

Conceptual Questions for Review

Chapter 1

- 1.1 Which vectors are linear combinations of $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (4, 3)$?
- 1.2 Compare the dot product of $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?
- 1.3 What is the cosine of the angle between \mathbf{v} and \mathbf{w} in Question 1.2? What is the cosine of the angle between the x -axis and \mathbf{v} ?

Chapter 2

- 2.1 Multiplying a matrix A times the column vector $\mathbf{x} = (2, -1)$ gives what combination of the columns of A ? How many rows and columns in A ?
- 2.2 If $A\mathbf{x} = \mathbf{b}$ then the vector \mathbf{b} is a linear combination of what vectors from the matrix A ? In vector space language, \mathbf{b} lies in the ____ space of A .
- 2.3 If A is the 2 by 2 matrix $\begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}$ what are its pivots?
- 2.4 If A is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ how does elimination proceed? What permutation matrix P is involved?
- 2.5 If A is the matrix $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ find \mathbf{b} and \mathbf{c} so that $A\mathbf{x} = \mathbf{b}$ has no solution and $A\mathbf{x} = \mathbf{c}$ has a solution.
- 2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?
- 2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?
- 2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB ? How many *column times row* products go into AB ? How many separate small multiplications are involved (the same for both)?
- 2.9 Suppose $A = \begin{bmatrix} \mathbf{I} & \mathbf{U} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

- 2.10 How can you find the inverse of A by working with $[A \ I]$? If you solve the n equations $A\mathbf{x} = \text{columns of } I$ then the solutions \mathbf{x} are columns of ____.
- 2.11 How does elimination decide whether a square matrix A is invertible?
- 2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U ?
- 2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?
- 2.14 What is the transpose of the inverse of AB ?
- 2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

- 3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?
- 3.2 If every column of A is a multiple of the first column, what is the column space of A ?
- 3.3 What are the two requirements for a set of vectors in \mathbf{R}^n to be a subspace?
- 3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?
- 3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $A\mathbf{x} = \mathbf{b}$?
- 3.6 From the row reduced form R , how would you decide the rank of A ?
- 3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.
- 3.8 Describe in words the complete solution to a linear system $A\mathbf{x} = \mathbf{b}$.
- 3.9 If $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every \mathbf{b} , what can you say about A ?
- 3.10 Give an example of vectors that span \mathbf{R}^2 but are not a basis for \mathbf{R}^2 .
- 3.11 What is the dimension of the space of 4 by 4 symmetric matrices?
- 3.12 Describe the meaning of *basis* and *dimension* of a vector space.
- 3.13 Why is every row of A perpendicular to every vector in the nullspace?

- 3.14 How do you know that a column \mathbf{u} times a row \mathbf{v}^T (both nonzero) has rank 1?
- 3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?
- 3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?
- 3.17 Describe a *pivot column* of A .
- 3.18 True? The vectors in the left nullspace of A have the form $A^T \mathbf{y}$.
- 3.19 Why do the columns of every invertible matrix yield a basis?

Chapter 4

- 4.1 What does the word *complement* mean about orthogonal subspaces?
- 4.2 If \mathbf{V} is a subspace of the 7-dimensional space \mathbf{R}^7 , the dimensions of \mathbf{V} and its orthogonal complement add to ____.
- 4.3 The projection of \mathbf{b} onto the line through \mathbf{a} is the vector ____.
- 4.4 The projection matrix onto the line through \mathbf{a} is $P =$ ____.
- 4.5 The key equation to project \mathbf{b} onto the column space of A is the *normal equation* ____.
- 4.6 The matrix $A^T A$ is invertible when the columns of A are ____.
- 4.7 The least squares solution to $A\mathbf{x} = \mathbf{b}$ minimizes what error function?
- 4.8 What is the connection between the least squares solution of $A\mathbf{x} = \mathbf{b}$ and the idea of projection onto the column space?
- 4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection \mathbf{p} appear in the graph?
- 4.10 If the columns of Q are orthonormal, why is $Q^T Q = I$?
- 4.11 What is the projection matrix P onto the columns of Q ?
- 4.12 If Gram-Schmidt starts with the vectors $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (1, 1)$, which two orthonormal vectors does it produce? If we keep $\mathbf{a} = (2, 0)$ does Gram-Schmidt always produce the same two orthonormal vectors?
- 4.13 True? Every permutation matrix is an orthogonal matrix.
- 4.14 The inverse of the orthogonal matrix Q is ____.

Chapter 5

- 5.1 What is the determinant of the matrix $-I$?
- 5.2 Explain how the determinant is a linear function of the first row.
- 5.3 How do you know that $\det A^{-1} = 1/\det A$?
- 5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?
- 5.5 Suppose the first row of A is 0, 0, 0, 3. What does the “big formula” for the determinant of A reduce to in this case?
- 5.6 Is the ordering (2, 5, 3, 4, 1) even or odd? What permutation matrix has what determinant, from your answer?
- 5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?
- 5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.
- 5.9 How does Cramer’s Rule give the first component in the solution to $I\mathbf{x} = \mathbf{b}$?
- 5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?
- 5.11 What is the connection between determinants and volumes?
- 5.12 Find the cross product of $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 1, 0)$ and its direction.
- 5.13 If A is n by n , why is $\det(A - \lambda I)$ a polynomial in λ of degree n ?

Chapter 6

- 6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?
- 6.2 If A is singular what does this say about its eigenvalues?
- 6.3 If A times A equals $4A$, what numbers can be eigenvalues of A ?
- 6.4 Find a real matrix that has no real eigenvalues or eigenvectors.
- 6.5 How can you find the sum and product of the eigenvalues directly from A ?
- 6.6 What are the eigenvalues of the rank one matrix $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$?
- 6.7 Explain the diagonalization formula $A = SAS^{-1}$. Why is it true and when is it true?

- 6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A ? Which might be larger?
- 6.9 Explain why the trace of AB equals the trace of BA .
- 6.10 How do the eigenvectors of A help to solve $d\mathbf{u}/dt = A\mathbf{u}$?
- 6.11 How do the eigenvectors of A help to solve $\mathbf{u}_{k+1} = A\mathbf{u}_k$?
- 6.12 Define the matrix exponential e^A and its inverse and its square.
- 6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?
- 6.14 What is the diagonalization formula when A is symmetric?
- 6.15 What does it mean to say that A is *positive definite*?
- 6.16 When is $B = A^T A$ a positive definite matrix (A is real)?
- 6.17 If A is positive definite describe the surface $\mathbf{x}^T A \mathbf{x} = 1$ in \mathbf{R}^n .
- 6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B ?
- 6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?
- 6.20 The SVD expresses A as a product of what three types of matrices?
- 6.21 How is the SVD for A linked to $A^T A$?

Chapter 7

- 7.1 Define a linear transformation from \mathbf{R}^3 to \mathbf{R}^2 and give one example.
- 7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.
- 7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?
- 7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M ?
- 7.5 Suppose our new basis is formed from the eigenvectors of a matrix A . What matrix represents A in this new basis?
- 7.6 If A and B are the matrices representing linear transformations S and T on \mathbf{R}^n , what matrix represents the transformation from \mathbf{v} to $S(T(\mathbf{v}))$?
- 7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A ?).