8 October 1997 **Profs. S. Lee and A. Kirillov**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam I for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **40**
2. **16**
3. **16**
4. **28**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$.

[16] **1a.** Give an LU-factorization of A.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}; \quad U = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

l_{21}: row $\underline{2}$ $-(2)\times$ row $\underline{1}$
l_{31}: row $\underline{3}$ $-(3)\times$ row $\underline{1}$
l_{41}: row $\underline{4}$ $-(3)\times$ row $\underline{1}$

l_{32}: row $\underline{3}$ $-(1)\times$ row $\underline{2}$
l_{42}: row $\underline{4}$ $-(1)\times$ row $\underline{2}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}.$$

[8] **1b.** Give a basis for the column space of A.

Pivot columns in A (column 1, column 3): $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 10 \\ 10 \end{bmatrix}$

[8] **1c.** Give a basis for the nullspace of A.

Special solution(s) to $Ux = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

[8] **1d.** Give the complete solution to $Ax = \begin{bmatrix} 3 \\ 4 \\ 7 \\ 7 \end{bmatrix}$. Recall that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 3 & 6 & 10 \end{bmatrix}$.

$$x_{\text{complete}} = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 4 & 7 & 4 \\ 3 & 6 & 10 & 7 \\ 3 & 6 & 10 & 7 \end{array}\right]}_{Ax=b} \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \underbrace{\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 9 \\ 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{Rx=d}.$$

The pivot variables are x_1 and x_3; the free variable is x_2.

$$x_1 \begin{bmatrix} \boxed{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ -2 \\ 0 \\ 0 \end{bmatrix}}_{d} \text{ gives } x_{\text{particular}} = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix}.$$

The complete solution is the particular solution plus all linear combinations of the special solution(s).

[8] **2a.** If possible, give a matrix A which has $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ as a basis for the column space, and $\begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}$ as a basis for its row space. If not possible, give your reason.

$A = \begin{bmatrix} 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -2 \\ 0 & 3 & 2 & -1 \end{bmatrix}$, `or any nonzero multiple of it.`

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 & -1 \\ 0 & 6 & 4 & -2 \\ 0 & 3 & 2 & -1 \end{bmatrix}.$$

[8] **2b.** Are the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ a basis for the vector space $\mathbf{R}^3$?
(More than 'yes' or 'no' is needed for full credit.)
Show your work, then briefly explain your answer.

`YES.`

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \underbrace{\begin{bmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{-1} \end{bmatrix}}_{r=3 \text{ pivot columns}}.$$

The pivot columns in A are a basis for the column space of A.
The column space of A is a $r = 3$-dimensional subspace of $\mathbf{R}^3$
(i.e., all of $\mathbf{R}^3$).

[16] **3.** Given that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}$, find A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 8 \\ 1 & 2 & 3 \\ 2 & 5 & 3 \end{bmatrix}.}$$

`Premultiply both sides of the original equation by` $P^{-1} = P^T$.

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}}_{A^{-1}} A = I\,.$$

4. Suppose $x = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is the only solution to $Ax = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$.

[12] **4a.** Fill in each (blank) with a number.
The columns of A span a ____(blank)____-dimensional subspace of the vector space $\mathbf{R}^{\text{(blank)}}$.

The columns of A span a 3-dimensional subspace of the vector space $\mathbf{R}^5$.

$$A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$ shows that A is a $m = 5$ by $n = 3$ matrix.

The nullspace of A has dimension $\underbrace{(n - r)} = 0$ since $x = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ is unique.

A has rank $r = n = 3$.
The column space is a subspace of dimension $r = 3$ in $\mathbf{R}^5$.

[16] **4b.** After applying elementary row operations to A, the <u>reduced</u> row echelon form will be $R =$ ____(give the matrix)____.

$$R = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each pivot column in the <u>reduced</u> row echelon form R has 1 as a pivot, with zeros below and above it. From 4a, recall that R has $r = 3$ pivot columns.