{I_{sat}}$ are linearly dependent, we can pick out a linearly independent subset to define a modified combinatorial optimization problem whose solutions must be solutions to subproblem (7). Then Lemma 1 applies to the modified optimization problem to yield the desired conclusion.

The modified optimization subproblem is defined by (7) with $I_{sat}$ replaced by an index set $I_{sat} \subset I_{sat}$ and $J$ in (7) redefined as the set of all subsets of $I_{sat}$ with cardinality less than or equal to $q$. For any $I \subset I_{sat}$, let $\phi(I)$ map an index set of active saturation constraints of subproblem (7) to the corresponding set of solutions when $I_{sat}$ is redefined to be $I$. Then $\phi(I_{sat}) = J^*$, where $J^* = \{j_1^*, j_2^*, \ldots, j_n^*\}$ $(\subset J)$ is the set of $w$ distinct optimal solutions to the original subproblem (7). Let $s := \max_{I \subset I_{sat}} |\phi(I)|$ be the maximum cardinality of all solutions to (7). We claim that there exists a set $\hat{I}^* \subset I_{sat}$ with $|\hat{I}^*| = s$, such that $\hat{I}^* \in \phi(\hat{I}_0)$, and $\hat{I}^* \in J^*$. If the claim holds, then Lemma 1 applied to the modified problem with $I_{sat} := \hat{I}_0$ proves the lemma.

To prove the claim, take any index set $I_0 \subset I_{sat}$ such that $|I_0| = s$ and $N_{I_0}$ is full rank. If $\hat{I}_0 \in \phi(\hat{I}_0)$, the first statement of the claim holds. Otherwise, any solution $\hat{I}_0^* \in \phi(\hat{I}_0)$ satisfy $|\hat{I}_0^*| < s$. We know that by progressively adding indices in $I_{sat} \setminus \hat{I}_0$ to the set $\hat{I}_0$ and redefining the modified optimization problem, we must have a solution with cardinality $s$ emerging at some point. Hence there must exist an index $j \in I_{sat} \setminus \hat{I}_0$ such that $\nabla h_{\sigma^*}^T f_{I_0} > 0$. The fact that $N_{I_0}$ is full rank means that the constraints indexed by $\hat{I}_0 \setminus \hat{I}_0^*$ are redundant for the modified problem. Hence one can replace one of the indices corresponding to a redundant constraint by $j$, and denote the modified index set by $\hat{I}_1$. The cardinality of solutions $\phi(\hat{I}_1)$ will then be increased by 1, while $|\hat{I}_1| = |\hat{I}_0|$. By repeating this process through $\hat{I}_i$, $i \in \{1, 2, \ldots\}$, we will reach a point where a solution $\hat{I}_k$ to the modified subproblem has cardinality $s$ for some $k$. The only way this can happen is for $\hat{I}_k \in \phi(\hat{I}_k)$, which proves the first statement of the claim with $\hat{I}^* := \hat{I}_k$. Moreover, we must have $N_{I_{sat} \setminus \hat{I}_k} f_{I_0} \leq 0$. Otherwise, the process can be continued to yield a solution with cardinality greater than $s$, a contradiction. Hence we have $\hat{I}^* \subset J^*$ as desired.

**REFERENCES**


