Analysis of gradient projection anti-windup scheme

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Analysis of Gradient Projection Anti-windup Scheme

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Abstract— **The *gradient projection anti-windup* (GPAW) scheme was recently proposed as an anti-windup method for *nonlinear* multi-input-multi-output systems/controllers, the solution of which was recognized as a largely open problem in a recent survey paper. This paper analyzes the properties of the GPAW scheme applied to an input constrained first order linear time invariant (LTI) system driven by a first order LTI controller, where the objective is to regulate the system state about the origin. We show that the GPAW compensated system is in fact a *projected dynamical system* (PDS), and use results in the PDS literature to assert existence and uniqueness of its solutions. The main result is that the GPAW scheme can only *maintain/enlarge* the *exact* region of attraction of the uncompensated system.**

I. Introduction

The *gradient projection anti-windup* (GPAW) scheme was proposed in [1] as an anti-windup method for *nonlinear* multi-input-multi-output (MIMO) systems/controllers. It was recognized in a recent survey paper [2] that anti-windup compensation for nonlinear systems remains largely an *open problem*. To this end, [3] and relevant references in [2] represent some recent advances. The GPAW scheme uses a continuous-time extension of the gradient projection method of nonlinear programming [4], [5] to extend the "stop integration" heuristic outlined in [6] to the case of nonlinear MIMO systems/controllers. Application of the GPAW scheme to some nominal controllers results in a *hybrid* GPAW compensated controller [1], and hence a hybrid closed loop system.

Here, we apply the GPAW scheme to a first order linear time invariant (LTI) system stabilized by a first order LTI controller, where the objective is to regulate the system state about the origin. This case is particularly insightful because the closed loop system is a planar dynamical system whose vector field is easily visualized, and is highly tractable because there is a large body of relevant work, eg. [7, Chapter 2] [8, Chapter 2] [9, Chapter 2]. Related literature on constrained planar systems include [10], [11].

After presenting the generalities in Section II, we address the existence and uniqueness of solutions to the GPAW compensated system. Due to *discontinuities* of the governing vector field of the GPAW compensated system on the saturation constraint boundaries, classical existence and uniqueness results based on Lipschitz continuity of vector fields [7]–[9] do not apply directly. We show that the GPAW compensated system is in fact a *projected dynamical system* (PDS) [12]–[14] in Section III. Observe that PDS is a significant line of independent research that has attracted the attention of economists and mathematicians, among others. The link to PDS thus enables cross utilization of ideas and methods, as demonstrated in [15]. Using results from the PDS literature, existence and uniqueness of solutions to the GPAW compensated system can thus be easily established, as shown in Section IV.

It is widely accepted as a rule that the performance of a control system can be enhanced by trading off its robustness [16, Section 9.1]. As such, we consider an anti-windup scheme to be valid only if it can provide performance enhancements *without reducing the system's region of attraction (ROA)*. The first question to be addressed is whether the GPAW scheme satisfy such a criterion, and is shown to be affirmative in Section V. Numerical results further illuminate this property of GPAW compensated systems.

II. Preliminaries

Let the system to be controlled be described by

$$\dot{x} = ax + b\,\mathrm{sat}(u), \tag{1}$$

where the saturation function is defined by

$$\mathrm{sat}(u) = \max\{\min\{u, u_{max}\}, u_{min}\},$$

and $x, u \in \mathbb{R}$ are the plant state and control input respectively, $a, b, u_{min}, u_{max} \in \mathbb{R}$ are constant plant parameters with u_{min}, u_{max} satisfying $u_{min} < 0 < u_{max}$. Let the *nominal* controller be

$$\dot{x}_c = \tilde{c}x_c + \tilde{d}x, \qquad u = \tilde{e}x_c, \tag{2}$$

where $x_c, u \in \mathbb{R}$ are the controller state and output respectively, $x \in \mathbb{R}$ is the measurement of the plant state, and $\tilde{c}, \tilde{d}, \tilde{e} \in \mathbb{R}$ are controller gains chosen to *globally* stabilize the *unconstrained* system, ie. when $u_{max} = -u_{min} = \infty$.

Remark 1: It is important that the output equation of the nominal controller, namely $u = \tilde{e}x_c$, depends only on the controller state x_c and be independent of measurement x. That is, if the output equation is $u = \tilde{e}x_c + \tilde{f}x$, then we require $\tilde{f} = 0$. This property ensures that full controller state-output consistency, ie. $\mathrm{sat}(u) = u$, can be maintained at "almost all" times (stated more precisely as Fact 1 below) when applying the GPAW scheme. See [17] when the nominal controller is of more general forms. □

A simple transformation of (2) yields the equivalent controller realization

$$\dot{u} = cx + du, \tag{3}$$

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with $c := \tilde{d}\tilde{e}$, $d := \tilde{c}$. Applying the GPAW scheme [1] to the preceding transformed nominal controller (3) yields the GPAW compensated controller [18, Appendix A]

$$\dot{u} = \begin{cases} 0, & \text{if } u \geq u_{max}, cx + du > 0, \\ 0, & \text{if } u \leq u_{min}, cx + du < 0, \\ cx + du, & \text{otherwise}, \end{cases} \tag{4}$$

which is similar to the "conditionally freeze integrator" method [19]. This similarity is expected since the GPAW scheme can be viewed as a generalization of this idea to MIMO nonlinear controllers. Observe that the first order GPAW compensated controller is independent of the GPAW tuning parameter Γ introduced in [1], which is true for all first order controllers. Furthermore, inspection of (4) reveals the following.

Fact 1 (Controller State-Output Consistency): If for some $T \in \mathbb{R}$, the control signal of the GPAW compensated controller (4) at time T satisfies $u_{min} \leq u(T) \leq u_{max}$, then $u_{min} \leq u(t) \leq u_{max}$ holds for all $t \geq T$. □

That is, the GPAW compensated controller maintains full controller state-output consistency, $\mathrm{sat}(u) = u$, for all future times once it has been achieved for any time instant. In particular, if the controller state is initialized such that $\mathrm{sat}(u(0)) = u(0)$, then $\mathrm{sat}(u(t)) = u(t)$ holds for all $t \geq 0$.

Remark 2: For nonlinear MIMO controllers whose output equation depends *only on the controller state*, the same result (state-output consistency of GPAW compensated controller) holds, as shown in [17, Theorem 1]. □

The *nominal* constrained closed-loop system, Σ_n, is described by (1) and (3), while the GPAW compensated closed-loop system, Σ_g, is described by (1) and (4). Each of these systems can be expressed in the form $\dot{z} = f(z)$ with $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. The representing functions (vector fields) for systems Σ_n and Σ_g will be denoted by f_n and f_g respectively. The following will be assumed.

Assumption 1: The controller parameters c, d satisfy

$$a + d < 0, \tag{5}$$

$$ad - bc > 0, \tag{6}$$

and $bc \neq 0$. □

Conditions (5) and (6) ensure that the origin is a globally exponentially stable equilibrium point for the nominal *unconstrained* system, ie. Σ_n with $u_{max} = -u_{min} = \infty$, while $bc \neq 0$ ensures that Σ_n is a *feedback* system.

We will need the following sets

$$\begin{aligned}
K &= \{(x,u) \in \mathbb{R}^2 \mid u_{min} < u < u_{max}\}, \\
K_+ &= \{(x,u) \in \mathbb{R}^2 \mid u > u_{max}\}, \\
K_- &= \{(x,u) \in \mathbb{R}^2 \mid u < u_{min}\}, \\
\partial K_+ &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{max}\}, \\
\partial K_- &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{min}\}, \\
\partial K_{+div} &= \{(x,u) \in \mathbb{R}^2 \mid u > u_{max}, cx + du = 0\}, \\
K_{+in} &= \{(x,u) \in \mathbb{R}^2 \mid u > u_{max}, cx + du < 0\}, \\
K_{+out} &= \{(x,u) \in \mathbb{R}^2 \mid u > u_{max}, cx + du > 0\},
\end{aligned}$$

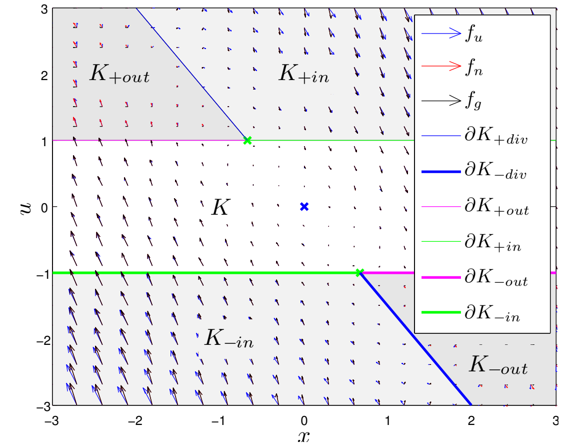


Fig. 1: Closed loop vector fields (f_n, f_g) of systems Σ_n, Σ_g and the *unconstrained* system (Σ_u, f_u), associated with an open loop *unstable* system (plant and controller parameters: $a = 1$, $b = 1$, $c = -3$, $d = -2$, $-u_{min} = u_{max} = 1$).

$$\begin{aligned}
\partial K_{-div} &= \{(x,u) \in \mathbb{R}^2 \mid u < u_{min}, cx + du = 0\}, \\
K_{-in} &= \{(x,u) \in \mathbb{R}^2 \mid u < u_{min}, cx + du > 0\}, \\
K_{-out} &= \{(x,u) \in \mathbb{R}^2 \mid u < u_{min}, cx + du < 0\}, \\
\partial K_{+in} &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{max}, cx + du_{max} < 0\}, \\
\partial K_{+out} &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{max}, cx + du_{max} > 0\}, \\
\partial K_{-in} &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{min}, cx + du_{min} > 0\}, \\
\partial K_{-out} &= \{(x,u) \in \mathbb{R}^2 \mid u = u_{min}, cx + du_{min} < 0\}, \\
\bar{K} &= K \cup \partial K_+ \cup \partial K_-,
\end{aligned}$$

and the points

$$z_+ = \left(-\tfrac{d}{c}u_{max}, u_{max}\right), \qquad z_- = \left(-\tfrac{d}{c}u_{min}, u_{min}\right).$$

These sets and associated vector fields are illustrated in Fig. 1 for an open-loop *unstable* plant.

Observe that $K_+ = K_{+in} \cup K_{+div} \cup K_{+out}$ and $\partial K_+ = \partial K_{+in} \cup \partial K_{+out} \cup \{z_+\}$, with analogous counterparts for K_- and ∂K_-. Observe further that on ∂K_{+in} and ∂K_{-in}, vector fields of systems Σ_n and Σ_g (f_n and f_g respectively) point into K. On ∂K_{+out}, f_n points into K_+ and f_g points into ∂K_+. On ∂K_{-out}, f_n points into K_- and f_g points into ∂K_-.

By inspection of the vector fields f_n and f_g *from their definitions*, we have the following.

Fact 2: The vector fields f_n and f_g coincide in

$$K \cup K_{+in} \cup K_{-in} \cup \partial K_{+div} \cup \partial K_{-div} \cup \partial K_{+in} \cup \partial K_{-in} \cup \{z_+, z_-\}.$$

That is, they coincide in $\mathbb{R}^2 \setminus (K_{+out} \cup K_{-out} \cup \partial K_{+out} \cup \partial K_{-out})$. □

Fact 3: Any solution of systems Σ_n or Σ_g can pass from K_+ to K if and only if it intersects the line segment ∂K_{+in}, and analogously with respect to K_- and ∂K_{-in}. □

Fact 4: Any solution of system Σ_n can pass from K to K_+ if and only if it intersects the line segment ∂K_{+out}, and analogously with respect to K_- and ∂K_{-out}. □

III. GPAW Compensated Closed Loop System as a Projected Dynamical System

Two of the most fundamental properties required for a meaningful study of dynamic systems is the existence and uniqueness of their solutions. As evident from the definition of the GPAW compensated controller (4), the vector field of the GPAW compensated system, f_g, is in general *discontinuous* on the saturation constraint boundaries ∂K_{+out} ($\subset \partial K_+$) and ∂K_{-out} ($\subset \partial K_-$). Classical results on the existence and uniqueness of solutions [7]–[9] rely on Lipschitz continuity of the governing vector fields, and hence do not apply to GPAW compensated systems. We will use results from the *projected dynamical system* (PDS) [12]–[14] literature to assert the existence and uniqueness of solutions to GPAW compensated systems. First, we show here that the GPAW compensated system, Σ_g, is in fact a PDS.

Observe that the set $\bar{K}$ is a closed convex set (in fact, a closed convex polyhedron). The interior and boundary of $\bar{K}$ are K and $\partial K_+ \cup \partial K_-$ respectively. Let $P\colon \mathbb{R}^2 \to \bar{K}$ be the projection operator [12] defined for all $y \in \mathbb{R}^2$ by

$$P(y) = \arg\min_{z\in\bar{K}} \|y - z\|,$$

with $\|\cdot\|$ as the Euclidean norm. It can be seen that for any $(x, u) \in \mathbb{R}^2$, $P((x, u)) = (x, \mathrm{sat}(u))$. Next, for any $y \in \bar{K}$, $v \in \mathbb{R}^2$, define the projection of vector v at y by [12], [13]

$$\pi(y, v) = \lim_{\delta\downarrow 0} \frac{P(y + \delta v) - y}{\delta}.$$

Note that the limit is one-sided in the above definition [13]. With f_n being the vector field of Σ_n, written explicitly as

$$f_n(x, u) = \begin{bmatrix} ax + bu \\ cx + du \end{bmatrix}, \qquad \forall (x, u) \in \bar{K},$$

we have the following, the corollary of which is the desired result.

Claim 1 ([18, Claim 1]): For all $(x, u) \in \bar{K}$, the vector field f_g of the GPAW compensated closed loop system Σ_g satisfy $f_g(x, u) = \pi((x, u), f_n(x, u))$.

Proof: If $(x, u) \in K$, the result follows from [13, Lemma 2.1(i)] and Fact 2. Next, consider a boundary point, $(x, u) \in \partial K_{+in} \cup \{z_+\}$. On this segment, we have $u = u_{max}$ and $cx + du_{max} \le 0$ from definition of the set $\partial K_{+in} \cup \{z_+\}$. Since $\mathrm{sat}(u_{max} + \delta\beta) = u_{max} + \delta\beta$ for $\beta \le 0$ and a sufficiently small $\delta > 0$, we have

$$\begin{aligned} P((x, u) + \delta f_n(x, u)) &= \begin{bmatrix} x + \delta(ax + bu) \\ \mathrm{sat}(u + \delta(cx + du)) \end{bmatrix}, \\ &= \begin{bmatrix} x + \delta(ax + bu) \\ u + \delta(cx + du) \end{bmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \pi((x, u), f_n(x, u)) &= \lim_{\delta\downarrow 0} \frac{P((x, u) + \delta f_n(x, u)) - (x, u)}{\delta}, \\ &= \begin{bmatrix} ax + bu \\ cx + du \end{bmatrix} = f_n(x, u) = f_g(x, u), \end{aligned}$$

for all $(x, u) \in \partial K_{+in} \cup \{z_+\}$, where the final equality follows from Fact 2.

Finally, consider a boundary point $(x, u) \in \partial K_{+out}$. On this segment, we have $u = u_{max}$ and $cx + du_{max} > 0$ from the definition of ∂K_{+out}. Since $\mathrm{sat}(u_{max} + \delta\beta) = u_{max}$ for $\beta > 0$ and a sufficiently small $\delta > 0$, we have

$$\begin{aligned} P((x, u) + \delta f_n(x, u)) &= \begin{bmatrix} x + \delta(ax + bu) \\ \mathrm{sat}(u + \delta(cx + du)) \end{bmatrix}, \\ &= \begin{bmatrix} x + \delta(ax + bu) \\ u \end{bmatrix}, \end{aligned}$$

so that

$$\begin{aligned} \pi((x, u), f_n(x, u)) &= \lim_{\delta\downarrow 0} \frac{P((x, u) + \delta f_n(x, u)) - (x, u)}{\delta}, \\ &= \begin{bmatrix} ax + bu \\ 0 \end{bmatrix} = f_g(x, u), \end{aligned}$$

for all $(x, u) \in \partial K_{+out}$. The above established the claim for all points on $\bar{K} \setminus \partial K_-$. The verification on the boundary ∂K_- is similar to that for ∂K_+. ■

Corollary 1 ([18, Corollary 1]): The GPAW compensated system Σ_g is a projected dynamical system [12] governed by

$$\dot{z} = f_g(z) = \pi(z, f_n(z)), \tag{7}$$

where $z = (x, u)$.

Corollary 1 will be used in the next section to assert the existence and uniqueness of solutions to system Σ_g. See [12]–[14] for a detailed development of PDS, and [15] for known relations to other system descriptions.

IV. Existence and Uniqueness of Solutions

As shown in [18, Claim 2], existence and uniqueness of solutions to Σ_n holds on $\mathbb{R}^2$, and the vector field f_n is globally Lipschitz, satisfying

$$\|f_n(z) - f_n(\tilde{z})\| \le (\|A\| + |b|)\|z - \tilde{z}\|, \quad \forall z, \tilde{z} \in \mathbb{R}^2, \tag{8}$$

where $A = \left[\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right]$. The following is the main result of this section.

Proposition 1 ([18, Proposition 1]): The GPAW compensated system Σ_g has a unique solution for all initial conditions $(x(t_0), u(t_0)) \in \mathbb{R}^2$ and all $t \ge t_0$.

Proof: By Corollary 1, Σ_g is a PDS governed by (7). Since $f_n\colon \mathbb{R}^2 \to \mathbb{R}^2$ is globally Lipschitz, it is Lipschitz in $\bar{K} \subset \mathbb{R}^2$. It follows from [12, Theorem 2] (see [18, Claim 3] and remark following [12, Assumption 1]) that Σ_g has a unique solution defined for all $t \ge t_0$ whenever the initial condition satisfies $(x(t_0), u(t_0)) \in \bar{K}$ (also recall Fact 1). To assert the existence and uniqueness of solutions for all initial conditions $(x(t_0), u(t_0)) \in \mathbb{R}^2$, it is sufficient to establish this outside $\bar{K}$, and if the solution enters $\bar{K}$, there will be a unique continuation in $\bar{K}$ for all future times from this result.

Consider the region $K_+ = K_{+in} \cup K_{+out} \cup \partial K_{+div}$. The proof for the region K_- is similar. For any $z_1, z_2 \in K_+$, there are three possible cases. Firstly, in the region $\hat{K}_{+out} := K_{+out} \cup \partial K_{+div}$, we get from the definition of f_g and $\hat{K}_{+out}$, that $f_g(z) = f_g(x, u) = (ax + bu_{max}, 0)$. Clearly, for any $z_1, z_2 \in \hat{K}_{+out}$, we have $\|f_g(z_1) - f_g(z_2)\| \le L_{out}\|z_1 -$

$z_2\|$ where $L_{out} = |a| < \infty$. Secondly, from Fact 2, f_g and f_n coincide in $\hat{K}_{+in} := K_{+in} \cup \partial K_{+div}$, so that f_g is also Lipschitz in $\hat{K}_{+in}$. For any $z_1, z_2 \in \hat{K}_{+in}$, we have $\|f_g(z_1) - f_g(z_2)\| \leq L_{in}\|z_1 - z_2\|$ where $L_{in} = \|A\| + |b| < \infty$ (see (8)). The last case corresponds to z_1 and z_2 being in *different* regions, $\hat{K}_{+in}$ and $\hat{K}_{+out}$. Without loss of generality, let $z_1 \in \hat{K}_{+in}$ and $z_2 \in \hat{K}_{+out}$. The straight line in $\mathbb{R}^2$ connecting z_1 and z_2 then contains a point $\tilde{z} \in \partial K_{+div}$ with the property that $\tilde{z} \in \hat{K}_{+in} \cap \hat{K}_{+out}$, $\|z_1 - \tilde{z}\| \leq \|z_1 - z_2\|$, and $\|z_2 - \tilde{z}\| \leq \|z_1 - z_2\|$. Then we have

$$\begin{aligned}\|f_g(z_1) - f_g(z_2)\| &= \|f_g(z_1) - f_g(\tilde{z}) + f_g(\tilde{z}) - f_g(z_2)\|,\\ &\leq \|f_g(z_1) - f_g(\tilde{z})\| + \|f_g(z_2) - f_g(\tilde{z})\|,\\ &\leq L_{in}\|z_1 - \tilde{z}\| + L_{out}\|z_2 - \tilde{z}\|,\\ &\leq (L_{in} + L_{out})\|z_1 - z_2\|,\end{aligned}$$

which, together with the first two cases, shows that f_g is Lipschitz in K_+. By [9, Theorem 3.1, pp. 18 – 19], Σ_g has a unique solution contained in K_+ whenever $(x(t_0), u(t_0)) \in K_+$. If the solution stays in K_+ for all $t \geq 0$, the claim holds. Otherwise, by [9, Theorem 2.1, pp. 17], the solution can be continued to the boundary of K_+, $\partial K_+ \subset \bar{K}$. In this case, the first part of the proof shows that there is a unique continuation in $\bar{K}$ for all $t \geq 0$. ■

Remark 3: Care is due when interpreting the existence and uniqueness results of Proposition 1. Let $\phi_n(t, z_0)$ be the unique solution of system Σ_n starting from $z_0 \in \mathbb{R}^2$ at time $t = 0$. For system Σ_n, existence and uniqueness of solution implies that no two different paths intersect [9, pp. 38], and

$$\phi_n(-t, \phi_n(t, z_0)) = z_0, \quad \forall t \in \mathbb{R}, \forall z_0 \in \mathbb{R}^2.$$

That is, proceeding forwards and then backwards in time by the same amount, the solution always reaches its starting point. This is not true for system Σ_g whenever the solution intersects ∂K_{+out} or ∂K_{-out}. Inspection of the vector field f_g reveals that in this case, all *forward* solutions either stay in ∂K_{+out} or ∂K_{-out} for all future times, or they eventually reach the points z_+ or z_-. Furthermore, traversing *backwards in time* from any point of ∂K_{+out} or ∂K_{-out}, the solution stays on these segments indefinitely. That is, ∂K_{+out} and ∂K_{-out} are *negative invariant sets* [9, pp. 47] for system Σ_g. If a *forward* solution of Σ_g intersects ∂K_{+out} or ∂K_{-out} starting from some *interior* point $z_0 \in K$, then traversing backwards in time, the solution will never reach z_0.

Existence and uniqueness of solutions of system Σ_g means that if two distinct trajectories, $\phi_g(t, z_1)$, $\phi_g(t, z_2)$, intersect at some time, then they will be identical for all future times, ie. if $\phi_g(T_1, z_1) = \phi_g(T_2, z_2)$ for some $T_1, T_2 \in \mathbb{R}$, then $\phi_g(t + T_1, z_1) = \phi_g(t + T_2, z_2)$ for all $t \geq 0$. Specifically, they can never diverge into two distinct trajectories. □

V. Region of Attraction

The purpose of anti-windup schemes is to provide performance improvements only in the presence of control saturation. It is widely accepted as a rule that the performance of a control system can be enhanced by trading off its robustness [16, Section 9.1]. To distinguish anti-windup schemes from conventional control methods, we consider an anti-windup scheme to be valid only if it can provide performance enhancements *without reducing the system's region of attraction (ROA)*. We show in this section that GPAW compensation can only maintain/enlarge the ROA of the nominal system Σ_n. In other words, the ROA of system Σ_n is *contained within* the ROA of Σ_g.

It was shown in [18, Claims 4, 5 and 6] that when either the open loop system (1) or nominal controller (3) is unstable, both systems Σ_n and Σ_g admits additional equilibria apart from the origin. Here, we are primarily interested in the ROA of the equilibrium point at the origin, $z_{eq0} := (0, 0) \in \mathbb{R}^2$. A distinguishing feature is that the results herein refers to the *exact ROA* in contrast to *ROA estimates* that is found in a significant portion of the literature on anti-windup compensation. We prove part of the main result (Proposition 2) using a series of intermediate claims, the proofs wherever not presented, are available in [18]. The main result is simply stated, whose proof is also in [18]. Some numerical examples will illustrate typical ROAs and show that the said ROA containment can hold strictly for some systems. In the sequel, we will state and prove results only for one side of the state space, namely with respect to $K_+ \cup \partial K_+$. The analogous results with respect to $K_- \cup \partial K_-$ can be readily extended, and will not be expressly stated.

Let $\phi_n(t, z_0)$ and $\phi_g(t, z_0)$ be the unique solutions of systems Σ_n and Σ_g respectively, both starting at initial state z_0 at time $t = 0$. The ROA of the origin z_{eq0} for systems Σ_n and Σ_g are then defined by [8, pp. 314]

$$\begin{aligned}R_n &= \{z \in \mathbb{R}^2 \mid \phi_n(t, z) \to z_{eq0} \text{ as } t \to \infty\},\\ R_g &= \{z \in \mathbb{R}^2 \mid \phi_g(t, z) \to z_{eq0} \text{ as } t \to \infty\},\end{aligned}$$

respectively. We recall the notion of ω *limit sets*.

Definition 1 ([7, Definition 2.11, pp. 44]): A point $z \in \mathbb{R}^2$ is said to be an ω *limit point* of a trajectory $\phi(t, z_0)$ if there exists a sequence of times $t_n, n \in \{1, 2, \ldots, \infty\}$ such that $t_n \uparrow \infty$ as $n \to \infty$ for which $\lim_{n\to\infty} \phi(t_n, z_0) = z$. The set of all ω limit points of a trajectory is called the ω *limit set* of the trajectory. □

For convenience, let the straight line connecting two points $\alpha, \beta \in \mathbb{R}^2$ be denoted by $l(\alpha, \beta)$ $(= l(\beta, \alpha))$, and defined by

$$l(\alpha, \beta) = \{z \in \mathbb{R}^2 \mid z = \theta\alpha + (1 - \theta)\beta, \forall\theta \in (0, 1)\}.$$

What follows is a series of intermediate claims to arrive at part of the main result, Proposition 2. Let the straight lines connecting the origin to the points z_+ and z_- be

$$\sigma_+ = l(z_{eq0}, z_+) \cup \{z_+\}, \quad \sigma_- = l(z_{eq0}, z_-) \cup \{z_-\},$$

respectively. Consider a point $z_0 \in \partial K_{+in}$ with the property that $z_0 \in R_n$ and $\phi_n(t, z_0) \notin K_+$ for all $t \geq 0$. In other words, z_0 is in the ROA of system Σ_n and its solution stays in $\bar{K} \cup K_-$ for all $t \geq 0$. As a consequence of Fact 4, $\phi_n(t, z_0)$ can never intersect ∂K_{+out} for all $t \geq 0$. Let

$$t_{int} = \inf\{t \in (0, \infty) \mid \phi_n(t, z_0) \in \sigma_+\}.$$

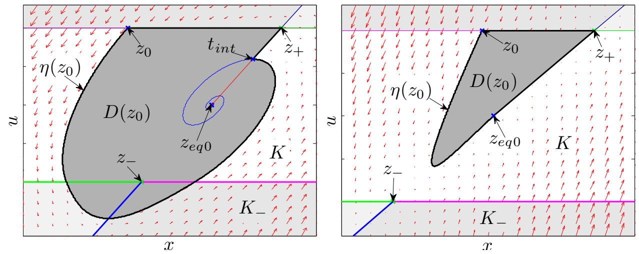


Fig. 2: Closed path $\eta(z_0)$ encloses region $D(z_0) \subset \bar{K} \cup K_-$. A case where the solution enters K_- and also intersects σ_+ is shown on the left, while a case where the solution never enters K_- and never intersects σ_+ is shown on the right.

That is, t_{int} is the first time instant that the solution starting from z_0 at $t = 0$ intersects σ_+, or ∞ if it does not intersect σ_+. If $t_{int} < \infty$, the path

$$\eta_{int}(z_0) = \{z \in \mathbb{R}^2 \mid z = \phi_n(t, z_0), \forall t \in [0, t_{int}]\} \\ \cup\, l(\phi_n(t_{int}, z_0), z_+) \cup \{z_+\} \cup l(z_0, z_+),$$

is well defined. Otherwise, the path

$$\eta_0(z_0) = \{z \in \mathbb{R}^2 \mid z = \phi_n(t, z_0), \forall t \geq 0\} \\ \cup \{z_{eq0}\} \cup \sigma_+ \cup l(z_0, z_+),$$

is well defined. Now, define the path $\eta(z_0) \in \mathbb{R}^2$ by

$$\eta(z_0) = \begin{cases} \eta_{int}(z_0), & \text{if } t_{int} < \infty, \\ \eta_0(z_0), & \text{otherwise}, \end{cases}$$

which can be verified to be closed and connected. Let the *open, bounded* region enclosed by $\eta(z_0)$ be $D(z_0)$, and its closure be $\bar{D}(z_0)$. The region $D(z_0)$ is illustrated in Fig. 2.

The following result states that $\bar{D}(z_0)$ is a *positive invariant set* [9, pp. 47], and it must contain the origin z_{eq0}.

Claim 2 ([18, Claim 7]): If there exists a point $z_0 \in \partial K_{+in}$ such that $z_0 \in R_n$ and $\phi_n(t, z_0) \in \bar{K} \cup K_-$ for all $t \geq 0$, then $\bar{D}(z_0) \subset \bar{K} \cup K_-$ is a positive invariant set for system Σ_n, and it must contain z_{eq0}, ie. $z_{eq0} \in \bar{D}(z_0)$.

Remark 4: The claim states specifically that under the assumptions, it is not possible for $\phi_n(t, z_0)$ to intersect σ_+ without having $\eta(z_0)$ enclose z_{eq0}, a case not illustrated in Fig. 2. □

Claim 3 ([18, Claim 8]): If there exists a point $z_0 \in \partial K_{+in}$ such that $z_0 \in R_n$ and $\phi_n(t, z_0) \in \bar{K}$ for all $t \geq 0$, then all points in $\bar{D}(z_0) \subset \bar{K}$ also lie in the ROA of system Σ_n, ie. $\bar{D}(z_0) \subset R_n$.

Remark 5: Specifically, the conclusion implies $z_+ \in \bar{D}(z_0) \subset R_n$. □

Proof: Since $\bar{K} \subset (\bar{K} \cup K_-)$, the hypotheses of Claim 2 are satisfied. Claim 2 then shows that $\bar{D}(z_0)$ is a positive invariant set. The condition $\phi_n(t, z_0) \in \bar{K}$ for all $t \geq 0$ implies $\bar{D}(z_0) \subset \bar{K}$. It was shown in [20, Section 6.2, pp. 353 – 363], [9, Theorem 1.3, pp. 55] that for *planar* dynamic systems with only a countable number of equilibria and with unique solutions, the ω limit set of any trajectory contained in any bounded region can only be of three types: equilibrium points, closed orbits, or *heteroclinic/homoclinic orbits* [21, pp. 45], which are unions of saddle points and the trajectories connecting them. It follows from [18, Claims 4 and 5] that the origin z_{eq0} is the *only* equilibrium point of Σ_n in $\bar{K}$, which must be a stable node or stable focus. Hence the ω limit set of any trajectory contained in $\bar{D}(z_0) \subset \bar{K}$ cannot be heteroclinic/homoclinic orbits. By Bendixson's Criterion [8, Lemma 2.2, pp. 67] and (5), region $\bar{D}(z_0)$ contains no closed orbits. As a result, the ω limit sets must consist of equilibrium points only, and it must be z_{eq0} since it is the only equilibrium point in $\bar{K}$. The conclusion follows by observing that $\bar{D}(z_0)$ is a positive invariant set, and any trajectory starting in it must converge to the ω limit set $\{z_{eq0}\}$ due to [8, Lemma 4.1, pp. 127]. ■

Claim 4 ([18, Claim 11]): If there exists a $z_0 \in \partial K_{+out} \cap R_n$, then for every $z \in l(z_0, z_+) \cup \{z_0\}$, there exists a $T(z) \in (0, \infty)$ such that the solution of system Σ_g satisfies $\phi_g(T(z), z) = z_+$ and $\phi_g(t, z) \in \partial K_{+out}$ for all $t \in [0, T(z))$.

Remark 6: Observe that under the assumptions, the solution of the GPAW compensated system $\phi_g(t, z_0)$ *slides* along the line segment ∂K_{+out} to reach z_+. Note that Fact 1 corroborates this observation. □

The next result shows that a solution of Σ_n converging to the origin can intersect ∂K_{+out} or ∂K_{-out} only in a specific way, namely that subsequent intersection points, if any, must steadily approach z_+ or z_-.

Claim 5 ([18, Claim 12]): If $z_0 \in \partial K_{+out} \cap R_n$ and there exists a $T \in (0, \infty)$ such that $\phi_n(T, z_0) \in \partial K_{+out}$, then $\phi_n(T, z_0) \in l(z_0, z_+)$.

The following is part of the main result. The proof amounts to using the solution of Σ_n to bound the solution of Σ_g.

Proposition 2 ([18, Proposition 2]): The part of the ROA of the origin of system Σ_n contained in $\bar{K}$, is itself contained within the ROA of the origin of system Σ_g, ie. $(R_n \cap \bar{K}) \subset R_g$.

Remark 7: The distinction between the solutions of systems Σ_n and Σ_g, namely $\phi_n(t, z)$ and $\phi_g(t, z)$, and their ROAs, R_n and R_g, should be kept clear when examining the proof below. □

Proof: The following argument will be used repeatedly in the present proof. If for some $z \in \bar{K}$, we have $\phi_n(t, z) \in \bar{K}$ for all $t \geq 0$, then Fact 4 implies that $\phi_n(t, z)$ cannot intersect ∂K_{+out} or ∂K_{-out}, ie. $\phi_n(t, z) \in \bar{K} \setminus (\partial K_{+out} \cup \partial K_{-out})$ for all $t \geq 0$. Fact 2 shows that f_n and f_g coincide in $\bar{K} \setminus (\partial K_{+out} \cup \partial K_{-out})$, which implies $\phi_g(t, z) = \phi_n(t, z)$ for all $t \geq 0$. If in addition, we have $\lim_{t \to \infty} \phi_n(t, z) = z_{eq0}$, then $\lim_{t \to \infty} \phi_g(t, z) = \lim_{t \to \infty} \phi_n(t, z) = z_{eq0}$. In summary, if $\phi_n(t, z) \in \bar{K}$ for all $t \geq 0$ and $z \in R_n$, then $z \in R_g$. For ease of reference, we call this the *coincidence argument*.

We need to show that if $z_0 \in R_n \cap \bar{K}$, then $z_0 \in R_g$. Let $z_0 \in R_n \cap \bar{K}$, so that $\phi_n(0, z_0) = z_0 \in \bar{K}$, and $\phi_n(t, z_0) \to z_{eq0}$ as $t \to \infty$. Consider the case where $\phi_n(t, z_0)$ stays in $\bar{K}$ for all $t \geq 0$. It follows from the *coincidence argument* that $z_0 \in R_g$.

Now, we let the solution $\phi_n(t, z_0)$ enter K_+ and consider all possible continuations. Due to Fact 4, $\phi_n(t, z_0)$ must intersect ∂K_{+out} at least once. If $\phi_n(t, z_0)$ intersects ∂K_{+out} multiple times, it can only intersect it for finitely many times. Otherwise, there is an infinite sequence of times $t_m, m \in \{1, 2, \ldots, \infty\}$ such that $t_m \uparrow \infty$ as $m \to \infty$ for which $\phi_n(t_m, z_0) \in \partial K_{+out}$. Since $z_0 \in R_n$, it follows that $\phi_n(t_m, z_0) \in \partial K_{+out} \cap R_n$ for every m. As a consequence of Claim 5, we have $\lim_{m\to\infty} \phi_n(t_m, z_0) = z_+$, which shows that z_+ is an ω limit point of $\phi_n(t, z_0)$. But this is impossible because $\lim_{t\to\infty} \phi_n(t, z_0) = z_{eq0} \neq z_+$. Similarly, if $\phi_n(t, z_0)$ intersects ∂K_{-out} multiple times, it can only intersect it for finitely many times.

Hence, let T_1 and T_2 be the first and last times for which $\phi_n(t, z_0)$ intersects ∂K_{+out}, and let T_3 be the (*only*) time after T_2 that $\phi_n(t, z_0)$ intersects ∂K_{+in}. Then we have $0 \le T_1 \le T_2 < T_3 < \infty$ and $\phi_n(t, z_0) \in K_+$ for all $t \in (T_2, T_3)$, $\phi_n(T_1, z_0), \phi_n(T_2, z_0) \in \partial K_{+out}$, and $\phi_n(T_3, z_0) \in \partial K_{+in}$, with behavior after T_3 to be specified. Let $z_1 = \phi_n(T_1, z_0) \in \partial K_{+out}$, $z_2 = \phi_n(T_2, z_0) \in \partial K_{+out}$ and $z_3 = \phi_n(T_3, z_0) \in \partial K_{+in}$. Since $z_0 \in R_n$, we have $z_1, z_2 \in \partial K_{+out} \cap R_n$ and $z_3 \in \partial K_{+in} \cap R_n$. It is clear that $\phi_g(t, z_0) = \phi_n(t, z_0)$ for all $t \in [0, T_1]$. By Claim 4, there exist a $\tilde{T}_1 < \infty$ such that $\phi_g(T_1 + \tilde{T}_1, z_0) = \phi_g(\tilde{T}_1, \phi_g(T_1, z_0)) = \phi_g(\tilde{T}_1, \phi_n(T_1, z_0)) = \phi_g(\tilde{T}_1, z_1) = z_+$. Because $\phi_n(t, z_0)$ cannot intersect ∂K_{+out} for all $t > T_2$, the only possible continuations from time T_3 ($> T_2$) onwards are

(i) $\phi_n(t, z_0)$ stays in $\bar{K}$ for all $t \ge T_3$, or
(ii) $\phi_n(t, z_0)$ enters K_- at some finite time.

Consider case (i), which implies $\bar{D}(z_3) \subset \bar{K}$. Claim 3 yields $z_+ \in \bar{D}(z_3) \subset R_n$, and Claim 2 shows that $\bar{D}(z_3)$ is a positive invariant set for system Σ_n. Then we have $\phi_n(t, z_+) \in \bar{D}(z_3) \subset \bar{K}$ for all $t \ge 0$. It follows from the *coincidence argument* that $z_+ \in R_g$. Because $\phi_g(t, z_+) = \phi_g(t, \phi_g(T_1 + \tilde{T}_1, z_0))$ for all $t \ge 0$, we have $z_0 \in R_g$, as desired.

Now, consider case (ii). Due to Fact 4, $\phi_n(t, z_0)$ must intersect ∂K_{-out} at least once. From the above discussion, $\phi_n(t, z_0)$ can intersect ∂K_{-out} only finitely many times. Let T_4 be the first time (after T_3) and T_5 be the last time for which $\phi_n(t, z_0)$ intersects ∂K_{-out}, and let T_6 be the (*only*) time after T_5 that $\phi_n(t, z_0)$ intersects ∂K_{-in}. Then $T_3 < T_4 \le T_5 < T_6 < \infty$ and $\phi_n(t, z_0) \in K_-$ for all $t \in (T_5, T_6)$, $\phi_n(T_4, z_0), \phi_n(T_5, z_0) \in \partial K_{-out}$, and $\phi_n(T_6, z_0) \in \partial K_{-in}$. Let $z_4 = \phi_n(T_4, z_0) \in \partial K_{-out}$, $z_5 = \phi_n(T_5, z_0) \in \partial K_{-out}$ and $z_6 = \phi_n(T_6, z_0) \in \partial K_{-in}$. Since $z_0 \in R_n$, we have $z_4, z_5 \in \partial K_{-out} \cap R_n$ and $z_6 \in \partial K_{-in} \cap R_n$. Now, the only possible continuation after T_6 is for $\phi_n(t, z_0) \in \bar{K}$ for all $t \ge T_6$. Recall the definition of $\eta(z)$ and $\bar{D}(z)$ for some $z \in \partial K_{+in} \cap R_n$, as illustrated in Fig. 2. It is clear that $z_+ \in \bar{D}(z_3)$. Claim 2 shows that $\bar{D}(z_3)$ (with a portion in K_-) is a positive invariant set for system Σ_n, so that $\phi_n(t, z_+) \in \bar{D}(z_3)$ for all $t \ge 0$. Recall also, that $\phi_g(T_1 + \tilde{T}_1, z_0) = z_+$ and we want to show that $z_+ \in R_g$. There are two possible ways for the solution $\phi_n(t, z_+)$ to continue. Either $\phi_n(t, z_+)$ stays in $\bar{D}(z_3) \cap \bar{K}$ for all $t \ge 0$, or it enters $\bar{D}(z_3) \cap K_-$ at some finite time. If $\phi_n(t, z_+) \in \bar{D}(z_3) \cap \bar{K}$ for all $t \ge 0$, then as in the proof of Claim 3, Bendixson's Criterion [8, Lemma 2.2, pp. 67] and the absence of saddle points in $\bar{D}(z_3) \cap \bar{K}$ means that $\{z_{eq0}\}$ is the ω limit set of $\phi_n(t, z_+)$ and hence $z_+ \in R_n$. By the *coincidence argument*, we have $z_+ \in R_g$. It follows from $\phi_g(t, z_+) = \phi_g(t, \phi_g(T_1 + \tilde{T}_1, z_0))$ for all $t \ge 0$, that $z_0 \in R_g$. Finally, consider when $\phi_n(t, z_+)$ enters $\bar{D}(z_3) \cap K_-$ at some finite time. By Fact 4, $\phi_n(t, z_+)$ must intersect ∂K_{-out} at least once. Let $\tilde{T}_2 < \infty$ be such that $\phi_n(\tilde{T}_2, z_+) \in \partial K_{-out}$ and $\phi_n(t, z_+) \in K$ for all $t \in (0, \tilde{T}_2)$, and let $\tilde{z}_2 = \phi_n(\tilde{T}_2, z_+) \in \partial K_{-out}$. Because the boundary of $\bar{D}(z_3)$ intersects ∂K_{-out} at z_4 and $\tilde{z}_2 \in \bar{D}(z_3) \cap \partial K_{-out}$, we have that $\tilde{z}_2 \in l(z_4, z_-)$. Since $z_4 \in \partial K_{-out} \cap R_n$, we have by (the analogous counterpart to) Claim 4 that there exists a $\tilde{T}_3 < \infty$ such that $\phi_g(\tilde{T}_3, \tilde{z}_2) = z_-$. Since $z_6 \in \partial K_{-in} \cap R_n$, it follows from (the analogous counterparts to) Claims 3 and 2 that $z_- \in \bar{D}(z_6) \subset R_n$, $\bar{D}(z_6)$ is a positive invariant set, and $\phi_n(t, z_-) \in \bar{D}(z_6) \subset \bar{K}$ for all $t \ge 0$. The *coincidence argument* then yields $z_- \in R_g$. Since $\phi_n(t, z_+) \in K \cup \{z_+\}$ for all $t \in [0, \tilde{T}_2)$, Fact 2 implies that $\phi_g(t, z_+) = \phi_n(t, z_+)$ for all $t \in [0, \tilde{T}_2]$. We can trace back the path to z_0 by observing that $\phi_g(t, z_-) = \phi_g(t, \phi_g(\tilde{T}_3, \tilde{z}_2)) = \phi_g(t + \tilde{T}_3, \tilde{z}_2) = \phi_g(t + \tilde{T}_3, \phi_n(\tilde{T}_2, z_+)) = \phi_g(t + \tilde{T}_3, \phi_g(\tilde{T}_2, z_+)) = \phi_g(t + \tilde{T}_3 + \tilde{T}_2, z_+) = \phi_g(t + \tilde{T}_3 + \tilde{T}_2, \phi_g(T_1 + \tilde{T}_1, z_0))$ for all $t \ge 0$. Since $z_- \in R_g$, we have $z_0 \in R_g$, as desired.

In similar manner, it can be shown that if $z_0 \in R_n \cap \bar{K}$ and the solution $\phi_n(t, z_0)$ enters K_- first, then $z_0 \in R_g$. ■

Observe that the *partial result* stated in Proposition 2 is practically meaningful because the controller state can usually be initialized in a manner such that the system state is in the unsaturated region.

A. Main Result

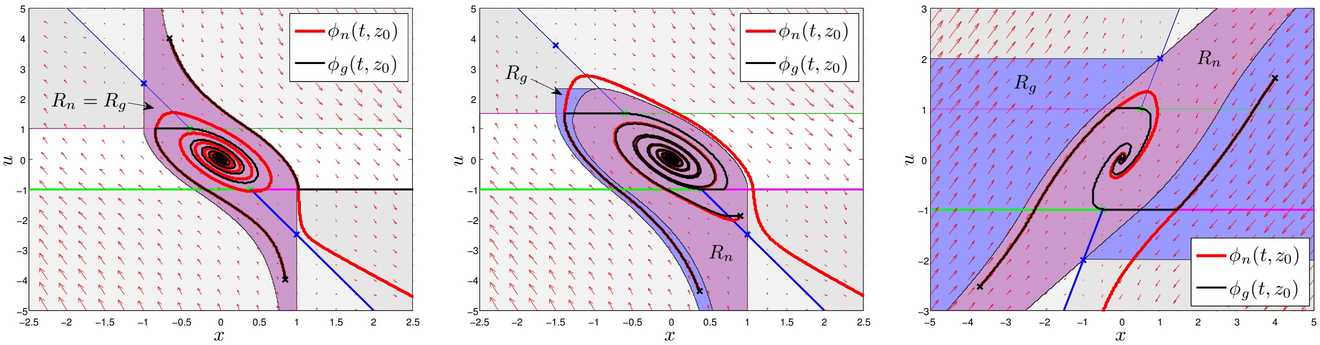
The following is the main result proved in [18]. It shows that the GPAW scheme can only maintain/enlarge the ROA of the uncompensated system, and establishes the GPAW scheme as a valid anti-windup method for this simple system.

Proposition 3 ([18, Proposition 4]): The ROA of the origin of system Σ_n is contained within the ROA of the origin of system Σ_g, ie. $R_n \subset R_g$.

B. Numerical Examples

Here, we show numerical results on the *exact* ROAs of systems Σ_n and Σ_g. The reader is reminded that in these figures, the ROAs are to be interpreted as *open* sets, since ROAs must be *open* [8, Lemma 8.1, pp. 314]. Fig. 3a shows the case where $R_n = R_g$ for an open loop unstable system, together with two pairs of representative solutions, when the saturation constraints are symmetric, ie. $u_{max} = -u_{min}$. When the same system is subjected to *asymmetric saturation constraints*, the ROAs are illustrated in Fig. 3b. Clearly, the set containment $R_n \subset R_g$ is strict. In Fig. 3c, the ROAs are illustrated for an open loop strictly stable system with the nominal controller parameter chosen to satisfy $d \in (0, -a)$. Again, the set containment $R_n \subset R_g$ is strict.

Remark 8: Observe that the case of asymmetric saturation constraints arises whenever the objective is to regulate about



(a) $a > 0, u_{max} = -u_{min}$ (b) $a > 0, u_{max} \neq -u_{min}$ (c) $a < 0, d \in (0, -a), u_{max} = -u_{min}$

Fig. 3: Numerical examples to illustrate the ROAs of systems Σ_n and Σ_g, which shows that the ROA containment $R_n \subset R_g$ of Proposition 3 can hold *strictly*. The vector field f_n is shown in the background, light purple regions represent R_n ($\subset R_g$), and light blue regions represent $R_g \setminus R_n$. In (a), the open loop system is unstable and the saturation limits are *symmetrical* ($a = 1$, $b = 1$, $c = -3$, $d = -1.2$, $u_{max} = -u_{min} = 1$), resulting in $R_n = R_g$. The pair of solutions starting at $z_0 = (0.85, -4) \in R_n \cap R_g$ converges to the origin, while the pair of solutions starting at $z_0 = (-0.66, 4) \notin R_n \cup R_g$ failed to converge to the origin. Cases (b) and (c) shows that $R_n \subset R_g$ holds *strictly*. Case (b) is identical with case (a), except with *asymmetric* saturation limits ($a = 1$, $b = 1$, $c = -3$, $d = -1.2$, $u_{max} = 1.5$, $u_{min} = -1$). Two pairs of solutions starting from $z_0 = (0.9, -1.9) \in R_n \cap R_g$ and $z_0 = (0.37, -4.37) \in R_g \setminus R_n$ are also included. A case where the open loop system is stable with an *unstable* controller is shown in (c) ($a = -1$, $b = 1$, $c = -1$, $d = 0.5$, $u_{max} = -u_{min} = 1$), together with two pairs of solutions starting from $z_0 = (-3.7, -2.54) \in R_n \cap R_g$ and $z_0 = (4, 1.6) \in R_g \setminus R_n$.

an equilibrium not lying in $\{(x, u) \in \mathbb{R}^2 \mid u = 0\}$, and the system state is transformed such that the resulting equilibrium lies at the origin. □

CONCLUSION

We analyzed the gradient projection anti-windup (GPAW) scheme when applied to a constrained first order LTI system driven by a first order LTI controller, where the objective is to regulate the system state about the origin. Existence and uniqueness of solutions are assured using results from the projected dynamical systems literature. The main result of this paper is that GPAW compensation applied to this simple system can only maintain/enlarge the system's region of attraction, which renders it a valid anti-windup method.

While these results are attractive, their applicability are severely limited. Extending these results to general MIMO nonlinear systems/controllers is a topic for future work.

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