18.06 - Spring 2005 - Problem Set 4

Solution to the MATLAB Problems

1. We have

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

The matrix C is singular, since the sum of the columns of C is the zero vector:

$$C \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The adjacency matrix of a hexagon is the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

An easy computation shows that $A^T A = C$.

2. Using MATLAB we find

$$\text{inv}(T) = \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and we may guess that the formula for $(i,j)-$entry of the $n \times n$ matrix T^{-1} is

$$\left(T^{-1}\right)_{ij} = n + 1 - \max\{i, j\}$$

To check that this really is the inverse of T, we compute $T^{-1}T$. The $j-$th column of this product, for $1 < j < n$, is equal to

$$2\begin{pmatrix} j \\ \vdots \\ j \\ j \\ j-1 \\ j-2 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} j+1 \\ \vdots \\ j+1 \\ j \\ j-1 \\ j-2 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} j-1 \\ \vdots \\ j-1 \\ j-1 \\ j-1 \\ j-2 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is in the $j-$th row. The first column of the product is

$$\begin{pmatrix} n \\ n-1 \\ n-2 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} n-1 \\ n-1 \\ n-2 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the last one is

$$2\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Thus indeed the guess above is correct.

3. Using Problem 43 of §2.5 part 2 with $M = K$ and $A = T$, and the fact that

$$K = T - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we find

$$\begin{aligned}
K^{-1} &= T^{-1} + \left(1 - \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)^{-1} T^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T^{-1} = \\
&= T^{-1} + \left(1 - \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}\right)^{-1} \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T^{-1} = \\
&= T^{-1} + \frac{1}{7} \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T^{-1} = \\
&= T^{-1} - \frac{1}{7} \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}
\end{aligned}$$

and thus

$$T^{-1} - K^{-1} = \frac{1}{7} \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$