18.06 Prof. Edelman

Solutions to Selected Problems from Quiz #2 November 9, 1998

(2)

(a) The columns of A are orthogonal and of norm 2. Hence

$$A = Q \cdot (2I), \text{ where } Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}A.$$

This means $R = 2I$.

(b) $A^{-1} = (Q(2I))^{-1} = (2I)^{-1}Q^{-1} = \frac{1}{2}I \cdot Q^T = \frac{1}{4}A^T = \frac{1}{4}A$ since $A^T = A$.

(4) A is an $n \times n$ matrix.

(a) $R(A) = \mathbb{R}^n \Rightarrow \text{rank } (A) = n \Rightarrow \dim C(A) = n \Rightarrow C(A) = \mathbb{R}^n$.

(b) $N(A) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z =$ zero vector space.

(c) $N(A^T) = \mathbb{R}^n \Rightarrow \text{rank } (A) = 0 \Rightarrow C(A) = Z$

(d) Example 1 : $A = [0]$.

Example 2:

$$A = \left[\begin{array}{c|c} 0 & 0_m \\ \hline I_k & 0 \end{array}\right]$$

I_k is the $k \times k$ identity matrix.
0_m is the $m \times m$ zero matrix.

(e)

$$\left.\begin{array}{l} C(A) \perp R(A) \\ N(A) = (R(A))^{\perp} \end{array}\right\} \Rightarrow C(A) \subseteq N(A)$$

Hence $r + r = \dim C(A) + \dim R(A) \leq \dim N(A) + \dim R(A) = n$,

i.e. $\boxed{2r \leq m}$.

Example 2 above with $k = \left[\frac{n}{2}\right]$, $m = n - k$ shows that $2r \leq n$ is the strongest inequality.

(f) From (e), $r \leq \frac{n}{2} < n \Rightarrow A$ is singular $\Rightarrow \det(A) = 0$.