18.06 Professor Strang Final Exam May 16, 2005

Your PRINTED name is: ____________________

Closed book / 10 wonderful problems.
Thank you for taking this course!

Grading
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1 (10 pts.) Suppose $P_1, \ldots, P_n$ are points in $\mathbf{R}^n$. The coordinates of P_i are $(a_{i1}, a_{i2}, \ldots, a_{in})$. We want to find a hyperplane $c_1x_1+\cdots+c_nx_n = 1$ that contains all n points P_i.

(a) What system of equations would you solve to find the c's for that hyperplane?

(b) Give an example in $\mathbf{R}^3$ where no such hyperplane exists (of this form), and an example which allows more than one hyperplane of this form.

(c) Under what conditions on the points or their coordinates is there **not** a unique interpolating hyperplane *with this equation*?

2 (10 pts.) (a) Find a complete set of "special solutions" to $Ax = 0$ by noticing the pivot variables and free variables (those have values 1 or 0).

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) and (c) Prove that those special solutions are a basis for the nullspace $\boldsymbol{N}(A)$. What two facts do you have to prove?? Those are parts (b) and (c) of this problem.

3 (10 pts.) (a) I was looking for an m by n matrix A and vectors b, c such that $Ax = b$ has no solution and $A^{\mathrm{T}}y = c$ has exactly one solution. *Why can I not find A, b, c?*

(b) In $\mathbf{R}^m$, suppose I gave you a vector b and a vector p and n linearly independent vectors $a_1, a_2, \ldots, a_n$. If I claim that p is the projection of b onto the subspace spanned by the a's, what tests would you make to see if this is true?

4 (10 pts.) (a) Find the determinant of

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

(b) Let A be the 5 by 5 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Find all five eigenvalues of A by noticing that $A - I$ has rank 1 and the trace of A is ______.

(c) ***Find the*** $\boldsymbol{(1,3)}$ ***and*** $\boldsymbol{(3,1)}$ ***entries of*** $\boldsymbol{A^{-1}}$***.***

5 (10 pts.) (a) Complete the matrix A (fill in the two blank entries) so that A has eigenvectors $x_1 = (3, 1)$ and $x_2 = (2, 1)$:

$$A = \begin{bmatrix} 2 & 6 \\ & \end{bmatrix}$$

(b) Find a different matrix B with those same eigenvectors x_1 and x_2, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. **What is $\boldsymbol{B}^{10}$?**

6 (10 pts.) We can find the four coefficients of a polynomial $P(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3$ if we know the values y_1, y_2, y_3, y_4 of $P(z)$ at the four points $z = 1, i, i^2, i^3$.

(a) What equations would you solve to find c_0, c_1, c_2, c_3 ?

(b) *Write down one special property of the coefficient matrix.*

(c) Prove that the matrix in those equations is invertible.

7 (10 pts.) Suppose $\mathbf{S}$ is a 4-dimensional subspace of $\mathbf{R}^7$, and P is the projection matrix onto $\mathbf{S}$.

(a) What are the seven eigenvalues of P?

(b) What are all the eigenvectors of P?

(c) If you solve $\dfrac{du}{dt} = -Pu$ (notice minus sign) starting from $u(0)$, the solution $u(t)$ approaches a steady state as $t \to \infty$. Can you describe that limit vector $u(\infty)$?

8 (10 pts.) Suppose my favorite $-1, 2, -1$ matrix swallowed extra zeros to become

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.$$

(a) Find a permutation matrix P so that

$$B = PAP^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

(b) What are the 4 eigenvalues of B? Is this matrix diagonalizable or not?

(c) How do you know that A has the same eigenvalues as B? Then A is positive definite—what function of u, v, w, z is therefore positive except when $u = v = w = z = 0$?

9 (10 pts.) (a) Describe all vectors that are orthogonal to the nullspace of this singular matrix A. You can do this without computing the nullspace.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 2 & 6 \\ 2 & 1 & 4 \end{bmatrix}.$$

(b) If you apply Gram-Schmidt to the columns of this A, what orthonormal vectors do you get?

(c) Find a "reduced" LU factorization of A, with only **2** columns in L and **2** rows in U. Can you write A as the sum of two rank 1 matrices?

10 (10 pts.) Suppose the singular value decomposition $A = U\Sigma V^{\mathrm{T}}$ has

$$U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad V = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^{\mathrm{T}}A$.

(b) Find a basis for the nullspace of A.

(c) Find a basis for the column space of A.

(d) Find a singular value decomposition of $-A^{\mathrm{T}}$.