18.06 Professor Strang Quiz 1 February 28, 2005

Grading
1
2
3
4

Your PRINTED name is: SOLUTIONS

1 (26 pts.) Suppose A is reduced by the usual row operations to

$$R = \begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the complete solution (if a solution exists) to this system involving the original A:

$$Ax = \text{ sum of the columns of } A.$$

Solution

The complete solution $x = x_{\text{particular}} + x_{\text{nullspace}}$ has

$$x_{\text{particular}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad x_{\text{nullspace}} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

The free variables x_2 and x_4 can take any values.

The two special solutions came from the nullspace of $R =$ nullspace of A.

The particular solution of 1's gives $Ax =$ sum of the columns of A.

Note: This also gives $Rx =$ sum of columns of R.

2 (18 pts.) Suppose the 4 by 4 matrices A and B have the *same column space*. They may not have the same columns!

(a) Are they sure to have the same number of pivots? YES NO WHY?

(b) Are they sure to have the same nullspace? YES NO WHY?

(c) If A is invertible, are you sure that B is invertible? YES NO WHY?

Solution

(a) YES. Number of pivots = **rank** = dimension of the column space.

This is the same for A and B.

(b) NO. The nullspace is not determined by the column space (unless we know that the matrix is symmetric.) Example with same column spaces but different nullspaces:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(c) YES. If A is invertible, its column space is the whole space $\mathbf{R}^4$. Since B has the same column space, B is also invertible.

3 (40 pts.) (a) Reduce A to an upper triangular matrix U and carry out the same elimination steps on the right side b:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} U & c \end{bmatrix}$$

Factor the 3 by 3 matrix A into $LU =$ (lower triangular)(upper triangular).

(b) If you change the last entry in A from 2 to ______ (what number gives $A_{\mathbf{new}}$?) then $A_{\mathbf{new}}$ becomes singular. *Describe its column space exactly.*

(c) In that singular case from part (b), what condition(s) on b_1, b_2, b_3 allow the system $A_{\mathbf{new}}x = b$ to be solved?

(d) *Write down the complete solution to* $A_{\mathbf{new}}x = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$ (the first column).

Solution

(a) $$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 3 & 5 & 1 & b_2 \\ -3 & 3 & 2 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 & b_1 \\ 0 & 2 & 0 & b_2 - b_1 \\ 0 & 0 & 3 & b_3 - 3b_2 + 4b_1 \end{bmatrix}$$

Here $$A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 1 \\ -3 & 3 & 2 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) If you change A_{33} from 2 to -1, the third pivot is reduced by 3 and A_{new} becomes singular. Its column space is the *plane in* $\mathbf{R}^3$ containing all combinations of the first columns $(3, 3, -3)$ and $(3, 5, 3)$.

(c) We need $b_3 - 3b_2 + 4b_1 = 0$ on the right side (since the left side is now a row of zeros).

(d) A_{new} gives $\begin{bmatrix} 3 & 3 & 1 & 3 \\ 3 & 5 & 1 & 3 \\ -3 & 3 & -1 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 3 & 1 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Certainly $x_{\text{particular}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Also $x_{\text{nullspace}} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

The complete solution is $x_{\text{particular}}+$ any vector in the nullspace.

4 (16 pts.) Suppose the columns of a 7 by 4 matrix A are linearly independent.

(a) After row operations reduce A to U or R, how many rows will be all zero (or is it impossible to tell)?

(b) What is the row space of A? Explain why this equation will surely be solvable:

$$A^{\mathrm{T}} y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solution

(a) The rank is 4, so there will be $7 - 4 = 3$ rows of zeros in U and R.

(b) The row space of A will be all of $\mathbf{R}^4$ (since the rank is 4). Then every vector c in $\mathbf{R}^4$ is a combination of the rows of A, which means that $A^{\mathrm{T}} y = c$ is solvable for every right side c.