18.06 Exam 1 #1 Solutions

1 a)

$$\vec{v} \cdot \vec{x} = 0 \Rightarrow x_1 + 2x_2 + x_3 = 0$$
$$\vec{w} \cdot \vec{x} = 0 \Rightarrow 2x_1 + 4x_2 + 3x_3 = 0$$

So the set to be found is the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$. The row echelon form of A is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The second variable, x_2, is free and the vector $(-2, 1, 0)$ is a basis of the nullspace.

b) Since the set in a) is the nullspace of the matrix A, it is a vector space. Generally to prove a set satisfying some property, say P, is a vector space, one needs to show:
(1) If $\vec{x}$ satisfies property P, then $c\vec{x}$ also satisfies property P, for any $c \in \mathbb{R}$.
(2) If $\vec{x}, \vec{y}$ satisfy property P, then $\vec{x} + \vec{y}$ also satisfies property P.

2 a)

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0 & 3 \\ -4 & 3 & -2 \\ 8 & 9 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 9 & 23 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 0 & 47 \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} L &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 3 & 1 \end{bmatrix} \\ U &= \begin{bmatrix} -2 & 0 & 3 \\ 0 & 3 & -8 \\ 0 & 0 & 47 \end{bmatrix} \end{aligned}$$

b) To solve $A\vec{x} = LU\vec{x} = \vec{b}$, it is equivalent to solve the two equations $L\vec{y} = \vec{b}$ and $U\vec{x} = \vec{y}$.

$$\begin{aligned} L\vec{y} &= \vec{b} \Rightarrow \begin{cases} y_1 &= 3 \\ 2y_1 + y_2 &= -1 \\ -4y_1 + 3y_2 + y_3 &= 13 \end{cases} \Rightarrow \vec{y} = \begin{bmatrix} 3 \\ -7 \\ 46 \end{bmatrix}. \\ U\vec{x} &= \vec{y} \Rightarrow \vec{y} = \begin{bmatrix} \frac{-3}{94} \\ \frac{13}{47} \\ \frac{46}{47} \end{bmatrix}. \end{aligned}$$

3 a) Denote $B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 5 & 2 & 6 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, Then

$$BA = P \Rightarrow P^{-1}BA = I \Rightarrow P^{-1}B = A^{-1}$$

Because P is a permutation matrix, $P^{-1} = P^T$. So

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 5 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 7 \\ 5 & 2 & 6 \\ 1 & 2 & 4 \end{bmatrix}.$$

b) i B, D have full column rank, so the nullspace of each is the zero vector. Now

$$BD\vec{x} = 0 \Rightarrow D\vec{x} \in N(B) = \{0\} \Rightarrow D\vec{x} = 0 \Rightarrow \vec{x} = 0.$$

Hence N(BD)=0.

ii This time only B has full column rank, that is, $N(B) = \{0\}$.

$$BD\vec{x} = 0 \Rightarrow D\vec{x} \in N(B) = \{0\} \Rightarrow D\vec{x} = 0 \Rightarrow \vec{x} \in N(D).$$

So $N(BD) \subseteq N(D)$. On the other hand,

$$D\vec{x} = 0 \Rightarrow BD\vec{x} = B0 = 0 \Rightarrow x \in N(BD) \Rightarrow N(D) \subseteq N(BD).$$

So $N(D) = N(BD)$, which is all we can say about $N(BD)$ without further assumptions on D.

iii $r < n$, implies B is not of full column rank and the nullspace of B contains an infinite number of vectors. $r < m$ implies the row echelon form of B has zero rows, so the equation $B\vec{x} = \vec{b}$ has no solutions for some $\vec{b}$. Furthermore, if there is a solution to $B\vec{x} = \vec{b}$, say $\vec{x_p}$, then there are infinitely many solutions since $\vec{x_p} + \vec{x_n}$ is a solution for any $\vec{x_n}$ in $N(B)$. The answer to the question is 0 or infinitely many.

4 a) Apply row operations on A and get the following matrix

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & c-3 & -3 \\ 0 & 0 & 2(c-3) & -8 \\ 0 & 0 & 0 & d-8 \end{bmatrix}.$$

- No values of c, d will make the rank of A equal to 2.
- if $c \neq 3, d \neq 8$, R is the row echelon form of A and A has rank 4.
- Any other combination of c, d will give rank 3, that is, the rank is 3 if $c = 3$ or $d = 8$.

b) substituting $c = 3, d = 8$ in the matrix R, one finds that the third column gives a free variable, and null space of A is spanned by $(-3, 0, 1, 0)$. Use the **augmented matrix** $\left[A \mid \vec{b}\right]$ (NOT $\left[R \mid \vec{b}\right]$) to find a particular solution of the equation $A\vec{x} = \vec{b}$, which is $(-1/2, 1/4, 0, 1/4)$. So the complete solution of the equation is $(-1/2, 1/4, 0, /1/4) + x_3(-3, 0, 1, 0)$.