

Your name is: _____

For full credit, carefully explain your reasoning, as always!

1 (36 pts.) Let

$$a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} -7 \\ 2 \\ 2 \end{bmatrix}.$$

- (a) Give d as a linear combination of a , b and c .
- (b) By using Gram-Schmidt, orthogonalize a , b and c to get orthonormal vectors q_1 , q_2 and q_3 .
- (c) Give the change of basis matrix from the basis a , b and c to the basis q_1 , q_2 and q_3 .

2 (36 pts.) Let A be the 3-by-4 matrix defined by

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 + x_4 \\ x_1 + 2x_2 - x_4 \\ x_1 + x_2 + 3x_3 - 3x_4 \end{bmatrix}.$$

- (a) Give A explicitly.
- (b) Find the nullspace matrix of A^T .
- (c) What is the dimension of the column space of A ?

3 (36 pts.) (a) Let

$$A = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}.$$

Find all diagonal matrices D for which DAD^{-1} is a Jordan matrix.

(b) Find all pairs $a, b \in \mathbf{R}$ such that

$$A = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ a & b \end{bmatrix}$$

is an orthogonal matrix.

(c) Find all orthogonal upper-triangular matrices of size 3-by-3.

- 4 (32 pts.) (a) Diagonalize the 3-by-3 symmetric matrix A that corresponds to the quadratic form

$$f(x, y, z) = 4xy + 3y^2 + 4z^2.$$

(In your diagonalization $A = S\Lambda S^{-1}$, choose S to be orthogonal.)

- (b) Replace the (1, 1)-entry of A with $a > 0$. For which values of a do you get a positive definite matrix?

5 (32 pts.) Let

$$A = QR = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & a \end{bmatrix}$$

(Q has orthonormal columns).

- (a) Give an orthonormal basis of $C(A)$.
- (b) For which values of a the rank of A is 2?

(c) Solve $A^T x = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$.

- (d) Let $a = 2$. Solve $Ay = b$ in the least squares sense for

$$b = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

6 (36 pts.) Let A be a symmetric matrix of size 3-by-3 with the 3rd column $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(a) Assume A has the cofactors $C_{1,3} = 0$ and $C_{23} = 1$ and

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Give A explicitly. Is A invertible?

(b) Compute the SVD of A .

(c) With the help of the SVD, give an orthonormal basis of the column space of A .