3 December 1997 **Profs. S. Lee and A. Kirillov**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam III for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **30**
2. **22**
3. **16**
4. **32**

TOTAL: 100

Show all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results.
Point values (total of 100) are marked on the left margin.

1. Let $A = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$.

[10] **1a.** Find the eigenvalues of A.

$$\boxed{\lambda_1 = 4 + i\,,\ \ \lambda_2 = 4 - i\,.}$$

`Find roots of quadratic equation:`
$\det(A - \lambda I) = (4-\lambda)(4-\lambda) - (-1) = \lambda^2 - 8\lambda + 17 = 0\,.$

[10] **1b.** Find an eigenvector for each eigenvalue of A.

$$\boxed{x_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix},\ \ x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}.}$$

$$\underbrace{\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}}_{(A-\lambda_1 I)}\underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \underbrace{\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}}_{(A-\lambda_2 I)}\underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[10] **1c.** Compute $x_1^H x_2$.
(Note: x_1 and x_2 are the complex eigenvectors that you obtained in **1b**.)

$$\boxed{x_1^H x_2 = 0.}$$

2. Let $A = \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$.

[12] **2a.** Find an invertible matrix S that makes $S^{-1}AS$ a diagonal matrix.

$$\boxed{S = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}.}$$

`The diagonal entries of` A `are its eigenvalues.`
`The columns of` S `are the eigenvectors of` A `for` $\lambda_1 = -2$, $\lambda_2 = 0$.

[10] **2b.** For the differential equation $\frac{du}{dt} = Au$, give a nonzero initial vector $u(0)$ such that $u(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, or any nonzero multiple of it.

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

Choose $c_2 = 0$ and $c_1 \neq 0$ for initial vector to approach $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

[16] **3.** Fill in the matrix $A = \begin{bmatrix} 0.5 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ so that A is a positive Markov matrix with the steady state vector $x_1 = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$.

(Recall that the limit of $A^k u_0$ is always a multiple of x_1.)

$$\boxed{A = \begin{bmatrix} 1/2 & 1/6 \\ 1/2 & 5/6 \end{bmatrix}.}$$

The first column of A adds to 1 when $a_{21} = 1/2$.

Next, we solve $A = \begin{bmatrix} 1/2 & a_{12} \\ 1/2 & a_{22} \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$ for the second column of A.

Along the first row: $(1/8) + (3/4)a_{12} = 1/4$, which gives $a_{12} = 1/6$.

The second column of A adds to 1 for $a_{22} = 5/6$.

Observe that the steady state vector satisfies $Ax_1 = (1)x_1$.

4. Each <u>independent</u> question refers to the matrix $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

In each case, find the value of d that makes the statement true (and show your work!).

[10] **4a.** Give a value for d such that $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector of A.

$d = 41/25$.

$$\begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 21 \\ 5d-4 \end{bmatrix} = (21/5) \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

In the second component, solve $5d - 4 = (21/5)$ for d.

[10] **4b.** Give a value for d such that 2 is one of the eigenvalues of A.

$d = -12$.

When 2 is an eigenvalue of A,

$A - 2I = \begin{bmatrix} 2 & 1 \\ d & -6 \end{bmatrix}$ must have linearly dependent columns.

[12] **4c.** Give a value for d such that A is a nondiagonalizable matrix.

Recall that $A = \begin{bmatrix} 4 & 1 \\ d & -4 \end{bmatrix}$.

$\boxed{d = -16.}$

The issue of nondiagonalizability only comes up for a matrix that has some repeated eigenvalues.
In this case, 0 is a twice repeated eigenvalue of A when $d = -16$.
The eigenvalue is repeated twice, but we only find one linearly independent eigenvector (via the special solution to $(A - 0I)x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).