18.06 Professor Edelman Quiz 3 November 20, 1998

Your name is ________________________________.

1. (a.) (10 pts) Find ALL the eigenvalues and ONE eigenvector of each of the matrices below:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ -3 & 0 & -2 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of A and B.

$\det(A - \lambda I) = (5-\lambda)\begin{vmatrix} -\lambda & -1 \\ -2 & 3-\lambda \end{vmatrix} = (5-\lambda)(\lambda^2 - 3\lambda + 2) \Rightarrow \lambda = -1, -2, 5$

B is lower triangular. The eigenvalues are on the diagonal: 1, 5, -2.

1. (b.) (10 pts) Find ONLY one eigenvalue of each of the matrices below: (This can be done with no arithmetic.)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

A is singular since column 1 + column 3 = 2 x column 2 . So A has an eigenvalue 0.

$$B \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

so 5 is an eigenvalue of B.

2. (20 pts) Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ (all nonzero) and corresponding eigenvectors $x_1, \ldots, x_n$ forming a basis for R^n. Let C be its cofactor matrix. (The answers to the questions below should be in terms of the λ_i.)

(a) (5 pts) What is trace(A^{-1})? det(A^{-1})?

(b) (15 pts) What is trace(C)? What is det(C)? (Hint: $A^{-1} = \frac{C^T}{\det A}$)

(a) A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda n}$.
trace$(A^{-1}) = \frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda n}$
det$(A^{-1}) = \frac{1}{\lambda_1}\frac{1}{\lambda_2} \cdots \frac{1}{\lambda n}$.
(b) The eigenvalues of C^T are the same as that of C or det$(A)\times$ those of A^{-1}.
Thus they are $\mu_i = \frac{\lambda_1 \lambda_2 \ldots \lambda_n}{\lambda_i}$
trace$(C) = \lambda_1 \lambda_2 \ldots \lambda_n (\frac{1}{\lambda_1} + \ldots + \frac{1}{\lambda_n})$
det$(C) = (\lambda_1 \ldots \lambda_n)^{n-1}$

3. (30 pts.) Suppose A is symmetric $(n \times n)$ with rank $r = 1$ and one eigenvalue equal to 7. Let the general solution to

$$\frac{du}{dt} = -Au$$

be written as $u(t) = M(t)u(0)$. (Note the minus sign!)

(a) (5 pts.) Write down an expression for $M(t)$ in terms of A and t.
(b) (15 pts.) Is it true that for all t, trace$(M(t)) \geq$ det$(M(t))$? Explain your answer by finding all the eigenvalues of $M(t)$.
(c) (5 pts.) Can $u(t)$ blow up when $t \to \infty$? Explain.
(d) (5 pts.) Can $u(t)$ approach 0 when $t \to \infty$? Explain.

(a) $M(t) = e^{-At}$
(b) $M(t)$ has one eigenvalue e^{-7t} and the rest are 1.
(c) No blow up. All eigs are ≤ 1.
(d) If $u(0)$ is the eigenvector corresponding to to e^{7t} then $u(t)$ approaches 0.

4. (30pts.) (a). If B is invertible prove that AB has the same eigenvalues as BA. (Hint: Find a matrix M such that $ABM = MBA$.)

$M = B^{-1}$ so $AB = MBAM^{1}$ is similar to BA.

(b). Find a diagonalizable matrix $A \neq 0$ that is similar to $-A$. Also find a nondiagonalizable matrix A that is similar to $-A$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$