18.06 Exam 2 #1 Solutions

1. The row echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So we find that a basis for $R(A^T)$ is

   $\{(1, 2, -1, 4), (0, 1, -2, 3)\}$, and a basis for $N(A)$ is $\{(-3, 2, 1, 0), (2, -3, 0, 1)\}$. Similarly, row

   echelon form of A^T is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So a basis for $C(A)$ is $\{(1, 0, -1), (0, 1, 2)\}$ and a basis

   for $N(A^T)$ is $\{(1, -2, 1)\}$.

2. a) Using the row operation $R4 - R1$ gives

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix}.$$

   Using a cofactor expansion about the fourth column gives

$$\begin{vmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ -0 & -3 & 0 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ 0 & 3 & 0 \end{vmatrix}$$

$$= (-1)(3) \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = -12.$$

   (here, we computed the 3×3 determinant by expanding about the third row.

   b) Using Cramer's rule,

$$A^{-1}(1, 4) = \frac{C_{4,1}}{\det(A)} = \frac{(-1)^{4+1} \begin{vmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}}{\det(A)} = \frac{-3}{-12} = \frac{1}{4}$$

   c)

$$\det(2A^2 A^T (A^{-1})^3) = \det(2I)\det(A)^2 \det(A^T)\det(A^{-1})^3$$
$$= 2^4 \cdot \det(A)^2 \det(A)\det(A)^{-3} = 16$$

3. A basis for the space in question is $\{(1, 1, 0, 0), (2, 0, 1, 0), (-1, 0, 0, 1)\}$. To get a orthogonal basis, we need to do the Gram-Schmidt algorithm. Start with $v_1 = (1, 1, 0, 0), v_2 = (2, 0, 1, 0), v_3 = (-1, 0, 0, 1)$,

$$\tilde{v}_1 = v_1 = (1, 1, 0, 0)$$
$$\tilde{v}_2 = v_2 - \frac{(\tilde{v}_1, v_2)}{|\tilde{v}_1|^2}\tilde{v}_1 = (1, -1, 1, 0)$$
$$\tilde{v}_3 = v_3 - \frac{(\tilde{v}_1, v_3)}{|\tilde{v}_1|^2}\tilde{v}_1 - \frac{(\tilde{v}_2, v_3)}{|\tilde{v}_2|^2}\tilde{v}_2 = (-\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 1)$$

Then $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ is a set of orthogonal basis, to make them orthonormal, just multiply each $\tilde{v}_i$ by the reciprocal of its norm. So an orthonormal basis for the subspace in the problem is $\{\frac{\tilde{v}_1}{|\tilde{v}_1|}, \frac{\tilde{v}_2}{|\tilde{v}_2|}, \frac{\tilde{v}_3}{|\tilde{v}_3|}\} = \{\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{\sqrt{3}}(1,-1,1,0), \frac{1}{\sqrt{42}}(-1,1,2,6)\}$.

4. a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}, \hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\hat{x}$ is the solution to the linear equation $A^T A \hat{x} = A^T b$, and the least squares line is $y = C + Dx$.

b) $P = B(B^T B)^{-1} B^T$.