18.06 Exam 3 Solutions

1. a) $M = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$.

   b) It is the eigenvector for M corresponding to eigenvalue 1, $(0.5, 0.3)$.

   c) After many years, the percentage of people drinking two kinds of coffee will converge to one that is proportional to the steady state vector. So people drinking regular coffee will be about $5/8 = 62.5\%$.

2. a) Column vectors of Q are normalized eigenvectors of A, denote column vectors of Q by v_1, v_2, v_3. Then

$$\begin{aligned} (A-(-2)I)v_1 &= 0 \Rightarrow v_1 = (0,1,0) \\ (A-4I)v_2 &= 0 \Rightarrow v_2 = (-1,0,2)/\sqrt{5} \\ (A-(-1)I)v_3 &= 0 \Rightarrow v_3 = (2,0,1)/\sqrt{5} \end{aligned}$$

$$Q = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   b) Λ is the matrix for L under the basis $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$

3. First we find out the eigenvalues of A^TA corresponding to w_i respectively.

$$\begin{aligned} A^TAw_1 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3w_1 \\ A^TAw_2 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = w_2 \\ A^TAw_3 &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \end{aligned}$$

   So $\lambda_1 = 3, \sigma_1 = \sqrt{3}, \lambda_2 = 1, \sigma_2 = 1$, and $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

   Column vectors of V are normalized eigenvectors, $v_1 = w_1/|w_1|, v_2 = w_2/|w_2|, v_3 = w_3/|w_3|$,
   $V = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

   Column vectors of U satisfies, $u_1 = Av_1/\sigma_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $u_2 = Av_2/\sigma_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

Finally the answer is

$$SVD(A) = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

4. a)

$$L(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = L(\frac{1}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \frac{1}{2}\begin{bmatrix} 8 \\ -6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix} = -8\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the matrix for L with respect to the standard basis for $\mathbf{R}^2$ is $A = \begin{bmatrix} 4 & 0 \\ 5 & -8 \end{bmatrix}$

b) The change of basis is equal to the inverse of basis matrix, i.e. $P = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$.

c)

$$B = P^{-1}AP = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 14 \\ \frac{5}{2} & -3 \end{bmatrix}$$